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Towards backstepping design for incremental stability

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Abstract— **The notion of incremental stability has been successfully used as a tool for the analysis and design of intrinsic observers, output regulation of nonlinear systems, frequency estimators, synchronization of coupled identical dynamical systems, symbolic models for nonlinear control systems, and bio-molecular systems. However, most of the existing controller design techniques provide controllers enforcing stability rather than incremental stability. Hence, there is a growing need for design methods for incremental stability. In this paper, we take a step in this direction by developing a backstepping design approach to incremental stability. The effectiveness of the proposed method is illustrated by synthesizing a controller rendering a magnetic levitator incrementally stable.**

I. Introduction

Stability is a property of dynamical systems comparing trajectories with an equilibrium point. Incremental stability is a stronger property comparing arbitrary trajectories with themselves, rather than with an equilibrium point. It is well-known that for linear systems incremental stability is equivalent to stability. However, incremental stability is a stronger property than stability for nonlinear systems.

The applications of incremental stability have grown in the past years. Examples include intrinsic observer design [AR03], consensus problems in complex networks [WS05], output regulation of nonlinear systems [PvdWN05], design of frequency estimators [SK08], synchronization of coupled identical dynamical systems [RdB09], construction of symbolic models for nonlinear control systems [PGT08], [PT09], [GPT09], and the analysis of bio-molecular systems [RdBS09]. Hence, there is a growing need for design methods providing controllers enforcing incremental stability. Most of the existing design methods guarantee stability rather than incremental stability.

In [Ang02] Angeli conducted a Lyapunov based study of incremental stability. He defined incremental global asymptotic stability (δ-GAS) and incremental input-to-state stability (δ-ISS), proposed a notion of δ-GAS and δ-ISS Lyapunov function and proved the Lyapunov characterizations of δ-GAS and δ-ISS properties. In addition to Lyapunov functions, δ-GAS and δ-ISS properties can be checked by using contraction metrics. The study of contraction properties in the context of stability theory goes back to more than 40 years before being used in control theory; see [Jou05], [PPvdWN04] for a historical discussion. In control theory, contraction analysis was popularized by the work of Slotine [LS98]. Recent work related to contraction analysis can be found in [AR03], [APS08], [PTS09], [ZPJT10], [Son10]. The descriptions of δ-GAS and δ-ISS properties, based on contraction metrics, were shown in [AR03] and [ZPJT10], respectively.

Backstepping design methods providing controllers enforcing δ-GAS for parametric-strict-feedback systems[1] were proposed in [JL02], and [SK09]. In this paper, we generalize these results by developing a backstepping design method providing controllers enforcing δ-ISS and not just δ-GAS. The generalization of these results by enlarging the class of control systems from parametric-strict-feedback to strict-feedback form[1] can be found in [ZT10]. The proposed approach was inspired by the original backstepping method described, for example, in [KKK95]. Like the original backstepping method, which provides a recursive way of constructing controllers as well as Lyapunov functions, the approach proposed in this paper provides a recursive way of constructing controllers as well as contraction metrics. Our design approach is illustrated by designing a controller rendering a magnetic levitator δ-ISS.

II. Control Systems and Stability Notions

A. Notation

The symbols $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ denote the set of real, positive, and nonnegative real numbers, respectively. The symbols I_m, and 0_m denote the identity and zero matrices on $\mathbb{R}^m$. Given a vector $x \in \mathbb{R}^n$, we denote by x_i the i–th element of x, and by $\|x\|$ the Euclidean norm of x; we recall that $\|x\| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$. Given a measurable function $f : \mathbb{R}_0^+ \to \mathbb{R}^n$, the (essential) supremum of f is denoted by $\|f\|_\infty$; we recall that $\|f\|_\infty :=$ (ess)sup$\{\|f\|, t \geq 0\}$; f is essentially bounded if $\|f\|_\infty < \infty$. For a given time $\tau \in \mathbb{R}^+$, define f_τ so that $f_\tau(t) = f(t)$, for any $t \in [0, \tau)$, and $f(t) = 0$ elsewhere; f is said to be locally essentially bounded if for any $\tau \in \mathbb{R}^+$, f_τ is essentially bounded. A continuous function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the map $\beta(r, s)$ belongs to class $\mathcal{K}_\infty$ with respect to r and, for each fixed r, the map $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.

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[1] See [KKK95] for a definition of parametric-strict-feedback and strict-feedback systems.

B. Control Systems

The class of control systems that we consider in this paper is formalized in the following definition.

Definition 2.1: A *control system* is a quadruple:

$$\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f),$$

where:

- $\mathbb{R}^n$ is the state space;
- $\mathsf{U} \subseteq \mathbb{R}^m$ is the input space;
- $\mathcal{U}$ is a subset of the set of all locally essentially bounded functions of time from intervals of the form $]a, b[\subseteq \mathbb{R}$ to U with $a < 0$, $b > 0$;
- $f : \mathbb{R}^n \times \mathsf{U} \to \mathbb{R}^n$ is a continuous map satisfying the following Lipschitz assumption: for every compact set $Q \subset \mathbb{R}^n$, there exists a constant $Z \in \mathbb{R}^+$ such that $\|f(x, u) - f(y, u)\| \leq Z\|x - y\|$ for all $x, y \in Q$ and all $u \in \mathsf{U}$.

A curve $\xi :]a, b[\to \mathbb{R}^n$ is said to be a *trajectory* of Σ if there exists $\upsilon \in \mathcal{U}$ satisfying:

$$\dot{\xi}(t) = f(\xi(t), \upsilon(t)), \tag{II.1}$$

for almost all $t \in\,]a, b[$. We also write $\xi_{x\upsilon}(t)$ to denote the point reached at time t under the input υ from initial condition $x = \xi_{x\upsilon}(0)$; this point is uniquely determined, since the assumptions on f ensure existence and uniqueness of trajectories [Son98]. We also denote an autonomous system Σ with no control inputs by $\Sigma = (\mathbb{R}^n, f)$. A control system Σ is said to be forward complete if every trajectory is defined on an interval of the form $]a, \infty[$. Sufficient and necessary conditions for a system to be forward complete can be found in [AS99]. A control system Σ is said to be smooth if f is an infinitely differentiable function of its arguments.

C. Stability notions

Here, we recall the notions of incremental global asymptotic stability (δ-GAS) and incremental input-to-state stability (δ-ISS).

Definition 2.2 ([Ang02]): A control system Σ is incrementally globally asymptotically stable (δ-GAS) if it is forward complete and there exists a $\mathcal{KL}$ function β such that for any $t \in \mathbb{R}_0^+$, any $x, x' \in \mathbb{R}^n$ and any $\upsilon \in \mathcal{U}$ the following condition is satisfied:

$$\|\xi_{x\upsilon}(t) - \xi_{x'\upsilon}(t)\| \leq \beta(\|x - x'\|, t). \tag{II.2}$$

Whenever the origin is an equilibrium point for Σ, δ-GAS implies global asymptotic stability (GAS).

Definition 2.3 ([Ang02]): A control system Σ is incrementally input-to-state stable (δ-ISS) if it is forward complete and there exist a $\mathcal{KL}$ function β and a $\mathcal{K}_\infty$ function γ such that for any $t \in \mathbb{R}_0^+$, any $x, x' \in \mathbb{R}^n$, and any υ, $\upsilon' \in \mathcal{U}$ the following condition is satisfied:

$$\|\xi_{x\upsilon}(t) - \xi_{x'\upsilon'}(t)\| \leq \beta(\|x - x'\|, t) + \gamma(\|\upsilon - \upsilon'\|_\infty). \tag{II.3}$$

By observing (II.2) and (II.3), it is readily seen that δ-ISS implies δ-GAS while the converse is not true in general. Moreover, if the origin is an equilibrium point for Σ, δ-ISS implies input-to-state stability (ISS).

D. Descriptions of incremental stability

One of the methods for checking δ-GAS and δ-ISS properties consists in using Lyapunov functions. The Lyapunov characterizations of δ-GAS and δ-ISS properties were developed in [Ang02]. In this paper we follow an alternative approach based on contraction metrics. The notion of contraction metric was popularized in control theory by the work of Slotine [LS98]. Before going through the next definition, we need to introduce variational systems and the notion of a Riemannian metric.

The variational system associated with a smooth autonomous system $\Sigma = (\mathbb{R}^n, f)$ is given by the differential equation:

$$\frac{d}{dt}(\delta\xi) = \left.\frac{\partial f}{\partial x}\right|_{x=\xi} \delta\xi, \tag{II.4}$$

where $\delta\xi$ is the variation[2] of a trajectory of Σ. Similarly, the variational system associated with a smooth control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ is given by the differential equation:

$$\frac{d}{dt}(\delta\xi) = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\xi \\ u=\upsilon}} \delta\xi + \left.\frac{\partial f}{\partial u}\right|_{\substack{x=\xi \\ u=\upsilon}} \delta\upsilon, \tag{II.5}$$

where $\delta\xi$ and $\delta\upsilon$ are variations of a state and an input trajectory of Σ, respectively.

A Riemannian metric $G : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a smooth map on $\mathbb{R}^n$ such that, for any $x \in \mathbb{R}^n$, $G(x)$ is a symmetric positive definite matrix [Lee03]. For any $x \in \mathbb{R}^n$ and smooth functions $I, J : \mathbb{R}^n \to \mathbb{R}^n$, one can define the scalar function $\langle I, J\rangle_G$ as $I^T(x)G(x)J(x)$. We will still use the notation $\langle I, J\rangle_G$ to denote $I^T G J$ even if G does not represent any Riemannian metric.

Definition 2.4 ([LS98]): Let $\Sigma = (\mathbb{R}^n, f)$ be a smooth autonomous system equipped with a Riemannian metric G. System Σ is said to be an exponential contraction with respect to the metric G, if there exists some $\lambda \in \mathbb{R}^+$ such that:

$$\langle X, X\rangle_F \leq -\lambda\langle X, X\rangle_G \tag{II.6}$$

for $F(x) = \left(\frac{\partial f}{\partial x}\right)^T G(x) + G(x)\frac{\partial f}{\partial x} + \frac{\partial G}{\partial x} f(x)$, any $X \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$, or equivalently:

$$X^T\left(\left(\frac{\partial f}{\partial x}\right)^T G(x) + G(x)\frac{\partial f}{\partial x} + \frac{\partial G}{\partial x} f(x)\right) X \leq -\lambda X^T G(x) X, \tag{II.7}$$

where the constant λ is called contraction rate.
Note that the inequality (II.6) or (II.7) implies:

$$\frac{d}{dt}\langle\delta\xi, \delta\xi\rangle_G \leq -\lambda\langle\delta\xi, \delta\xi\rangle_G, \tag{II.8}$$

where $\delta\xi$ is the variation of a state trajectory of the autonomous system Σ.

We say that a smooth autonomous system Σ is an exponential contraction if there exists a Riemannian metric G

[2] The variation $\delta\xi$ can be formally defined by considering a family of trajectories $\xi_{x\upsilon}(t, \epsilon)$ parametrized by $\epsilon \in \mathbb{R}$. The variation of the state is then $\delta\xi = \frac{\partial \xi_{x\upsilon}}{\partial \epsilon}$.

such that Σ is an exponential contraction with respect to the metric G.

The following theorem shows that the inequality (II.6) implies δ-GAS.

Theorem 2.5: If a smooth autonomous system $\Sigma = (\mathbb{R}^n, f)$ is an exponential contraction, then it is δ-GAS.

Different variations of this result appeared in [LS98], and [AR03]; see [AR03] for a concise proof.

The following definition is a generalization of Definition 2.4 for control systems.

Definition 2.6 ([ZPJT10]): Let $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ be a smooth control system on $\mathbb{R}^n$ equipped with a Riemannian metric G. Control system Σ is said to be an exponential contraction with respect to the metric G, if there exists some $\lambda \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}_0^+$ such that:

$$\langle X, X\rangle_F + 2\left\langle \frac{\partial f}{\partial u} Y, X\right\rangle_G \leq -\lambda \langle X, X\rangle_G + \alpha \langle X, X\rangle_G^{\frac{1}{2}} \langle Y, Y\rangle_{I_m}^{\frac{1}{2}} \tag{II.9}$$

for $F(x,u) = \left(\frac{\partial f}{\partial x}\right)^T G(x) + G(x)\frac{\partial f}{\partial x} + \frac{\partial G}{\partial x} f(x,u)$, any $X \in \mathbb{R}^n$, and $Y \in \mathbb{R}^m$, or equivalently:

$$\begin{aligned} X^T\left(\left(\frac{\partial f}{\partial x}\right)^T G(x) + G(x)\frac{\partial f}{\partial x} + \frac{\partial G}{\partial x} f(x,u)\right) X \\ +2Y^T\left(\frac{\partial f}{\partial u}\right)^T G(x) X \leq -\lambda X^T G(x) X \\ +\alpha (X^T G(x) X)^{\frac{1}{2}} (Y^T Y)^{\frac{1}{2}}, \end{aligned} \tag{II.10}$$

where the constant λ is called contraction rate.

Note that the inequality (II.9) or (II.10) implies:

$$\frac{d}{dt}\langle \delta\xi, \delta\xi\rangle_G \leq -\lambda \langle \delta\xi, \delta\xi\rangle_G + \alpha \langle \delta\xi, \delta\xi\rangle_G^{\frac{1}{2}} \langle \delta\upsilon, \delta\upsilon\rangle_{I_m}^{\frac{1}{2}}, \tag{II.11}$$

where $\delta\xi$ and $\delta\upsilon$ are variations of a state and an input trajectory of the control system Σ.

We say that a smooth control system Σ is an exponential contraction if there exists a Riemannian metric G such that Σ is an exponential contraction with respect to the metric G.

The following theorem shows that the inequality (II.9) implies δ-ISS.

Theorem 2.7 ([ZPJT10]): If a smooth control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ is an exponential contraction, then it is δ-ISS.

In the next section, we propose a backstepping design procedure to make control systems incrementally stable.

III. Backstepping Design Procedure

The method described here was inspired by the original backstepping described, for example, in [KKK95]. Consider the class of control systems $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathbb{R}, f)$ with f of the parametric-strict-feedback form [KKK95]:

$$\begin{aligned} f_1(x,u) &= h_1(x_1) + b_1 x_2, \\ f_2(x,u) &= h_2(x_1,x_2) + b_2 x_3, \\ &\vdots \\ f_{n-1}(x,u) &= h_{n-1}(x_1,\cdots,x_{n-1}) + b_{n-1}x_n, \\ f_n(x,u) &= h_n(x) + g(x)u, \end{aligned} \tag{III.1}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}$ is the control input. The functions $h_i : \mathbb{R}^i \to \mathbb{R}$, for $i = 1, \ldots, n$, and $g : \mathbb{R}^n \to \mathbb{R}$ are smooth, $g(x) \neq 0$ over the domain of interest, and $b_i \in \mathbb{R}$, for $i = 1, \ldots, n$, are nonzero constants. We can now state one of the main results, describing a backstepping controller for the control system (III.1).

Theorem 3.1: For any control system $\Sigma = (\mathbb{R}^n, \mathbb{R}, \mathcal{U}, f)$ with f of the form (III.1) and for any $\lambda \in \mathbb{R}^+$, the state feedback control law:

$$k(x) = \frac{1}{g(x)}\Big[k_n(x) - h_n(x)\Big], \tag{III.2}$$

where

$$\begin{aligned} k_l(x) &= -b_{l-1}\left(x_{l-1} - \phi_{l-2}(x)\right) - \frac{\lambda}{2}\left(x_l - \phi_{l-1}(x)\right) \\ &\quad + \frac{\partial \phi_{l-1}}{\partial x} f(x,k(x)), \text{ for } l = 1, \cdots, n, \quad \text{(III.3)} \\ \phi_l(x) &= \frac{1}{b_l}\Big[k_l(x) - h_l(x)\Big], \text{ for } l = 1, \cdots, n-1, \\ \phi_{-1}(x) &= \phi_0(x) = 0 \ \forall x \in \mathbb{R}^n, \ b_0 = 0, \text{ and } x_0 = 0, \end{aligned}$$

renders the control system Σ δ-GAS with contraction rate λ.

Proof: Consider the following system:

$$\Sigma_l : \begin{cases} \dot{\eta}_l = F_l(\eta_l) + B_l \xi_l, \\ \dot{\xi}_l = k_l(\eta_l, \xi_l), \end{cases} \tag{III.4}$$

where $\eta_l = [\xi_1, \cdots, \xi_{l-1}]^T$, $B_l = [0, \cdots, 0, b_{l-1}]^T \in \mathbb{R}^{l-1}$, $z_l = [y_l^T \ x_l]^T \in \mathbb{R}^l$ is the state of Σ_l, $y_l = [x_1, \cdots, x_{l-1}]^T$, and $F_l(y_l) = [f_1(x,u), \cdots, f_{l-2}(x,u), h_{l-1}(x_1, \cdots, x_{l-1})]^T$. By using induction on l, we show that the system (III.4) is an exponential contraction with respect to the contraction metric G_l, defined by:

$$G_l(y_l) = \begin{bmatrix} G_{l-1}(y_{l-1}) + \left(\frac{\partial \phi_{l-1}}{\partial y_l}\right)^T \frac{\partial \phi_{l-1}}{\partial y_l} & -\left(\frac{\partial \phi_{l-1}}{\partial y_l}\right)^T \\ -\frac{\partial \phi_{l-1}}{\partial y_l} & 1 \end{bmatrix}, \tag{III.5}$$

where the contraction rate is λ, and $G_1(y_1) = 1$. For $l = 1$, it can be easily checked that $G_1(y_1) = 1$ is a contraction metric with the contraction rate λ for the scalar system:

$$\Sigma_1 : \dot{\xi}_1 = k_1(\xi_1) = -\frac{\lambda}{2}\xi_1.$$

Assume that the system Σ_{k-1}, for some $3 \leq k \leq n$, is an exponential contraction with respect to the contraction metric G_{k-1} and with contraction rate λ. This implies:

$$\begin{aligned} \Big[Y^T \ X\Big]\Bigg(\left(\frac{\partial(F_k + B_k\phi_{k-1})}{\partial y_k}\right)^T G_{k-1}(y_{k-1}) \\ + G_{k-1}(y_{k-1})\frac{\partial(F_k + B_k\phi_{k-1})}{\partial y_k} \\ + \frac{\partial G_{k-1}}{\partial y_k}(F_k + B_k\phi_{k-1})\Bigg)\begin{bmatrix} Y \\ X \end{bmatrix} \\ \leq -\lambda\Big[Y^T \ X\Big] G_{k-1}(y_{k-1})\begin{bmatrix} Y \\ X \end{bmatrix}, \end{aligned} \tag{III.6}$$

for any $Y \in \mathbb{R}^{k-2}$, and $X \in \mathbb{R}$. Since the metric G_{k-1} is only a function of $y_{k-1} = [x_1, \cdots, x_{k-2}]^T$, and the vector B_k has zero entries except the last entry, it can be easily shown that $\frac{\partial G_{k-1}}{\partial y_k} B_k = 0_{k-1}$, and the inequality (III.6) reduces to:

$$\begin{aligned} &\left[Y^T \; X\right] \left(\left(\frac{\partial (F_k + B_k \phi_{k-1})}{\partial y_k} \right)^T G_{k-1}(y_{k-1}) \right. \\ &\left. + G_{k-1}(y_{k-1}) \frac{\partial (F_k + B_k \phi_{k-1})}{\partial y_k} + \frac{\partial G_{k-1}}{\partial y_k} F_k \right) \begin{bmatrix} Y \\ X \end{bmatrix} \\ &\leq -\lambda \left[Y^T \; X\right] G_{k-1}(y_{k-1}) \begin{bmatrix} Y \\ X \end{bmatrix}. \end{aligned} \tag{III.7}$$

Now, we show that:

$$G_k(y_k) = \begin{bmatrix} G_{k-1}(y_{k-1}) + \left(\frac{\partial \phi_{k-1}}{\partial y_k}\right)^T \frac{\partial \phi_{k-1}}{\partial y_k} & -\left(\frac{\partial \phi_{k-1}}{\partial y_k}\right)^T \\ -\frac{\partial \phi_{k-1}}{\partial y_k} & 1 \end{bmatrix}, \tag{III.8}$$

is a contraction metric for the system Σ_k. Positive definiteness of $G_k(y_k)$ follows from:

$$\begin{aligned} &\left[Y^T \; X\right] G_k(y_k) \begin{bmatrix} Y \\ X \end{bmatrix} = \left[Y^T \; X\right] \cdot \\ &\begin{bmatrix} G_{k-1}(y_{k-1}) + \left(\frac{\partial \phi_{k-1}}{\partial y_k}\right)^T \frac{\partial \phi_{k-1}}{\partial y_k} & -\left(\frac{\partial \phi_{k-1}}{\partial y_k}\right)^T \\ -\frac{\partial \phi_{k-1}}{\partial y_k} & 1 \end{bmatrix} \\ &\cdot \begin{bmatrix} Y \\ X \end{bmatrix} = Y^T G_{k-1}(y_{k-1}) Y + \left(\frac{\partial \phi_{k-1}}{\partial y_k} Y - X\right)^2. \end{aligned}$$

Since G_{k-1} is a metric and $[Y^T \; X]^T \in \mathbb{R}^k$ is a nonzero vector, the metric G_k is positive definite. Using the inequality (III.7), the long algebraic manipulations in (III.9) show that G_k satisfies (II.7) with the contraction rate λ. Hence, the metric G_k is a contraction metric for the system Σ_k. Therefore, for any $l \leq n$, the system (III.4) is an exponential contraction with respect to the contraction metric G_l and with the contraction rate λ.

The proposed control law (III.2), transforms the control system of the form (III.1) to:

$$\Sigma_n : \begin{cases} \dot{\eta}_n = F_n(\eta_n) + B_n \xi_n, \\ \dot{\xi}_n = k_n(\eta_n, \xi_n), \end{cases} \tag{III.10}$$

which, as shown by induction, is an exponential contraction with respect to the contraction metric G_n and with the contraction rate λ. By using Theorem 2.5, we conclude that the control system of the form (III.1), equipped with the state feedback control law (III.2), is δ-GAS. The δ-GAS condition (II.2), as shown in [AR03], is given by:

$$d_{G_n}\left(\xi_{xv}(t), \xi_{x'v}(t)\right) \leq e^{-\frac{\lambda}{2}t} d_{G_n}(x, x'), \tag{III.11}$$

where $d_{G_n}(x, x')$ is the distance induced by the metric G_n between the points x and x'. ■

Remark 3.2: The contraction metric for the control system (III.1), equipped with the state feedback control law (III.2), is given by (III.12), where $y_l = [x_1, \cdots, x_{l-1}]^T$, for $l = 2, \cdots, n$, and the contraction rate is λ.

Remark 3.3: It can be checked that the function

$$V(x) = \frac{1}{2} \sum_{l=0}^{n-1} \left(x_{l+1} - \phi_l(x)\right)^2,$$

is a GAS Lyapunov function [Kha96] for the control system (III.1), equipped with the state feedback control law (III.2). Moreover, the hessian of $V(x)$ is equal to the contraction metric G_n, defined in (III.12).

Control law (III.2) can be modified to enforce also δ-ISS.

Theorem 3.4: For any control system $\Sigma = (\mathbb{R}^n, \mathbb{R}, \mathcal{U}, f)$ with f of the form (III.1) and for any $\lambda \in \mathbb{R}^+$, the state feedback control law:

$$k(x, \widehat{u}) = \frac{1}{g(x)} \Big[k_n(x) - h_n(x)\Big] + \frac{1}{g(x)} \widehat{u}, \tag{III.13}$$

where

$$\begin{aligned} k_l(x) &= -b_{l-1}\left(x_{l-1} - \phi_{l-2}(x)\right) - \frac{\lambda}{2}\left(x_l - \phi_{l-1}(x)\right) \\ &\quad + \frac{\partial \phi_{l-1}}{\partial x} f(x, k(x, \widehat{u})), \text{ for } l = 1, \cdots, n, \quad \text{(III.14)} \\ \phi_l(x) &= \frac{1}{b_l}\Big[k_l(x) - h_l(x)\Big], \text{ for } l = 1, \cdots, n-1, \\ \phi_{-1}(x) &= \phi_0(x) = 0 \quad \forall x \in \mathbb{R}^n, \; b_0 = 0, \text{ and } x_0 = 0, \end{aligned}$$

renders the control system Σ δ-ISS with respect to the input $\widehat{\upsilon}$ and with contraction rate λ.

Proof: The proof of this result is similar to the proof of Theorem 3.1. The interested reader can find a complete proof in [ZT10].

The δ-ISS condition (II.3), as shown in [ZPJT10], is given by:

$$d_{G_n}\left(\xi_{x\widehat{\upsilon}}(t), \xi_{x'\widehat{\upsilon}'}(t)\right) \leq e^{-\frac{\lambda}{2}t} d_{G_n}(x, x') + \frac{2}{\lambda} \|\widehat{\upsilon} - \widehat{\upsilon}'\|_\infty, \tag{III.15}$$

where $d_{G_n}(x, x')$ is the distance induced by the metric G_n between the points x and x'. ■

Remark 3.5: The contraction metric for the control system of the form (III.1), equipped with the state feedback control law (III.13), is given by (III.12).

Remark 3.6: It can be shown that the function

$$V(x) = \frac{1}{2} \sum_{l=0}^{n-1} \left(x_{l+1} - \phi_l(x)\right)^2, \tag{III.16}$$

is an ISS Lyapunov function [Kha96] with respect to $\widehat{\upsilon}$ for the control system of the form (III.1), equipped with the state feedback control law (III.13). Moreover, the hessian of $V(x)$ is equal to the contraction metric G_n, defined in (III.12).

As described in [ZT10], Theorems 3.1 and 3.4 can also be generalized to systems in strict-feedback form:

$$\begin{aligned} f_1(x, u) &= h_1(x_1) + g_1(x_1) x_2, \\ f_2(x, u) &= h_2(x_1, x_2) + g_2(x_1, x_2) x_3, \\ &\vdots \\ f_{n-1}(x, u) &= h_{n-1}(x_1, \cdots, x_{n-1}) \\ &\quad + g_{n-1}(x_1, \cdots, x_{n-1}) x_n, \\ f_n(x, u) &= h_n(x) + g_n(x) u, \end{aligned} \tag{III.17}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}$ is the control input. The functions $h_i : \mathbb{R}^i \to \mathbb{R}$, and $g_i : \mathbb{R}^i \to \mathbb{R}$, for $i = 1, \ldots, n$, are smooth, and $g_i(x_1, \cdots, x_i) \neq 0$ over the domain of interest.

$$
\left[Y^T\ X\right]\left(\left(\frac{\partial\left[F_k^T+B_k^T x_k\ \ k_k(x)\right]^T}{\partial z_k}\right)^T G_k(y_k)+G_k(y_k)\frac{\partial\left[F_k^T+B_k^T x_k\ \ k_k(x)\right]^T}{\partial z_k}+\dot{G}_k(y_k)\right)\begin{bmatrix} Y \\ X \end{bmatrix}= \tag{III.9}
$$

$$
\left[Y^T\ X\right]\left(\begin{bmatrix} \frac{\partial F_k}{\partial y_k} & B_k \\ (F_k+B_k x_k)^T\frac{\partial^2\phi_{k-1}}{\partial {y_k}^2}+\frac{\partial\phi_{k-1}}{\partial y_k}\frac{\partial F_k}{\partial y_k}+\frac{\lambda}{2}\frac{\partial\phi_{k-1}}{\partial y_k}-B_k^T G_{k-1}(y_{k-1}) & -\frac{\lambda}{2}+\frac{\partial\phi_{k-1}}{\partial y_k}B_k \end{bmatrix}^T\cdot\right.
$$

$$
\begin{bmatrix} G_{k-1}(y_{k-1})+\left(\frac{\partial\phi_{k-1}}{\partial y_k}\right)^T\frac{\partial\phi_{k-1}}{\partial y_k} & -\left(\frac{\partial\phi_{k-1}}{\partial y_k}\right)^T \\ -\frac{\partial\phi_{k-1}}{\partial y_k} & 1 \end{bmatrix}+\begin{bmatrix} G_{k-1}(y_{k-1})+\left(\frac{\partial\phi_{k-1}}{\partial y_k}\right)^T\frac{\partial\phi_{k-1}}{\partial y_k} & -\left(\frac{\partial\phi_{k-1}}{\partial y_k}\right)^T \\ -\frac{\partial\phi_{k-1}}{\partial y_k} & 1 \end{bmatrix}\cdot
$$

$$
\begin{bmatrix} \frac{\partial F_k}{\partial y_k} & B_k \\ (F_k+B_k x_k)^T\frac{\partial^2\phi_{k-1}}{\partial {y_k}^2}+\frac{\partial\phi_{k-1}}{\partial y_k}\frac{\partial F_k}{\partial y_k}+\frac{\lambda}{2}\frac{\partial\phi_{k-1}}{\partial y_k}-B_k^T G_{k-1}(y_{k-1}) & -\frac{\lambda}{2}+\frac{\partial\phi_{k-1}}{\partial y_k}B_k \end{bmatrix}+
$$

$$
\left.\begin{bmatrix} \frac{\partial G_{k-1}}{\partial y_k}(F_k+B_k x_k)+\frac{\partial^2\phi_{k-1}}{\partial {y_k}^2}(F_k+B_k x_k)\frac{\partial\phi_{k-1}}{\partial y_k}+\left(\frac{\partial\phi_{k-1}}{\partial y_k}\right)^T(F_k+B_k x_k)^T\frac{\partial^2\phi_{k-1}}{\partial {y_k}^2} & -\frac{\partial^2\phi_{k-1}}{\partial {y_k}^2}(F_k+B_k x_k) \\ -(F_k+B_k x_k)^T\frac{\partial^2\phi_{k-1}}{\partial {y_k}^2} & 0 \end{bmatrix}\right)\cdot
$$

$$
\begin{bmatrix} Y \\ X \end{bmatrix}=\left[Y^T\ X\right]\cdot
$$

$$
\begin{bmatrix} \left(\left(\frac{\partial(F_k+B_k\phi_{k-1})}{\partial y_k}\right)^T G_{k-1}(y_{k-1})+G_{k-1}(y_{k-1})\frac{\partial(F_k+B_k\phi_{k-1})}{\partial y_k}+\frac{\partial G_{k-1}}{\partial y_k}F_k\right)-\lambda\left(\frac{\partial\phi_{k-1}}{\partial y_k}\right)^T\frac{\partial\phi_{k-1}}{\partial y_k} & \lambda\left(\frac{\partial\phi_{k-1}}{\partial y_k}\right)^T \\ \lambda\frac{\partial\phi_{k-1}}{\partial y_k} & -\lambda \end{bmatrix}\cdot
$$

$$
\begin{bmatrix} Y \\ X \end{bmatrix}\leq-\lambda\left[Y^T\ X\right]G_k(y_k)\begin{bmatrix} Y \\ X \end{bmatrix}.
$$

$$
G_n(y_n)= \tag{III.12}
$$

$$
\begin{bmatrix} \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} [1]+\left(\frac{\partial\phi_1}{\partial y_2}\right)^T\frac{\partial\phi_1}{\partial y_2} & -\left(\frac{\partial\phi_1}{\partial y_2}\right)^T \\ -\frac{\partial\phi_1}{\partial y_2} & 1 \end{bmatrix}+\left(\frac{\partial\phi_2}{\partial y_3}\right)^T\frac{\partial\phi_2}{\partial y_3} & -\left(\frac{\partial\phi_2}{\partial y_3}\right)^T \\ -\frac{\partial\phi_2}{\partial y_3} & 1 \end{bmatrix}+\cdots \\ \vdots \end{bmatrix}+\left(\frac{\partial\phi_{n-1}}{\partial y_n}\right)^T\frac{\partial\phi_{n-1}}{\partial y_n} & -\left(\frac{\partial\phi_{n-1}}{\partial y_n}\right)^T \\ -\frac{\partial\phi_{n-1}}{\partial y_n} & 1 \end{bmatrix},
$$

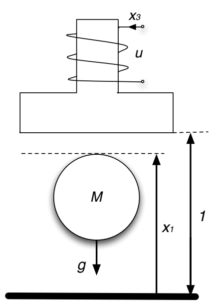

Fig. 1. A magnetic levitator.

IV. Backstepping Controller Design for a Magnetic Levitator

We illustrate the results in this paper on a magnetic levitator [JS09] shown in Figure 1.

We have the following model for the levitator:

$$
\Sigma:\begin{cases} \dot{\xi}_1=\frac{\xi_2}{M}, \\ \dot{\xi}_2=\frac{\xi_3}{2\alpha}-Mg, \\ \dot{\xi}_3=-\frac{2R}{\alpha}(1-\xi_1)\xi_3+2\sqrt{\xi_3}\upsilon, \end{cases} \tag{IV.2}
$$

where $x=[x_1\ x_2\ x_3]^T$ is the state of Σ, x_1 is the ball displacement, x_2 is its momentum, x_3 is the square of the flux linkage associated with the electromagnet, u is the voltage applied to the electromagnet, g is gravity's acceleration, M is the mass of the ball, R is the coil resistance, and α is a positive constant that depends on the number of coil turns. By using the results in Theorem 3.4 for the control system (IV.2) and for $\lambda=2$, we have:

$$
\begin{aligned}
k_1(x) &= -x_1, \\
\phi_1(x) &= -Mx_1, \\
k_2(x) &= -\left(M+\frac{1}{M}\right)x_1-2x_2, \\
\phi_2(x) &= -2\alpha\left(M+\frac{1}{M}\right)x_1-4\alpha x_2+2\alpha Mg, \\
k_3(x) &= -\left(\frac{M}{2\alpha}+2\alpha\left(M+\frac{1}{M}\right)\right)x_1 \\
&\quad -\left(\frac{1}{2\alpha}+4\alpha+\frac{2\alpha}{M}\left(M+\frac{1}{M}\right)\right)x_2 \\
&\quad -3x_3+6\alpha Mg.
\end{aligned}
$$

$$G(x) = \begin{bmatrix} \begin{bmatrix} 1+\left(\frac{\partial\phi_1}{\partial y_1}\right)^T\frac{\partial\phi_1}{\partial y_1} & -\left(\frac{\partial\phi_1}{\partial y_1}\right)^T \\ -\frac{\partial\phi_1}{\partial y_1} & 1 \end{bmatrix} + \left(\frac{\partial\phi_2}{\partial z_2}\right)^T\frac{\partial\phi_2}{\partial z_2} & -\left(\frac{\partial\phi_2}{\partial z_2}\right)^T \\ -\frac{\partial\phi_2}{\partial z_2} & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1+M^2+4\alpha^2\left(M+\frac{1}{M}\right)^2 & M+8\alpha^2\left(M+\frac{1}{M}\right) & 2\alpha\left(M+\frac{1}{M}\right) \\ M+8\alpha^2\left(M+\frac{1}{M}\right) & 1+16\alpha^2 & 4\alpha \\ 2\alpha\left(M+\frac{1}{M}\right) & 4\alpha & 1 \end{bmatrix}, \quad \text{(IV.1)}$$

Therefore, the state feedback control law:

$$\begin{aligned} k(x,\widehat{u}) &= \frac{1}{g(x)}\left[k_3(x)-h_3(x)\right]+\frac{1}{g(x)}\widehat{u} \qquad \text{(IV.3)} \\ &= \frac{1}{2\sqrt{x_3}}\Bigg[-\left(\frac{M}{2\alpha}+2\alpha\left(M+\frac{1}{M}\right)\right)x_1 \\ &\quad -\left(\frac{1}{2\alpha}+4\alpha+\frac{2\alpha}{M}\left(M+\frac{1}{M}\right)\right)x_2 \\ &\quad -3x_3+6\alpha Mg+\frac{2R}{\alpha}\left(1-x_1\right)x_3\Bigg] \\ &\quad +\frac{1}{2\sqrt{x_3}}\widehat{u}, \end{aligned}$$

makes the control system (IV.2) δ-ISS with respect to the input $\widehat{\upsilon}$. The corresponding contraction metric for the control system (IV.2) is given by (IV.1), where $z_2^T = [x_1\ x_2]^T$.

The δ-ISS condition in (II.3) is as follows:

$$\|\xi_{x\widehat{\upsilon}}(t)-\xi_{x'\widehat{\upsilon}'}(t)\| \leq \frac{\sqrt{\lambda_{\max}}}{\sqrt{\lambda_{\min}}}e^{-t}\|x-x'\|+\frac{1}{\sqrt{\lambda_{\min}}}\|\widehat{\upsilon}-\widehat{\upsilon}'\|_\infty,$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are maximum and minimum eigenvalues of the metric G, respectively.

V. Discussion

In this paper we extended the backstepping procedure to the design of controllers enforcing incremental stability. The proposed backstepping procedure provides a recursive way of constructing controllers as well as contraction metrics. An example was provided to illustrate the proposed technique.

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