Maximizing Available Diversity in a MIMO Fading Channel

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***Abstract*—The effects of Rayleigh fading on a wireless communications link and MIMO technique to combat fading is explored in this paper. In harnessing spatial diversity, the use of Quasi-Orthogonal space-time block code (QO-STBC) is proposed to combat fading. QO-STBC codes are able to achieve a unity rate with more than two antennas. The concept of constellation rotation for QO codes to maximize the available diversity is demonstrated. QO codes are shown by simulation to be able to achieve full diversity, despite the fact that the codes are non-orthogonal.**

I. Introduction

Communication systems have an extremely important role in the conduct of any military operation. The heart of network centric warfare (NCW) lies in the Command, Control, Communications, Computer and Intelligence (C4I) system. Therefore, reliable and robust communication links are the key to the success of a NCW concept.

Communication systems can either be wired or wireless. Wired systems are employed where extensive infrastructure can be built up beforehand and are required over a long time. These are typically used in command and control (C2) systems with strategic importance. Some examples of wired communication systems include C2 command centers, operating bases, and key communications infrastructure such as network and communication backbones. Wireless systems, on the other hand, are used where mobility is required or when time and accessibility to an area of operations are limited and do not allow for the setup of a wired communications architecture. Therefore, tactical communications links such as those found in operating units, sensor, and weapon systems typically use wireless communications link.

Wireless communications present a distinct set of challenges that affect their link reliability. This is predominantly due to the fact that as the wireless signal travels through the air, it suffers severe attenuation (due to the unguided nature of the medium), interference (from competing sources), various atmospheric effects (such as diffraction, refraction and scattering), as well as other geometric effects (movement of transmitters and receivers or the environment, interference due to multipath). All of these effects result in a random fluctuation of the received signals strength. This fluctuation can result in periods when the signal strength falls below the detectable threshold, which lead to a complete loss of communication. This fluctuation is called fade [1]–[3], and periods of complete outage are called deep fade. A wireless communications link that is experiencing fading can perform poorly, even when the bit error rate (BER) performance predicted by a simple additive white Gaussian noise (AWGN) channel is very good. Fading is a unique phenomenon experienced by a wireless channel that is typically not found in wired communications.

Fading can be overcome with diversity. Diversity is a technique that sends multiple copies of a message from the transmitter to the receiver in order to improve the reliability of the transmission. This is based on the premise that simultaneous outages will not occur on the multiple copies. The message can be replicated in the time, frequency, or spatial domains.

Since Alamouti's landmark paper [4] on transmit diveristy, there has been considerable research into harnessing transmit diversity from all domains, including the spatial domain [5]–[12], space-frequency domain [13]–[16] and space-time-frequency domain [17], [18]. In each of these papers, the researchers address the various challenges of harnessing the available diversity given the available channel conditions.

The objective of this paper is to address fading using spatial diversity. In harnessing spatial diversity, the use of space-time coding is proposed to combat fading. Several codes of diversity two and four are explored, including Quasi-Orthogonal (QO) codes that are able to achieve a unity rate. The concept of constellation rotation for QO codes to maximize the available diversity is studied. QO codes are shown by simulation to be able to achieve full diversity, despite the fact that the codes are non-orthogonal.

The paper is organized as follows: Section II describes the theory of spatial diversity technique and Section III evaluates the performance of the space-time codes. Lastly, we conclude in Section IV.

II. Spatial Diversity

Since Alamouti's landmark paper on a class of space-time block codes with simple decoding complexity [4], there has been considerable research into designing codes with higher orders of diversity as well as exploiting diversity in the spatial, time, and frequency domains simultaneously (for example, [5]–[22]). The performance of a few space-time block codes that harness spatial diversity with varying degrees of diversity is explored in this section.

A. Space-Time Block Codes

Given a quasi-static, frequency-flat fading MIMO system with n transmit antenna and m receive antenna, a space-time block code can exploit the spatial diversity with a diversity order equivalent to nm provided that each spatial path from each pair of transmit-receive antennas experiences independent fading characteristics from one another. A MISO system can be seen as a special case of MIMO, when $m = 1$, i.e., there is only one receiving antenna on the receiver end. This is usually the case for portable communications devices, which communicate with a base station, when the need to maintain a small device limits the ability to implement multiple antennas with uncorrelated spatial paths. However, the base station does not have such limitations and can house multiple antennas spaced sufficiently far apart to meet the requirements for uncorrelated fading paths, which are typically in the region of 10λ, where λ is the wavelength of the carrier. For example, given a typical C band communications system operating at 5 GHz with $\lambda = 60\ mm$, each antenna has to be spaced at least 60 cm apart. This is easily achieved at a base station but is not practical in a portable device.

The scope of this paper is restricted to the performance evaluation to MISO systems with four transmit antennas and one receive antenna, thereby enabling up to a maximum diversity order of four. Three distinct codes, two of which are orthogonal and one quasi-orthogonal, are studied with three different modulation schemes. The three codes are a rate 1 Alamouti 2x diversity code, a rate $\frac{1}{2}$ orthogonal 4x diversity code, and a rate 1 quasi-orthogonal 4x diversity code. The modulation schemes used are QPSK, 16-QAM and 64-QAM. BPSK has the same bit error performance as QPSK and is not explicitly simulated. All cases assume a quasi-static flat fading environment where the channel taps remain constant for the duration of the code block, i.e., for four sample durations in the case of a 4x4 code.

B. Orthogonal Space-Time Block Codes

The term orthogonal space-time code borrows the mathematical concept of an orthogonal matrix. With reference to [5], the definition of an orthogonal code is given as follows:

Definition 1*: An orthogonal space-time block code with a code matrix* $\boldsymbol{G}$ *of dimension* $T \times n$*, where T is the number of time slots and n the number of transmit antenna has a Gramian matrix* $\boldsymbol{G}^{\mathcal{H}}\boldsymbol{G}$ *such that* $\boldsymbol{G}^{\mathcal{H}}\boldsymbol{G} = \left(|s_1|^2 + |s_2|^2 + ... + |s_k|^2\right) I$ *where* s_i *are the entries in the code matrix* $\boldsymbol{G}$ *corresponding to the symbols being transmitted. Such a code is said to have code rate of* k/T.

Orthogonal codes have the important property that the transmit sequences are orthogonal to one another. This implies that all the transmit symbols can be decoupled from one another. This results in a simple maximum likelihood (ML) decoder that allows the use of independent decision statistics for every symbol transmitted. The two orthogonal codes presented below illustrate this.

1) Rate 1 Orthogonal 2x Diversity Code (Alamouti Code): The Alamouti code was the first space-time block code that achieved full spatial diversity while maintaining orthogonality between symbols, which leads to a simple decoder implementation. It is also the only code to achieve full orthogonal diversity with a code rate of 1. That is, one symbol is transmitted per time slot. Denoting $\mathbf{G}$ as the code matrix, where the row index corresponds to the time index and the column index corresponds to the antenna, we have

$$\mathbf{G} = \begin{bmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{bmatrix} \tag{1}$$

where s_1 and s_2 are the modulated symbols and a^* denotes the complex conjugate of a. Antenna 1 transmits s_1 and $-s_2^*$ at time slots 1 and 2, respectively.

2) Rate $\frac{1}{2}$ Orthogonal 4x Diversity Code: It has been proven that no orthogonal codes beyond diversity two can achieve full unity rate for a complex constellation [5]. However, it is possible to construct any arbitrary rate $\frac{1}{2}$ code for diversity beyond two. To evaluate the performance of a rate $\frac{1}{2}$ orthogonal code, an example is taken from [1] with the code matrix given as

$$\mathbf{G} = \begin{bmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2 & s_1 & -s_4 & s_3 \\ -s_3 & s_4 & s_1 & -s_2 \\ -s_4 & -s_3 & s_2 & s_1 \\ s_1^* & s_2^* & s_3^* & s_4^* \\ -s_2^* & s_1^* & -s_4^* & s_3^* \\ -s_3^* & s_4^* & s_1^* & -s_2^* \\ -s_4^* & -s_3^* & s_2^* & s_1^* \end{bmatrix} \tag{2}$$

C. Quasi-Orthogonal Space-Time Code

There exists another class of codes known as the quasi-orthogonal space-time block codes [6]–[11], sometimes abbreviated as QO-STBC. As described in Section II-B2, no orthogonal codes with full unity rate exist beyond a diversity order of two when using any modulation schemes with a complex constellation. Therefore, the main motivation behind the research on QO-STBC is to exploit higher orders of diversity beyond two while maintaining full unity code rate.

Surprisingly, there does not seem to exist an official definition of quasi-orthogonal codes based on existing literature; although, there is an implicit understanding that QO-STBC are STBCs that have $\mathbf{H}$ matrices (the matrix of channel taps) that are not orthogonal, i.e., $\mathbf{H}^{\mathcal{H}}\mathbf{H} \neq \|h\|^2\,\mathbf{I}$, where $\mathbf{I}$ is the identity matrix . From the proposed definition in [6], we define a QO-STBC as follows:

Definition 2*: A QO-STBC of dimension* $N \times N$ *has a* $\boldsymbol{H}$ *matrix such that* $\boldsymbol{H}^{\mathcal{H}}\boldsymbol{H}$ *is a sparse matrix with* $\|h\|^2$ *on its main diagonal, at least* $N^2/2$ *zero entries on its off-diagonal,* *and the magnitude for the rest of the entries being some value bounded by* $\pm \|h\|^2$.

This definition of a QO-STBC will become clear in an example in the following subsection.

1) Rate 1 Quasi-Orthogonal (QO) 4x Diversity Code: Given the QO-STBC in [8], the code matrix is

$$\mathbf{G} = \begin{bmatrix} s_1 & s_2 & s_3 & s_4 \\ s_2^* & -s_1^* & s_4^* & -s_3^* \\ s_3 & -s_4 & -s_1 & s_2 \\ s_4^* & s_3^* & -s_2^* & -s_1^* \end{bmatrix}. \tag{3}$$

Given the same mapping sequence used for the Alamouti code, the system model is given as $\mathbf{Y} = \sqrt{\frac{\rho}{4}}\mathbf{Hs} + \mathbf{N}$, where $\mathbf{s}$ is the modulated symbols vector $[s_1\ s_2\ s_3\ s_4]^T$, $\mathbf{N}$ is the normalized noise vector with variance of two and $\mathbf{H}$ is given by

$$\mathbf{H} = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 \\ -h_2^* & h_1^* & -h_4^* & h_3^* \\ -h_3 & h_4 & h_1 & -h_2 \\ -h_4^* & -h_3^* & h_2^* & h_1^* \end{bmatrix}. \tag{4}$$

The detailed derivation for $\mathbf{H}$ is left out due to space limitation. Note that $\mathbf{H}$ is not orthogonal as

$$\mathbf{H}^{\mathcal{H}}\mathbf{H} = \begin{bmatrix} \|\mathbf{h}\|^2 & 0 & a & 0 \\ 0 & \|\mathbf{h}\|^2 & 0 & -a \\ -a & 0 & \|\mathbf{h}\|^2 & 0 \\ 0 & a & 0 & \|\mathbf{h}\|^2 \end{bmatrix} \tag{5}$$

where $a = h_3h_1^* - h_1h_3^* + h_2h_4^* - h_4h_2^* = 2j\ Im(h_3h_1^* + h_2h_4^*)$ and j denotes the imaginary number $j = \sqrt{-1}$. From (5), it is clear that s_1 is orthogonal to s_2 and s_4 but not s_3 (columns 2 and 4 on row 1 are zero but column 3 is non-zero). Similarly, s_2 is orthogonal to s_1 and s_3 but not s_4 (columns 1 and 3 on row 2 are zero but column 4 is non-zero). This implies that the symbol pair (s_1, s_3) is orthogonal to (s_2, s_4), but each of the symbols within the symbol pair contributes an interference factor of $\pm a$ to each other. The interference term $2j\ Im(h_3h_1^* + h_2h_4^*)$ is bounded by $\pm j \|h\|^2$. An equivalent way of saying this is that $a/\|h\|^2$ is bounded by $\pm j$. These bounds become intuitively clear by letting $|h_i| = 1$ for all channel taps; then $a/\|h\|^2 = 2j(1+1)/4 = j$. Also note that the matrix in (5) satisfies Definition 2 as there is a sparse matrix with $\|h\|^2$ on its diagonal, eight zero entries and four non-zero entries bounded by $\pm j \|h\|^2$ in the off-diagonal.

From the above analysis, it is clear that the single-symbol ML decoding scheme used in the orthogonal codes will not work for a quasi-orthogonal code. However, a *pairwise* ML decoding for the symbol pair (s_1, s_3) and (s_2, s_4) can be done separately. This is still simpler than decoding all four symbols simultaneously. This increases the decoding complexity from $\mathcal{O}(m)$ to $\mathcal{O}(m^2)$, where m is the number of constellation points, as the pairwise ML decoder needs to try every possible *pair* of symbols.

2) Constellation Rotation for QO-STBC to Achieve Full Diversity: The above QO-STBC scheme does not achieve full diversity if the symbol pairs are chosen from the same constellation [8]–[11]. In the high SNR region, where wireless links typically operate, high order diversity has been shown to be much more important to the BER performance of the wireless link as the gradient of the BER versus SNR curves, which are steeper with high orders of diversity [3], [9], [13], [14]. Several references [8]–[11] propose that the constellation for one set of the symbols be chosen from a rotated copy of the other symbol set in order to maximize diversity. The optimum angle to rotate the constellation depends on the actual constellation being used. From [9] and [12] and given two distinct codeword matrices $\mathbf{C}$ and $\tilde{\mathbf{C}}$ such that $\left\{\mathbf{C}, \tilde{\mathbf{C}}\right\} \in \mathbf{G}$, where $\mathbf{G}$ is a valid QO-STBC code matrix, the pairwise error probability (PEP) $P(\mathbf{C} \to \tilde{\mathbf{C}})$ in a Rayleigh fading channel is given by

$$P(\mathbf{C} \to \tilde{\mathbf{C}}) \le \frac{1}{2}\left(\prod_{i=1}^{r} \lambda_i\right)^{-m} \left(\frac{\rho}{4n}\right)^{-rm} \tag{6}$$

where λ_i are the non-zero eigenvalues of the matrix $\mathbf{A}$ given by $\mathbf{A} = (\mathbf{C}\ \text{-}\tilde{\mathbf{C}})^{\mathcal{H}}(\mathbf{C}\ \text{-}\tilde{\mathbf{C}})$, r is the rank of the matrix $\mathbf{B}$ given by $\mathbf{B} = (\mathbf{C}\ \text{-}\tilde{\mathbf{C}})$, ρ is the SNR, n is the number of transmit antenna, and m is the number of receive antenna. Furthermore, (6) is only valid if $\mathbf{B}$ is of full rank, i.e., for every $\mathbf{C} \neq \tilde{\mathbf{C}}$, the hyperpoint spanned by $(\mathbf{C}\ \text{-}\tilde{\mathbf{C}})$ is a unique point within the entire hyperspace of all possible permutations of $(\mathbf{C}\ \text{-}\tilde{\mathbf{C}})$. In other words, no two distinct $\{\mathbf{C}, \tilde{\mathbf{C}}\}$ codeword pairs can result in the same received signal $\mathbf{Y}$. Since the purpose is to design QO-STBC that are of full diversity, $\mathbf{B}$ must be of full rank.

From (6), it is clear that in order to minimize the PEP of the code matrix $\mathbf{G}$, the rank of matrix $\mathbf{B}$ and the product of the eigenvalues of $\mathbf{A}$ must be maximized. Therefore, the well-known *rank and determinant criteria* in [12] are as follows:

Rank Criterion (also called the Diversity Criterion) states that the minimum rank of the code difference matrix $\mathbf{B}|_{\mathbf{C} \neq \tilde{\mathbf{C}}} = (\mathbf{C}\ \text{-}\tilde{\mathbf{C}})$ must be as large as possible in order to achieve maximum diversity. This implies that $\mathbf{B}$ must be of full-rank and, thus, invertible.

Determinant Criterion (also called the Product Criterion) states that the product of the non-zero eigenvalues of the matrix $\mathbf{A} = \mathbf{B}^{\mathcal{H}}\mathbf{B}$ must be as large as possible to achieve the maximum coding gain. Note that if $\mathbf{B}$ is of full rank, then $\mathbf{A}$ will be of full rank as well, and the product of the eigenvalues $\prod_{i=1}^{r} \lambda_i$ of $\mathbf{A}$ is simply the determinant of $\mathbf{A}$.

If $\mathbf{A}$ and $\mathbf{B}$ are of full rank, then $\prod_{i=1}^{r} \lambda_i = det(\mathbf{A})$. Substituting this into (6), we get

$$\begin{aligned} P(\mathbf{C} \to \tilde{\mathbf{C}}) &\le \frac{1}{2}\left(det(\mathbf{A})\right)^{-m} \left(\frac{\rho}{4n}\right)^{-rm} \\ &= \frac{1}{2}\left(\left(det(\mathbf{A})\right)^{1/r} \frac{\rho}{4n}\right)^{-rm} \end{aligned}$$

$$= \frac{1}{2}\left(\frac{(det(\mathbf{A}))^{1/r}}{4n}\rho\right)^{-rm}$$
$$= \frac{1}{2}\left(\zeta^2_{\{\mathbf{C},\tilde{\mathbf{C}}\}}\rho\right)^{-rm} \qquad (7)$$

where $\zeta^2_{\{\mathbf{C},\tilde{\mathbf{C}}\}} = (det(\mathbf{A}))^{1/r}/4n$ is defined for a specific codeword pair $\{\mathbf{C},\tilde{\mathbf{C}}\}$. Considering the PEP across all possible permutations of the codeword pair $\{\mathbf{C},\tilde{\mathbf{C}}\}$, we see that the worst performance is dictated by the minimum determinant of $\mathbf{A}$ obtained by substituting all possible permutations of $\{\mathbf{C},\tilde{\mathbf{C}}\}$, i.e.,

$$\zeta^2 = \frac{1}{4n}\min_{\mathbf{C}\neq\tilde{\mathbf{C}}}\left|det\left[(\mathbf{C}\text{ -}\tilde{\mathbf{C}})^{\mathcal{H}}(\mathbf{C}\text{ -}\tilde{\mathbf{C}})\right]\right|^{1/r}. \qquad (8)$$

The term ζ is known as the *diversity product* and can be calculated by taking the square root of (8). The union bound on the PEP of the any given code $\mathbf{G}$ is given as follows:

$$P(\mathbf{C}\rightarrow\tilde{\mathbf{C}}) \leq \frac{1}{2}\left(\zeta^2\rho\right)^{-rm}. \qquad (9)$$

Now, consider the QO-STBC given in (3) assuming that there is only one receiver antenna. As analyzed in Section II-C1, the symbol pairs (s_1, s_3) and (s_2, s_4) are orthogonal to each other, but the individual symbols within the pair are not orthogonal. In order to maximize the diversity of the QO-STBC, the encoder must maximize the rank and determinant of the codeword difference matrix ($\mathbf{C}$ -$\tilde{\mathbf{C}}$) of the symbol pair. This is equivalent to maximizing the diversity product given in (8). Since (s_1, s_3) and (s_2, s_4) are orthogonal to each other, the minimum achievable diversity is at least of order two since the rank of $\mathbf{B}$ is at least two. To achieve the maximum diversity of four for the QO-STBC, the code design only needs to consider the non-orthogonal elements of the QO-STBC, i.e., either (s_1, s_3) or (s_2, s_4). By considering either the symbol pair (s_1, s_3) or (s_2, s_4) in $\mathbf{G}$, we can break the matrix into four different sub-matrices as follows:

$$\mathbf{G}_{13} = \begin{bmatrix} s_1 & s_3 \\ s_3 & -s_1 \end{bmatrix} \; \mathbf{G}_{24} = \begin{bmatrix} s_2 & s_4 \\ -s_4 & s_2 \end{bmatrix}$$
$$\mathbf{G}_{1^*3^*} = \begin{bmatrix} -s_1^* & -s_3^* \\ s_3^* & -s_1^* \end{bmatrix} \; \mathbf{G}_{2^*4^*} = \begin{bmatrix} s_2^* & s_4^* \\ s_4^* & -s_2^* \end{bmatrix}. \qquad (10)$$

The above non-orthogonal sub-matrices are derived from $\mathbf{G}$ by simply deleting the appropriate rows and columns such that we are left with either (s_1, s_3) or (s_2, s_4). The ranks of these matrices are either zero or two depending on the actual symbols being transmitted. Therefore, from the rank and determinant criterion, the objective of maximizing the diversity in $\mathbf{G}$ is equivalent to maximizing the diversity product of its sub-matrices. In addition, it does not matter which of the sub-matrices is used to evaluate the diversity product as $det(\mathbf{G}_{13}^{\mathcal{H}}\mathbf{G}_{13}) = det(\mathbf{G}_{1^*3^*}^{\mathcal{H}}\mathbf{G}_{1^*3^*}) = det(\mathbf{G}_{24}^{\mathcal{H}}\mathbf{G}_{24}) = det(\mathbf{G}_{2^*4^*}^{\mathcal{H}}\mathbf{G}_{2^*4^*})$ if the same symbol pairs are sent using any of the four sub-matrices. Since all possible permutations of the symbol pairs will be evaluated, the diversity product for all four sub-matrices must be the same as well.

Consider the sub-matrix $\mathbf{G}_{13}$, the determinant of its Gramian matrix $\mathbf{G}_{13}^{\mathcal{H}}\mathbf{G}_{13}$ expressed as

$$det(\mathbf{G}_{13}^{\mathcal{H}}\mathbf{G}_{13}) = (s_1^2 + s_3^2)((s_1^*)^2 + (s_3^*)^2). \qquad (11)$$

Let $\mathbf{C}_{13}$ and $\tilde{\mathbf{C}}_{13}$ be two distinct code words of the sub-matrix $\mathbf{G}_{13}$, and define the Gramian of the code difference matrix $\mathbf{A}_{13} = (\mathbf{C}_{13} - \tilde{\mathbf{C}}_{13})^{\mathcal{H}}(\mathbf{C}_{13} - \tilde{\mathbf{C}}_{13})$. The determinant of the $\mathbf{A}_{13}$ can be derived from (11) by inspection as

$$det(\mathbf{A}_{13}) = ((s_1-\tilde{s}_1)^2+(s_3-\tilde{s}_3)^2)((s_1^*-\tilde{s}_1^*)^2+(s_3^*-\tilde{s}_3^*)^2). \qquad (12)$$

Note that if $(s_1, \tilde{s}_1)$ and $(s_3, \tilde{s}_3)$ are selected from the same constellation, then $det(\mathbf{A}_{13})$ can potentially be zero, e.g., when $(s_1 - \tilde{s}_1) = j(s_3 - \tilde{s}_3)$. The only situation where $\mathbf{A}_{13}$ can be of full rank when $(s_1, \tilde{s}_1)$ and $(s_3, \tilde{s}_3)$ are chosen from the same constellation is when the constellation consists of only one basis vector, e.g., in ASK where constellation points are purely real. Therefore, in order to achieve full diversity, $(s_1, \tilde{s}_1)$ must be chosen from a different constellation from $(s_3, \tilde{s}_3)$ to eliminate the possibility of $det(\mathbf{A}_{13})$ being zero.

Let $\mathcal{A}$ and $\mathcal{B}$ be the signal constellations for $(s_1, \tilde{s}_1)$ and $(s_3, \tilde{s}_3)$, respectively, and $\mathcal{B}$ is a rotated version of $\mathcal{A}$ given by $\mathcal{B} = \mathcal{A}e^{j\theta}$. With a properly chosen rotation angle, $\mathbf{A}_{13}$ can be of full rank, and the problem now is to maximize the diversity product in accordance with the determinant criterion. Combining (8) with (12), we get

$$\zeta^2 = \frac{1}{4n}\min_{\mathbf{C}\neq\tilde{\mathbf{C}}}\left|((s_1-\tilde{s}_1)^2+(s_3-\tilde{s}_3)^2)((s_1^*-\tilde{s}_1^*)^2+(s_3^*-\tilde{s}_3^*)^2)\right|^{1/r} \qquad (13)$$

where $(s_1, s_3) \in \mathbf{C}$ and $(\tilde{s}_1, \tilde{s}_3) \in \tilde{\mathbf{C}}$ and $(s_1, \tilde{s}_1) \in \mathcal{A}$ and $(s_3, \tilde{s}_3) \in \mathcal{B}$. Given that n = 4 and r = 2 for our QO-STBC and taking the square root of (13) produces

$$\zeta = \frac{1}{4}\min_{\mathbf{C}\neq\tilde{\mathbf{C}}}\left|((s_1-\tilde{s}_1)^2+(s_3-\tilde{s}_3)^2)\right|^{1/2}. \qquad (14)$$

With (14), a numerical search for the rotation angle for constellation $\mathcal{B}$ can be performed to maximize ζ. Reference [9] provides an analytical proof that the optimal rotation angle for rectangular constellations is 45^o. This claim is validated in Section III.

III. Performance Evaluation

We begin with the evaluation of the optimum rotation angle for three different modulation schemes (QPSK, 16-QAM and 64-QAM) as well as the simulation results using various angles of constellation rotation. The constellation maps used for the modulation, based on the IEEE 802.16-2009, are located on page 631 of [23].

A. Evaluation of Optimal Angle of Rotation

From (14), the diversity product is plotted against rotation angle for all modulation schemes as shown in Figure 1.

There are several interesting features of the diversity product plot. Firstly, it validates that 45^o is indeed an optimum

TABLE I
MAXIMUM DIVERSITY PRODUCTS AND CORRESPONDING ROTATION ANGLES.

	Maximum diversity product, ζ_{max}	Valid range of angles for ζ_{max}
QPSK	0.3536	$30^o - 60^o$
16-QAM	0.1581	$30^o - 31.13^o$, $42.61^o - 47.39^o$, $58.87^o - 60^o$
64-QAM	0.0772	$30^o - 30.07^o$, $30.96^o - 31.13^o$, 42.61^o, $44.59^o - 45.41^o$, 47.39^o, $58.87^o - 59.04^o$, $59.93^o - 60^o$

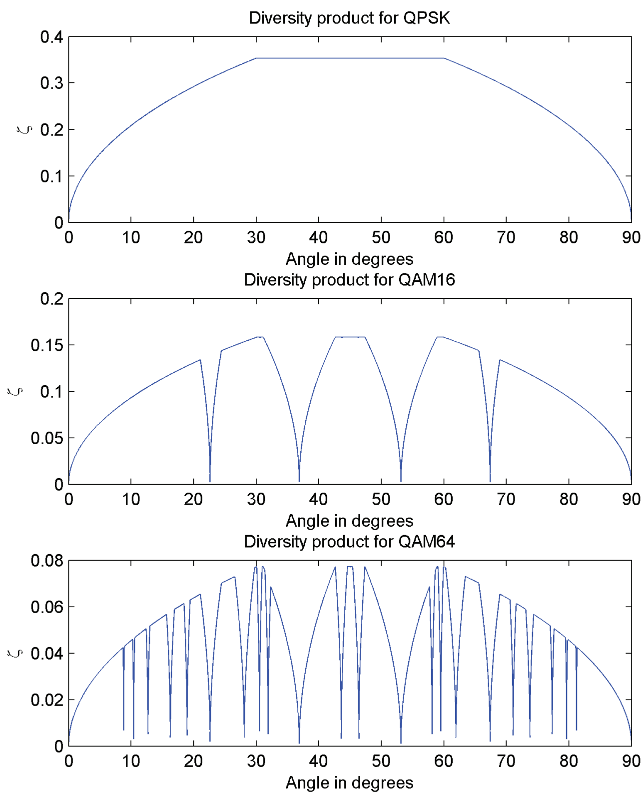

Fig. 1. Diversity product against angle for three different modulation schemes.

TABLE II
CHOSEN ROTATION ANGLES AND THEIR CORRESPONDING DIVERSITY PRODUCT.

	Diversity product, ζ			
Rotation angle	0^o	13.28^o	45^o	Custom angle
QPSK	0	0.2397	0.3536	0.3536 (Angle: 34.35^o)
16-QAM	0	0.1072	0.1581	0.1581 (Angle: 30.96^o)
64-QAM	0	0.0523	0.0772	0.0419 (Angle: 59.36^o)

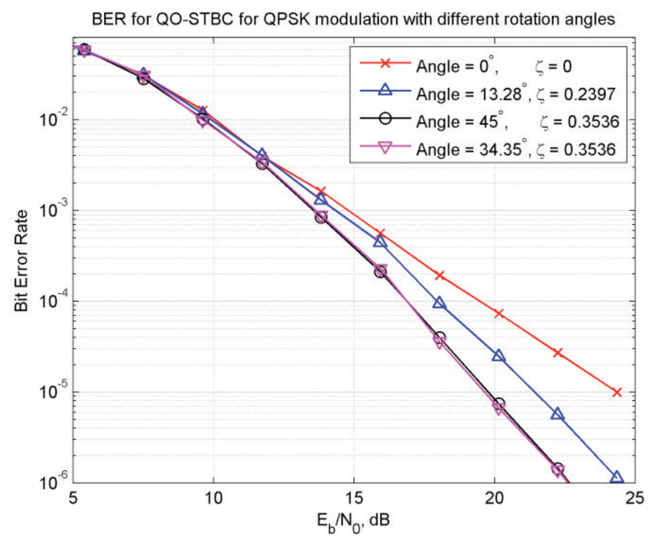

Fig. 2. BER for QO-STBC with QPSK for various rotation angles.

rotation angle for all three modulation schemes (which all have a rectangular constellation) as mentioned in [9]. However, there also exist several ranges of angles where diversity is maximum, all of which are as tabulated in Table I. In addition, the plot of the diversity product, with respect to the rotation angle, is clearly symmetrical about the 45^o line, as can be seen in Figure 1.

B. Simulation and Analysis for BER of QO-STBC

To evaluate the bit error rate (BER) performance for the QO-STBC given in Section II-C1, four different angles to rotate the constellation for symbols s_3 and s_4 for each of the modulation schemes are chosen. Three of the angles chosen are fixed at 0^o, 13.28^o and 45^o to represent the range of possible diversity, from the worst to the best. A custom angle other than 45^o was also chosen to test the ability to reach maximum diversity at other angles and their relationship with the diversity product. The chosen angles and their diversity product are shown in Table II.

With each of the modulation schemes, simulations were done using Matlab to test the effects of the diversity scheme with various angles of rotation across a range of E_b/N_o, and the results are shown in Figures 2, 3 and 4.

For angles 0^o, 13.28^o and 45^o, the behavior across all three modulations is the same, where 0^o exhibited the worst BER performance with a diversity order of 2, 45^o exhibited the best BER performance with a diversity order of 4, and 13.28^o had a BER performance between diversity two and four. This is as predicted by the diversity product as it increases from 0^o to 13.28^o to 45^o, as seen in Table II and Figure 1.

For QPSK and 16-QAM, the custom angle chosen has a diversity product that is the maximum for the corresponding modulation scheme, i.e., same ζ as that of 45^o. From Figures 2 and 3, we can see that its BER performance is the same as that of 45^o. This establishes that 45^o is indeed not the only optimum point in which maximum diversity is achievable.

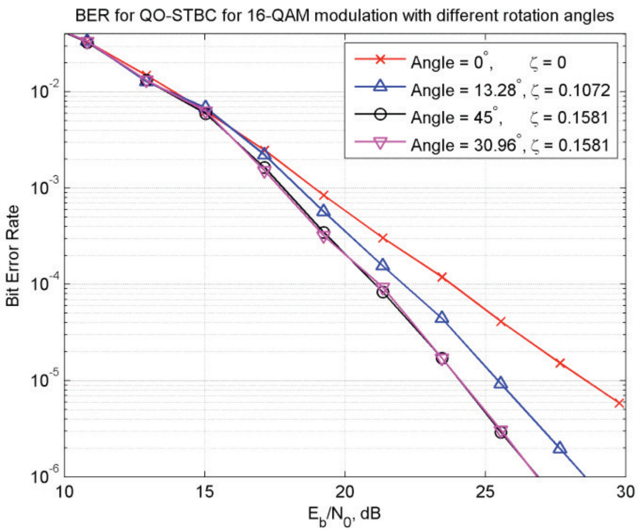

Fig. 3. BER for QO-STBC with 16-QAM for various rotation angles.

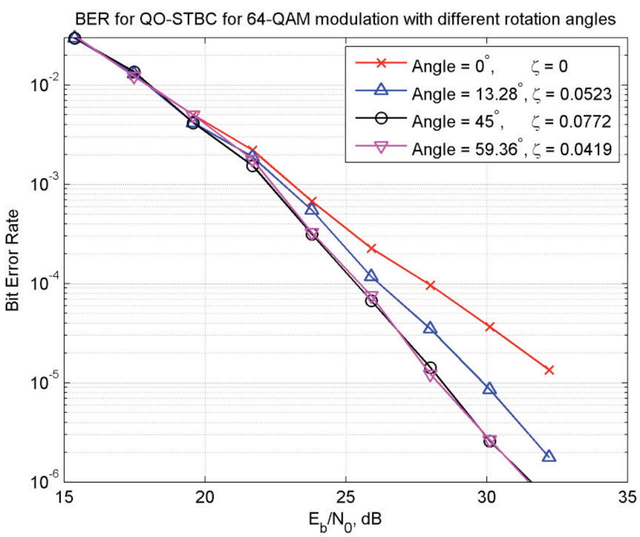

Fig. 4. BER for QO-STBC with 64-QAM for various rotation angles.

Things get more interesting with 64-QAM. As illustrated in Figure 4, the performance of the custom angle at 59.36^o closely tracks that of 45^o with no perceivable difference across the entire simulated E_b/N_o range. However, the diversity product at 59.36^o is lower than that of 45^o (in fact, it is also lower than that of 13.28^o). This seems to imply that the minimum diversity product predicted by (14) is not the only factor affecting the overall performance of the QO-STBC code.

A zoomed-in view of the plot of ζ with respect to rotation angle for 64-QAM is shown in Figure 5. The angle of 59.36^o lies in a narrow "valley" between 59^o and 60^o, which both have maximum diversity product (refer to Table I on the range of angles with maximum diversity and Figure 1 for how ζ varies with angle across the entire range of 0^o to 90^o rotation).

To further explore this phenomenon, a computer numerical

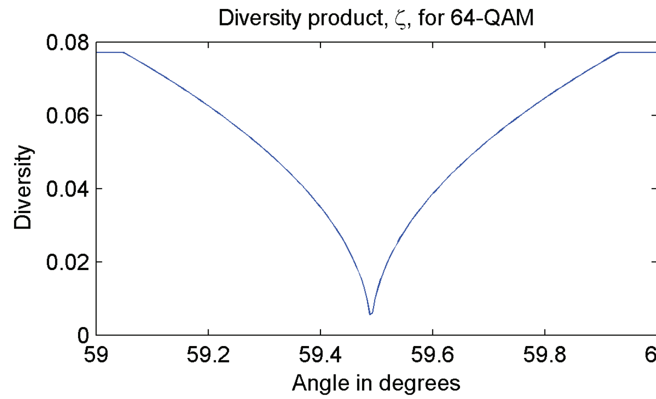

Fig. 5. Zoomed in view of ζ for 64-QAM from 59^o to 60^o.

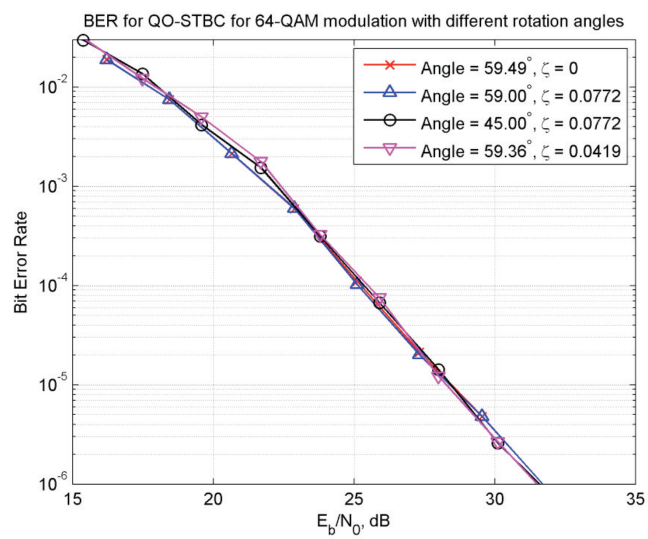

Fig. 6. BER for QO-STBC with 64-QAM using a different set of rotation angles.

search was performed to determine the rotation angle of the valley point. It was found that the angle of $tan^{-1}(56/33) \approx 59.4897^o$ gave a lowest possible ζ of 7.88×10^{-9}. Within the limits of numerical rounding error due to the limited precision of the computer, it can be considered that the calculated $\zeta = 0$.

The simulation results using 64-QAM for four different angles of rotations are shown in Figure 6. Three of the chosen angles come from around the "valley region" but have different diversity products, while the 45^o plot is left as a benchmark.

It is obvious from Figure 6 that while different rotations angles have very different diversity products, the BER performance of all four curves is practically the same. In fact, for the 59.49^o rotation, the QO-STBC should have been rank-deficient, and should only have achieved a performance with diversity of order two similar to that of 0^o rotation. Clearly, this is not the case illustrated in Figure 6. Therefore, we hypothesize that while the rank and determinant criteria are sufficient conditions for maximum diversity, they do not seem to be necessary.

Given that the above hypothesis is true, it then naturally follows that the union bound predicted by (9) will not be a useful bound since the performance of the QO-STBC is not a monotonic function of the diversity product. For example, in the above simulation, for the angle of 59.49^o, the predicted error bound with ζ is 0.5 across the entire E_b/N_o range, which clearly is not very useful. Furthermore, this bound is the same bound for the QO-STBC with no rotation, but its BER performance is very different from that of the QO-STBC with 59.49^o rotation.

IV. Conclusion

Based on our simulations, rotation angle for QO-STBC does not have to be 45^o in order to reach maximum diversity. In addition, we hypothesize that while the rank and determinant criteria are sufficient conditions for maximum diversity, they do not seem to be necessary. Further work will be done to evaluate the performance of QO-STBC and compare it orthogonal STBC.

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