

# On the Usage of Derived Quantities in Tracking: Energy and Other Estimators

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*Abstract*—Typically, the tracking and estimation community has been concerned with the direct problem of tracking a threat in a near optimal fashion. The usage of secondary quantities or derived quantities from state estimates that can be used for other purposes than tracking have been ignored. An example of a derived quantity that is useful directly for tracking and classification is the energy associated with the threat. We discuss a number of such informative priors that can be useful to pure tracking applications as well as how dissipation of energy and ballistic coefficients can be estimated indirectly.

1. Introduction
2. Derived Quantities
3. Energy Dissipation and Ballistic Estimators
4. Conclusion
5. Appendix

*Keywords*—Track Filtering, Energy Estimators, Energy Dissipation, Ballistic Estimators

## I. INTRODUCTION

Typically, the tracking and estimation community has been concerned with the direct problem of tracking a threat in a near optimal fashion. Hence the evolution of usage of track filters from the alpha-beta filter through the Kalman filter to the interactive multiple model filter ([1] [5]). What has largely been ignored is the usage of secondary quantities or derived quantities from state estimates that can be used for purposes other than tracking. The only secondary quantity that is widely recognized is the ballistic coefficient associated with ballistic missile slowdown, which is typically incorporated as a state in the Kalman filter. An example of a derived quantity that is useful directly for tracking and classification is the energy associated with the threat. An example which shows the efficacy of the energy comes about by considering the physical process of damaging a threat. It produces an observable effect by changing the motion characteristics of the threat. Thus, any means of determining the cause of acceleration may be indicative of damage. Furthermore, energy dissipation can also be indicative of a maneuver, as well as upper bounds on that

maneuver can be forecast. Such informative priors that can be useful to pure tracking applications. In the next section we discuss a number of such derived quantities that are related to a number of simple variables that are derived from a track filter. We then discuss how dissipation of energy and ballistic coefficients can be estimated as well as how they can be used.

## II. DERIVED QUANTITIES

There are several examples of quantities that are derived from track data that are useful to characterizing system performance. Such quantities are prediction factors that help with understanding the limitations of one's system that can be of interest to the tracking community. One of the important factors that provides a means for characterizing automated weapon intercept system performance is the  $K$  factor

$$K = \frac{V_T}{V_W}, \quad (1)$$

where  $V_T$  is the magnitude of velocity of the threat and  $V_W$  is the average speed of the self-defense weapon. It is this factor which allows one to determine the importance of the system prediction algorithms.  $K$  becomes less important as the speed of the weapon brings the intercept closer to "now", in which case intercept becomes impossible for a tail-chase situation. As  $K$  approaches infinity, intercept is possible only by target seduction, in which case prediction is irrelevant. It is easy to do a very simple analysis based on  $K$  to determine the effect of velocity uncertainty on  $K$ . The larger  $K$ , the more uncertainty can cause a system design to degrade. We consider analysis that shows how quickly a system design can fall apart as uncertainty grows.

Consider a system that can track and engage targets with velocities in the range of 50 – 1500 m/s and average speed of the interceptor missile being 500 m/s. Then  $K$  falls in the range of .1 to 3. The uncertainty ( $\delta$ ) in  $K$  is

$$\delta K = \frac{\delta V_T}{V_W} - \frac{V_T \delta V_W}{V_W^2} \quad (2)$$

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which can be written as

$$\delta K = K \left( \frac{\delta V_T}{V_T} + \rho \right) \quad (3)$$

where  $\rho = \pm \frac{\delta V_W}{V_W}$  or (3) can also be written as

$$\frac{\delta K}{K} = \left( \frac{\delta V_T}{V_T} + \rho \right). \quad (4)$$

The uncertainty in weapon speed is dependant on the weapon used. For a laser  $\rho = 0$ , while a rocket motor could produce a variable rocket motor burn so  $\rho$  can vary between .02 - .1. Guns can have an even higher value of  $\rho$ . If we have a system with a fixed upper value for  $K$ , we can see how it increases beyond the acceptable level as the velocity uncertainty increases, or equivalently, how the upper limit on the velocity one can track effectively decreases for a fixed  $K$  value.

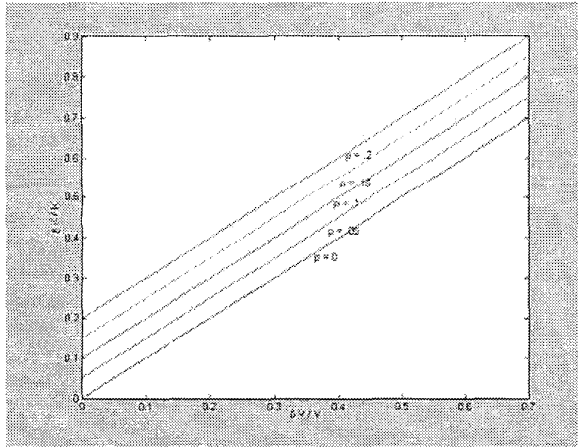


Figure 1 -  $\frac{\delta K}{K}$

Another means of characterizing  $\delta V_T$  is based on a simple  $\alpha - \beta$  filter characterization in the appendix where one can argue that the upper bound for uncertainty in the velocity is a combination of the noise reduction performance of the filter and lag due to modeling uncertainty based on the formulas in the appendix for the alpha-beta filter, one has

$$\begin{aligned} \delta V_T &= \sqrt{P_v \sigma_n + l_v a_0 T} \\ &= \sqrt{\frac{-2\beta^2}{\alpha T^2(-4+2\alpha+\beta)} \sigma_n + \left(\frac{\alpha}{\beta} - \frac{1}{2}\right) a_0 T}. \end{aligned} \quad (5)$$

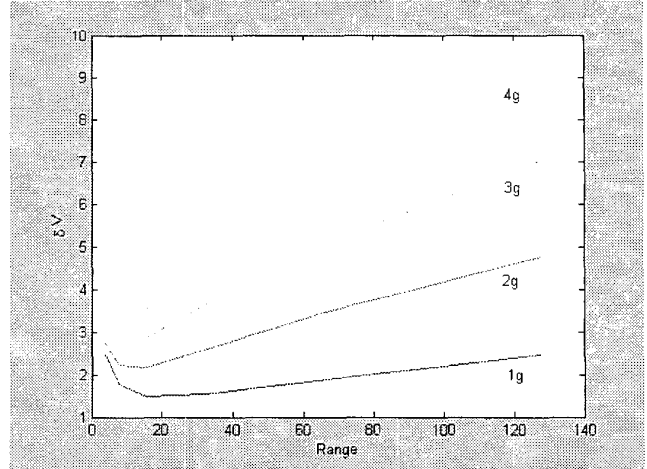


Figure 2 -  $\delta V_T$

Then the uncertainty in  $K$  becomes

$$\frac{\delta K}{K} = \left( \frac{\sqrt{\frac{-2\beta^2}{\alpha T^2(-4+2\alpha+\beta)} \sigma_n + \left(\frac{\alpha}{\beta} - \frac{1}{2}\right) a_0 T}}{V_T} + \rho \right). \quad (6)$$

Similiar analysis can be done using the alpha-beta-gamma filter based on the formulas in the appendix.

There are several different ways a calculation of time can be used in sensor related applications. One of the possible time parameters one can consider is based on the target velocity time-to-impact  $T_I$ , which is defined as

$$T_I = \frac{R_{TS}}{V_T} \quad (7)$$

where  $R_{TS}$  is the range of the threat to the ship and  $V_T$  is the speed of the threat in the direction of the ship. The larger that  $T_I$  is, the longer one has to engage the threat and the more prediction of the threats actions becomes important. Also, the larger  $T_I$  is, the more likely it is that weapons performance can be optimized by selecting an intercept region that maximizes the probability of kill. There are a number of simple parameterization that can characterize operational performance such as *effective operational range*. Intercept planning, prediction of intercept, and the associated performance can be viewed as perturbations of  $T_I$ . If we are using  $T_I$  for a prediction of intercept time using a weapon, then the uncertainty in  $T_I$  should be estimated as intercept time approaches in order to forecast whether or not the intercept has a chance of succeeding. Explicitly, the uncertainty in the time of predicted intercept is

$$\delta T_I = \frac{\delta R_{TS}}{V_T} + \frac{R_{TS} \delta V_T}{V_T^2}$$

so one has

$$\begin{aligned} \frac{\delta T_I}{T_I} &= \frac{\delta R_{TS}}{R_{TS}} + \frac{\delta V_T}{V_T} \\ &= \frac{\delta R_{TS}}{R_{TS}} + \frac{\delta K}{K} + \rho \end{aligned} \quad (8)$$

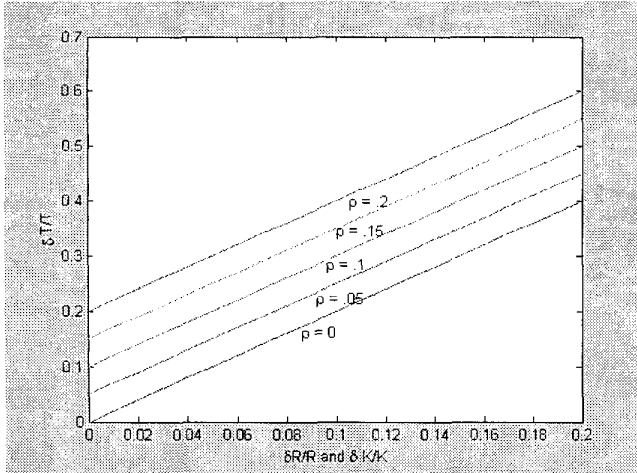


Figure 3 - Comparison of Uncertainties

Another means of characterizing  $\delta R_{TS}$  is based on a simple  $\alpha - \beta$  filter characterization in the appendix where one can argue that the upper bound for uncertainty in the position is a combination of the noise reduction performance of the filter and lag due to modeling uncertainty based on the formulas in the appendix for the alpha-beta filter, one has

$$\frac{\delta T_I}{T_I} = \frac{\left(\frac{1-\alpha}{\beta}\right) a_0 T^2 + \sqrt{\frac{-2\alpha^2 + \beta(3\alpha-2)}{\alpha(-4+2\alpha+\beta)}} \sigma_n}{R_{TS}} + \frac{\sqrt{\frac{-2\beta^2}{\alpha T^2(-4+2\alpha+\beta)}} \sigma_n + \left(\frac{\alpha}{\beta} - \frac{1}{2}\right) a_0 T}{V_T}. \quad (9)$$

Similar analysis can be done using the alpha-beta-gamma filter based on the formulas in the appendix. One can then use this to estimate when the predicted intercept time has exceeded a budgeted error allowance and start preparations for a refire.

When a threat has been damaged so that it loses engines or aerodynamics, one can estimate the time it takes a particle to fall a distance  $z$  in a constant gravitational field which in the absence of air is given by

$$T(z) = \frac{z}{v_z} = \frac{z}{\sqrt{2\left(\frac{E}{m} - gz\right)}} \quad (10)$$

where  $\frac{E}{m}$  is the normalized energy of the object being tracked and  $g$  is the acceleration due to gravity. Games can be played with the vacuum case to compare with the case of a trajectory through air to estimate the degree of ballistic behavior. When both  $E$  and  $T$  are functions of time  $t$ , one can discuss energy dissipation as is dealt with in the next section.

Another derived quantity is the normalized energy of threat

$$E_N(z) = \frac{E}{m} = \frac{1}{2}v^2 + gz, \quad (11)$$

where  $z$  is the altitude. During a non-thrusting maneuver, the energy cannot change, so an energy change may indicate thrust. Also, this can be monitored for a high altitude

flyer to determine how much delta velocity it is using for a dive. Estimates of these observables can be used to indicate trajectory changes, which is necessary for intercept prediction. Also,  $E_N(z)$  can be used as a means to classify threats. Note the data from Janes shown in the table shows in the subsonic region, the energy is similar for anti-ship type missiles. However knowing the energy combined with knowing the maximum maneuver the threat can pull does allow one to limit what the potential target choices are for a threat. In the supersonic case, there sufficient energy usages that might prove useful for both threat typing and threat intention forecasting.

Country	Missile Type	Average Speed	Altitude Range (m)	Maximum Range
France				
	Exocet 38/39	Mach .9	100-5	43-50 km
	Exocet 40 Block 1/2	Mach .9	100-2	70-75 km
	Shadow	Mach .9	30-2	400-600 km

Table 1.a - Cruise Missile Capabilities (France)

Country	Missile Type	Average Speed	Altitude Range (m)	Maximum Range
Russia	SS-N-2a-c	.9 Mach	100-350	40-80 km
	SS-N-2d	1.3 Mach	100-350	80 km
	SS-N-12	1.7 Mach	4 km -100 m	550 km
	SS-N-19	1.6 Mach	20 km	445 km

Table 1.b - Cruise Missile Capabilities (Russia)

Country (China)	Missile Type	Average Speed	Altitude Range (m)	Maximum Range
	YJ-1	Mach .9	50-20	42 km
	YJ-2	Mach .9	50-20	120 km
	HN-2C	Mach .9	100-10	1400 km
	'Silkworm'	Mach .9	100-8	85-95

Table 1.c - Cruise Missile Capabilities (China)

The situation is different for TBM type threats. In Figure 5 (fig in ppt pres), one observes that there three types of possible trajectories a missile can follow. Each has unique features that be quantified by considering the trajectories after the rocket motor burnout. For post-burnout trajectories, one can simplify considerations by considering the  $x - z$  plane only. Note, for the simplified two dimensional case that  $z(t)$  after burnout (denoted  $^b$ ) is given by

$$z(t) = z^b + v_z^b t - \frac{1}{2} g t^2 \quad (12)$$

where the vertical velocity at burnout is ( $V_R^b$  is the radial velocity at burnout)

$$v_z^b = V_R^b \sin \theta^b. \quad (13)$$

The horizontal position after burnout is given by

$$x(t) = x^b + v_x^b t \quad (14)$$

while the horizontal velocity at burnout is given by

$$v_x^b = V_R^b \cos \theta^b. \quad (15)$$

For maximum range,  $\theta^b$  should be between  $45^\circ - 48^\circ$  at burnout. For lofted trajectories, the range of  $\theta$  should be between  $55^\circ - 80^\circ$ , while for depressed trajectories  $\theta$  should be between  $25^\circ - 40^\circ$ . The ballistic object comes to rest at a time  $T_R$

$$0 = z^b + v_z^b T_R - \frac{1}{2} g T_R^2 \quad (16)$$

or

$$T_R = \frac{-v_z^b \pm \sqrt{(v_z^b)^2 + 2g z^b}}{g}, \quad (17)$$

so the maximum range is

$$x(T_R) = x^b + \frac{-v_z^b + \sqrt{(v_z^b)^2 + 2g z^b}}{g} v_x^b \quad (18)$$

Note  $z(t)$  is a maximum when  $\dot{z}(t) = 0$  or

$$\frac{v_z^b}{g} = T_M, \quad (19)$$

so the maximum altitude is

$$z_M = z^b + \frac{(v_z^b)^2}{2g} \quad (20)$$

while the vertical component of the velocity is zero. The total energy at the maximum altitude is

$$E_M = \frac{1}{2} (V_R^b)^2 + g z^b. \quad (21)$$

As one gains confidence about the missile velocity in the post-burnout phase, one observes that there would be significant differences between a depressed, maximum range and lofted trajectory. This allows one to partially classify the threat type as well estimate impact points.

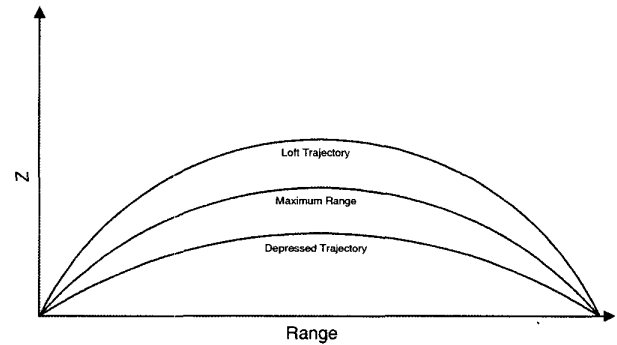


Figure 5 - Theatre Ballistic Missile (TBM) Trajectories

For Anti-Air Warfare (AAW), the sensitivity of the operational range is quite variable due to maneuvers, and "intent of the threat". The intent of the threat is the driver of performance in any multi-ship environment. One cannot assume in this situation that one's ship or a nearby defended asset is the intended target. There are multiple targets available and the dynamics of the threat do not necessarily preclude the possibility of one target versus another. So one is led to ask what qualities allow the ability to rule out certain types of intent. To answer this, one has to consider combinations of observables. For a subsonic threat, speed precludes it from maneuvers of 1-2 g's for low speeds, and 3-5 g's the nearer to the speed of sound. Also, the height above sea level factors in as well. For supersonic threats, high g maneuvers are limited by altitude. Thus one can decrease the maximum maneuver as a function of altitude. Above 30 km, limited maneuvers are possible (1-2 g's at most). For the lower atmosphere, a working rule of thumb is Mach number times 10 is a good estimate of the maximum g level for all AAW threats. Thus energy combined with altitude allows us to rule out some types of maneuvers as well as place a priori bounds on threat maneuver as the maneuver starts to occur.

Another estimator of target performance is radar cross section (RCS). The larger the RCS, the more likely a threat

is to be manned. This places an upper limit on the threat maneuverability.

For Theatre Ballistic Missile (TBM) applications, all of the observables that occur in AAW can also be used in TBM applications with some different interpretations of the range of the parameters. As a TBM reenters the atmosphere, all of the following parameters: acceleration, energy, ballistic coefficient, and RCS can be estimated. From these, one can derive some indicator of the threat dynamics. Each of these parameters can be used to determine different aspects of the threat trajectory. These can then be used to estimate the operational range of the TBM and the best location to attempt an intercept. Estimates of these parameters can be used to indicate trajectory changes, which are necessary for intercept prediction.

### III. ENERGY DISSIPATION AND BALLISTIC ESTIMATORS

There are two physical parameters that are related to dissipation that can be estimated from derived quantities or can be estimated directly using a Kalman filter. Both examples are related to the ability to calculate dissipation of energy, which is of interest to kill evaluation or threat damage. An example which shows the efficacy of the energy comes about by considering the physical process of damaging a threat. Damage to a threat produces an observable effect by changing the motion characteristics of the threat. Thus, any form of acceleration may be indicative of damage to the threat. Damage to a threat could also be indicative of a maneuver. (To distinguish between the two requires an estimate of the two components that contribute to the acceleration vector, which is discussed further in this section.) The underlying cause for the acceleration, when it is due to damage, is a change in the ballistic characteristics of the threat. Note that this comment even applies if the threat is an endo-atmospheric TBM threat which is not aerodynamically balanced in the same sense that air vehicles are. When a threat is no longer under control, it does not control its response to aerodynamic forces so it can be treated as a body that is only subject to aerodynamic forces.

One can use energy dissipation to characterize the loss in energy of a threat that is being tracked. This energy dissipation allows one to estimate the damage done to a threat after a missile intercept. Another application of an energy dissipation algorithm would be threat identification, where the amount of energy the threat dissipates during a given time interval is estimated and compared with a tabulated set of values to establish identification. A third application would be to determine threat intent. If an excessive amount of energy is being bled off, the threat is exerting energy to 'reach' an objective. Additionally, an estimator can be derived from it to determine the ballistic coefficient from tracking data based on the following observation.

One starts with the observation: *Any dispersive system that can be written as ([4], [11])*

$$\frac{d}{dt}(T + V) = \frac{d}{dt}E_c = F'v \quad (22)$$

where  $F'$  is the portion not derivable from a potential,  $T + V$  is kinetic plus potential energy or the total conservative energy  $E_c$ , and  $F$  is the portion of the force not derivable from a potential or the power delivered by an effective force  $F'$ . For example,  $E_N$ , the normalized  $E_c$ , is

$$E_N = \frac{1}{2}v^2 + gz \quad (23)$$

So any change in  $E_N$  can be used as a kill indicator. However, there are several possibilities for the effective force, depending on the circumstances. An estimator can be determined independent of the details of what  $F'(v)$  functional form provided. It is assumed that

$$F'(v) = k\rho f(v) \quad (24)$$

The method is general enough to work for any dispersive system of the type we are discussing.

$$d(\frac{1}{2}v^2 + gz) = k\rho v f(v)dt. \quad (25)$$

To derive an estimator from Eq. (15), the derivatives and the variables are made discrete.

$$d \rightarrow \Delta_i = (\delta_i - \delta_{i-1}),$$

so Eq. (15) becomes

$$E_i - E_{i-1} = k_i \rho_i v_i f(v_i)(t_i - t_{i-1}) \quad (26)$$

Summing over  $i$  gives

$$E_n - E_0 = \sum_{i=1}^n k_i \rho_i v_i f(v_i)(t_i - t_{i-1}). \quad (27)$$

If one defines  $D$  as the negative of the right hand side of Eq. (17), and recognizes that the variables are measurements, the energy loss estimator LE is formed

$$L\hat{E} = \hat{E}_n - \hat{E}_0 + \hat{D} \quad (28)$$

which can be used as ones' test statistic for a hypothesis test. Both (16) and (17) can also be used to estimate the ballistic coefficient.

The ballistic coefficient is another example of a derived quantity that can be estimated using a Kalman filter or as a derived quantity in the same manner as the energy dissipation. An estimator of the ballistic coefficient can be obtained in a manner similar to that in Berginfeld (internal to Lockheed Martin). First assume a flat earth model, then the solution is generalized for the round earth case. The derivation is based on the following two assumptions:

1. The effect of drag is to slow down the target along the zero drag trajectory.
2. Target dive angle is constant.

The equation of motion of a ballistic object subject to gravity and drag can be written

$$\begin{aligned}\frac{d\vec{v}}{dt} &= \frac{g\rho(z)}{2\beta} V^2 \hat{v} - g\hat{u}_g \\ &= \frac{g\rho(z)}{2\beta} V\vec{v} - g\hat{u}_g\end{aligned}\quad (29)$$

where

$V$  -magnitude of the velocity  
 $\vec{v}$  -velocity vector  
 $\hat{v}$  -unit velocity vector  
 $\hat{u}_g$  -unit vector for direction of gravity

Introduce a horizontal unit vector that obeys the two conditions:

$$\vec{v} \cdot \hat{v}_h = V_h \quad (30)$$

$$\hat{v}_h \cdot \hat{u}_g = 0 \quad (31)$$

Then, the horizontal component of the velocity becomes

$$\frac{dV_h}{dt} = \frac{g\rho(z)}{2\beta} V V_h \quad (32)$$

$$\frac{dV_h}{V_h} = \frac{g\rho(z)}{2\beta} V dt \quad (33)$$

$$\frac{dV_h}{V_h} = \frac{g\rho(z)}{2\beta} V V_z^{-1} dz \quad (34)$$

$$\frac{dV_h}{V_h} = \frac{g\rho(z)}{2\beta \sin \gamma} dz \quad (35)$$

where  $V_z = \frac{dz}{dt}$ ,  $\sin \gamma = \frac{V_z}{V}$ , and  $\gamma$  is the target flight path angle. From the hydrostatic equation the pressure is  $P = \int_{z_0}^z g\rho(z) dz$ , one can obtain an expression for the horizontal velocity by integrating both sides of the previous equation (assuming  $\sin \gamma$  is constant) to give

$$V_h = V_{h_0} \exp(k(P - P_0)) \quad (36)$$

where  $k = \frac{1}{2\beta \sin \gamma}$ .  $P$  is the current pressure and  $P_0$  is the pressure at the beginning of the time interval.

From the equations of motion, the current horizontal velocity is equal to the initial horizontal velocity,  $V_{h_0}$ , multiplied by an exponential term representing drag. The zero drag components of horizontal and vertical velocity are:

$$\begin{aligned}V_h &= V_{h_0} \\ V_z &= V_{z_0} - gt\end{aligned}\quad (37)$$

where  $V_{z_0}$  is the initial value of vertical velocity. These expressions can be used to obtain the  $V_z$  component of velocity with drag. Integrate the equations for zero drag with respect to  $t$  and solving for  $t$  gives a quadratic equation

$$t^2 - \frac{2V_{z_0}t}{g} + \frac{2}{g}(z - z_0) = 0, \quad (38)$$

which can be solved for  $T$ . Substitute  $T$  into the zero drag equations allows one to obtain the zero drag vertical velocity  $V_z$ ;

$$V_z = \sqrt{V_{z_0}^2 - 2g(z_0 - z)}. \quad (39)$$

Then it follows from the assumptions that  $\tan \gamma = \frac{V_z}{V_h}$ , where  $\gamma$  is the zero drag flight path angle; therefore, we have

$$V_z = V_h \frac{V_z}{V_h}. \quad (40)$$

Substitute this into to obtain  $V_z$ , the vertical velocity with drag;

$$V_z = \sqrt{V_{z_0}^2 + 2g(z_0 - z) \exp(k(P - P_0))} \quad (41)$$

The total velocity is

$$V = \sqrt{V_h^2 + V_z^2} = \sqrt{V_0^2 + 2g(z_0 - z) \exp(k(P - P_0))}. \quad (42)$$

Defining

$$\begin{aligned}V_p &= \sqrt{V_0^2 + 2g(z_0 - z)} \\ V_0 &= \sqrt{V_{h_0}^2 + V_{z_0}^2}\end{aligned}\quad (43)$$

we have the total velocity in the presence of drag:

$$V = V_p \exp(k(P - P_0)) = V_p \exp\left(\frac{P - P_0}{2\beta \sin \gamma}\right) \quad (44)$$

Solving for the ballistic coefficient  $\beta$ , we arrive at the flat earth solution for the estimator (without using a Kalman filter)

$$\beta = \frac{P - P_0}{2 \sin \gamma \ln \left( \frac{V}{V_p} \right)}. \quad (45)$$

Another method is to solve the equation of motion for the drag coefficient, obtaining

$$\delta = V^{-2z/h} \left[ -\frac{dV}{dt} - g \gamma \right] \quad (46)$$

The change in threat speed  $\Delta V$  is observed a number  $n$  of times along the TBM trajectory, resulting in a set of values

$$\Delta V_j, \quad \Delta t_j, \quad j = 1, \dots, n \quad (47)$$

At each observation an estimate of the drag coefficient

$$\hat{\delta}_j = \delta + V_j^{-2z_j/h} \left[ -\frac{\Delta V_{ej}}{\Delta t_j} \right] \quad (48)$$

is obtained, where  $\delta$  is the true value and  $\Delta V_{ej}$  are the errors in the observed speed increments with standard error  $\sigma_{\Delta V}$ . The weighted mean of these estimates is

$$\bar{\delta} = \frac{\sum \hat{\delta}_j V_j^{2z_j/h} \Delta t_j}{K} = \delta - \frac{\sum \Delta V_{ej}}{K}, \quad (49)$$

$$K = \sum V_j^{2z_j/h} \Delta t_j. \quad (50)$$

This weighting provides an unbiased estimate of the drag coefficient.

To find the standard error of this estimate it is necessary to find the standard error in the speed estimate. This is cumbersome in general even when the error in the estimates of the threat state are assumed Gaussian. However, usually the threat speed greatly exceeds its error. Accordingly, it is assumed that

$$|\vec{V} - \langle \vec{V} \rangle| \ll |\vec{V}| \quad (51)$$

where the angle brackets indicate the expected value. The threat speed  $V = |\vec{V}|$  is expressed by

$$V = \sqrt{[(\vec{V}) + (\vec{V} - \langle \vec{V} \rangle)]^T [(\vec{V}) + (\vec{V} - \langle \vec{V} \rangle)]} \quad (52)$$

where the magnitude of the vector within the parentheses is assumed small. To first order in the small quantity this becomes

$$V = \langle V \rangle \sqrt{1 + 2\langle V \rangle^{-2} \langle \vec{V} \rangle^T (\vec{V} - \langle \vec{V} \rangle)} = \frac{\vec{V}^T \vec{V}}{\langle V \rangle} \quad (53)$$

and

$$\langle V \rangle = |\langle \vec{V} \rangle| \quad (54)$$

Thus, since  $V$  is a linear combination of the components of  $\vec{V}$ ,  $V$  is Gaussian if  $\vec{V}$  is Gaussian. The error in the speed estimates is

$$\sigma_V^2 = \langle [V - \langle V \rangle]^2 \rangle = \frac{\vec{V}^T C \vec{V}}{\langle V \rangle^2} \quad (55)$$

where  $C$  is the error covariance of the velocity estimates,

$$C = \langle [\vec{V} - \langle \vec{V} \rangle][\vec{V} - \langle \vec{V} \rangle]^T \rangle \quad (56)$$

Estimates of the speed increments are not independent even when the speed errors are independent. If one increment is overestimated, the following increment is likely to be underestimated. However, the error in a speed increment is independent of the errors in all the non-contiguous speed increments. The correlation between contiguous increment errors has little effect, and the relation between them and those of the speed estimates is

$$\sigma_{\Delta V}^2 = 2\sigma_V^2 \quad (57)$$

The error in the drag estimate is then

$$\begin{aligned} \sigma_\delta^2 &= \langle [\delta - \langle \delta \rangle]^2 \rangle = \left\langle \left[ -\frac{\sum \Delta V_{ej}}{K} \right]^2 \right\rangle \\ &= K^{-2} \sigma_{\Delta V}^2 = \frac{2\vec{V}^T C \vec{V}}{K^2 \langle V \rangle^2} \end{aligned} \quad (58)$$

The foregoing analysis is based on a number of assumptions. One is that the air density is a simple exponential function of altitude. A more refined approach would be to take the actual density profile existent at the place and time. Another is that the speed errors are independent of time (and altitude), whereas in actuality these errors are likely to increase as the air density and drag increase. Still another is that the expected velocity varies little during the time the observations are made.

## IV. CONCLUSION

We have discussed the usefulness of using quantities derived from estimated quantities rather than estimating them directly. Both simple time estimates, energy and some dimensionless quantities are useful in some applications. In addition, we discussed how energy dissipation can be characterized and used in radar system applications.

## APPENDIX

The usage of tracking filters associated with a tracking radar dates back to work by Sklansky (1957) [27]. He proposed performance measures of performance including stability, transient response, noise and maneuver error as a function of the dynamic parameters  $\alpha$  and  $\beta$ . All of the work was based on a frequency domain or z-transform analysis. Subsequent work by Benedict-Bordner, 1962 [7] proposed a relationship between  $\alpha$  and  $\beta$  based on a pole-matching technique that combined transient performance and noise reduction capability. Subsequent analysis was performed by Simpson (1963) [26], Neal, and Benedict (1967) [24] for the  $\alpha - \beta - \gamma$  filter. By this time, the Kalman filter was becoming well known in the radar community. Thereafter, the tendency was to discuss the  $\alpha - \beta$  and  $\alpha - \beta - \gamma$  filters as steady state solution to the Kalman filter. Subsequent papers by Friedland (1973) [12], Fitzgerald (1980) [9], and Kalata (1984) [20] exploited this formal similarity to derive many results that can be used to characterize tracking performance in a multi-tracking environment. The basis for the analysis of performance used internally with the Aegis community is summarized in the internal manual entitled "The Working Engineers Guide To  $\alpha - \beta$  and  $\alpha - \beta - \gamma$  Filters" by Reiffer and Solomon (1982) [25]. Later, much of this work was summarized in the open literature by Kalata [20]. A summary of subsequent developments in the literature to 1992 is found in Kalata [21] with some additional work since then found in Gray (1998) ([15], [16]), as well as the open literature.

Contrary to the approach that is usually taken in the literature, we view that the more 'natural' viewpoint is to introduce the constant gain filter and an entity that is independent of the Kalman filter. One can derive the information that characterizes the filter performance without regard to the Kalman filter design criteria. One can then show how the performance criteria generalize naturally the Kalman filter or to a variation of the Kalman filter that replaces the process noise with a bias reduction criteria. While variations on the filters discussed are currently in use within naval systems, it is likely that they will be replaced with much more advanced estimation techniques such as the interacting multiple model (IMM) filter. Exploring the ability to bound filter performance is a necessary part of the redesign to replace existing filters with advanced filter architectures.

The  $\alpha - \beta$  filter has found application when large numbers of objects are to be tracked. By clever selection of the gains, and careful design, variable gain  $\alpha - \beta$  filters combine sufficient elements of the Kalman filter ([13],[1]) so that there is not significant performance degradation.

Thus, there is useful information to be gained by a detailed performance characterization of the filter.

The tracking equations for the  $\alpha - \beta$  filter consist of two parts: prediction equations, which are given by

$$x_p(k) = x_s(k-1) + v_s(k-1)T \quad (\text{a1})$$

$$v_p(k) = v_s(k-1) \quad (\text{a2})$$

and smoothing equations, which are given by

$$x_s(k) = x_p(k) + \alpha(x_m(k) - x_p(k)) \quad (\text{a3})$$

$$v_s(k) = v_p(k) + \frac{\beta}{T}(x_m(k) - x_p(k)) \quad (\text{a4})$$

where

- $x_s(k)$  = smoothed position at the k-th interval
- $x_p(k)$  = predicted position at the k-th interval
- $x_m(k)$  = measured position at the k-th interval
- $v_s(k)$  = smoothed velocity at the k-th interval
- $v_p(k)$  = predicted velocity at the k-th interval
- $T$  = radar update interval or period
- $\alpha, \beta$  = filter weighing coefficients.

The tracking equations for the  $\alpha - \beta - \gamma$  filter consists of two parts: prediction equations, which are given by

$$x_p(k) = x_s(k-1) + v_s(k-1)T + \frac{T^2}{2}a_s(k-1) \quad (\text{a5})$$

$$v_p(k) = v_s(k-1) + Ta_s(k-1) \quad (\text{a6})$$

$$a_p(k) = a_s(k-1) \quad (\text{a7})$$

and smoothing equations, which are given by

$$x_s(k) = x_p(k) + \alpha(x_m(k) - x_p(k)) \quad (\text{a8})$$

$$v_s(k) = v_p(k) + \frac{\beta}{T}(x_m(k) - x_p(k)) \quad (\text{a9})$$

$$a_s(k) = a_p(k) + \frac{\gamma}{T^2}(x_m(k) - x_p(k)) \quad (\text{a10})$$

where

- $x_s(k)$  = smoothed position at the k-th interval
- $x_p(k)$  = predicted position at the k-th interval
- $x_m(k)$  = measured position at the k-th interval
- $v_s(k)$  = smoothed velocity at the k-th interval
- $v_p(k)$  = predicted velocity at the k-th interval
- $a_s(k)$  = smoothed acceleration at the k-th interval
- $a_p(k)$  = predicted acceleration at the k-th interval
- $T$  = radar update interval or period
- $\alpha, \beta, \gamma$  = filter weighing coefficients.

The filter gains,  $\alpha$  and  $\beta$  satisfy the following relation

$$0 < \beta \leq \alpha < 1. \quad (\text{a11})$$

There are three commonly used relationships between  $\alpha$  and  $\beta$ . The first is the Kalata relation, which is obtained from steady state Kalman filter theory assuming zero mean white noise in the position and velocity state equations [20],

$$\beta = 2(2 - \alpha) - 4\sqrt{1 - \alpha}. \quad (\text{a12})$$

The second is the Benedict-Bordner relation, which is derived based on good noise reduction and good tracking through maneuvers,

$$\beta_{BB} = \frac{\alpha^2}{2 - \alpha}. \quad (\text{a13})$$

The third is the Continuous White Noise (CTWN) relation,

$$\alpha = \sqrt{2\beta + \frac{\beta^2}{12}} - \frac{\beta}{2}. \quad (\text{a14})$$

For the  $\alpha - \beta - \gamma$ , the commonly used relationship between  $\alpha$  and  $\beta$  is the Kalata relationship, which is obtained from steady state Kalman filter theory assuming zero mean white noise in the position and velocity state equations [20]. An additional relation between  $\gamma$  and  $\alpha - \beta$  is needed. The most common is

$$\gamma = \frac{\beta^2}{2\alpha} \quad (\text{a15})$$

which is known as the Neal-Simpson relation[24].

The equations for the  $\alpha - \beta$  filter can be written as

$$|x_s\rangle_k = F_\beta |x_s\rangle_{k-1} + G_\beta x_m(k) \quad (\text{a16})$$

$$|x_p\rangle_{k+1} = Q_\beta |x_s\rangle_k \quad (\text{a17})$$

where

$$F_\beta = \begin{bmatrix} 1 - \alpha & (1 - \alpha)T \\ -\frac{\beta}{T} & 1 - \beta \end{bmatrix}, \quad (\text{a18})$$

$$|x_s\rangle_k = \begin{bmatrix} x_s(k) \\ v_s(k) \end{bmatrix}, \quad (\text{a19})$$

$$G_\beta = \begin{bmatrix} \alpha \\ \frac{\beta}{T} \end{bmatrix}, \quad (\text{a20})$$

and

$$Q_\beta = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}. \quad (\text{a21})$$

These filter equations are one-dimensional, but can be extended to three dimensions by substituting successively  $y$  and  $z$  for  $x_s$  in Eq. (a16). The filter equations are usually analyzed in one dimension and the resulting analysis is usually extended to three dimensions with the assumption that similar results are given.

The  $\alpha - \beta - \gamma$  filter can be written in matrix form as

$$|x_s\rangle_k = F_\gamma |x_s\rangle_{k-1} + G_\gamma x_m(k) \quad (\text{a22})$$

$$|x_p\rangle_{k+1} = Q_\gamma |x_s\rangle_k \quad (\text{a23})$$

where

$$|x_s\rangle_k = \begin{bmatrix} x_s(k) \\ v_s(k) \\ a_s(k) \end{bmatrix}, \quad (\text{a24})$$

$$|x_p\rangle_k = \begin{bmatrix} x_p(k) \\ v_p(k) \\ a_p(k) \end{bmatrix}, \quad (\text{a25})$$



$$Q_\gamma = \begin{bmatrix} 1 & T & \frac{T^2}{2} \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}, \quad (\text{a26})$$

$$G_\gamma = \begin{bmatrix} \alpha \\ \frac{\beta}{T} \\ \frac{\gamma}{T^2} \end{bmatrix}, \quad (\text{a27})$$

and

$$F_\gamma = \begin{bmatrix} 1-\alpha & (1-\alpha)T & (1-\alpha)\frac{T^2}{2} \\ -\frac{\beta}{T} & 1-\beta & \left(1-\frac{\beta}{2}\right)T \\ -\frac{\gamma}{T^2} & -\frac{\gamma}{T} & 1-\frac{\gamma}{2} \end{bmatrix}, \quad (\text{a29})$$

For the class of problems when this occurs, the filter can be viewed as a constant gain filter which is nothing more than a matrix difference equation. This equation can then be solved regardless of the measurement model provided the model is deterministic. The general solution can then be used to compute the covariance matrix under very general assumptions about the noise. This is an alternative and slightly more general to work done by Fitzgerald [9],[10], in the early eighties. In this report, the general case will be solved first, and the  $\alpha - \beta$  filter [7] will be solved as an illustrative example. Previously [14], the  $z$ -transform or frequency domain method was used, but here the direct methods that have become more fashionable in recent years will be used. The solutions are independent of the particular relationship between  $\alpha$  and  $\beta$  that are discussed in [7] and [20].

If one assumes that  $x_m$  is quadratic in  $k$ , then the measurement model  $x_m$  can be written as  $x_k = \frac{a_0 k^2}{2}$ . Then, the solution to the acceleration motion model simplifies to a non-transient smoothed position solution (as has been noted in [23])

$$\begin{bmatrix} x_{ss} \\ v_{ss} \end{bmatrix} = \begin{bmatrix} \frac{T^2 k^2}{2} \\ Tk \end{bmatrix} a_0 - \begin{bmatrix} \frac{l_p T^2}{2} \\ -l_v T \end{bmatrix} a_0, \quad (\text{a30})$$

where  $l_p$  is the position lag in the response due to the acceleration input

$$l_p = \left( \frac{1-\alpha}{\beta} \right), \quad (\text{a31})$$

and  $l_v$  is the velocity lag in the response due to the acceleration input

$$l_v = \left( \frac{\alpha}{\beta} - \frac{1}{2} \right). \quad (\text{a32})$$

For a cubic jerk model, the non-transient solution to the  $\alpha - \beta - \gamma$  filter in the inhomogeneous solution for the non-transient case simplifies to

$$\begin{bmatrix} x_{ss} \\ v_{ss} \\ a_{ss} \end{bmatrix} = j_0 \begin{bmatrix} \frac{1}{6} T^3 k^3 \\ \frac{1}{2} T^2 k^2 \\ Tk \end{bmatrix} + j_0 \begin{bmatrix} \frac{(1-\alpha)T^3}{\left(\frac{\alpha-\frac{\beta}{2}+\frac{\gamma}{12}\right)T^2} \\ \frac{\gamma}{\left(\frac{\beta}{\gamma}+\frac{1}{2}\right)T} \end{bmatrix} \quad (\text{a33})$$

where  $l_p$  is the position lag,  $l_v$  is the velocity lag, and  $l_a$  is the acceleration lag due to the jerk input  $j_0 T^3/6$ .

The elements of this covariance matrix can be determined from the general solution to the filter equations which allows us to determine the noise reduction ratios for position ( $x$ ), velocity ( $v$ ), and cross term position-velocity ( $xv$ ) for the  $\alpha - \beta$  filter as

$$P_x^\beta = \frac{-2\alpha^2 + \beta(3\alpha - 2)}{\alpha(-4 + 2\alpha + \beta)}, \quad (\text{a34})$$

$$P_v^\beta = \frac{-2\beta^2}{\alpha T^2(-4 + 2\alpha + \beta)}, \quad (\text{a35})$$

which when multiplied by  $\sigma_n^2$  gives the covariances. For the  $\alpha - \beta - \gamma$  filter, recall the elements of this covariance matrix reveal the noise reduction ratios for position ( $x$ ), velocity ( $v$ ), acceleration ( $a$ ), position-velocity ( $xv$ ), and velocity-acceleration ( $va$ )

$$P_x^\gamma = \frac{(2\alpha(2\alpha\beta + \alpha\gamma - 2\gamma) - \beta^2(6\alpha - 4) + \alpha\beta\gamma)}{h_{\alpha\beta\gamma}}, \quad (\text{a36})$$

$$P_v^\gamma = \frac{-(2\gamma^2(2 - \alpha) + 4\beta^2(\beta - \gamma))}{T^2 h_{\alpha\beta\gamma}}, \quad (\text{a37})$$

$$P_a^\gamma = \frac{4\beta\gamma^2}{T^4 h_{\alpha\beta\gamma}}, \quad (\text{a38})$$

where

$$h_{\alpha\beta\gamma} = (2\alpha\beta + \alpha\gamma - 2\gamma)(4 - 2\alpha - \beta). \quad (\text{a39})$$

The  $P$ 's when multiplied by  $\sigma_n^2$  gives the covariances. (Note when  $\gamma = 0$ , these reduce to the  $\alpha - \beta$  filter case.)

Independent of which of the three relationships between  $\alpha$  and  $\beta$  one assumes, each relationship can be shown to obey the common constraint due to Kalata [20]

$$\Gamma^2 = \frac{\beta^2}{1 - \alpha}, \quad (\text{a40})$$

where the variable  $\Gamma$  is commonly known in the naval community as the Kalata tracking index,

$$\Gamma^2 = \frac{T^4 \sigma_a^2}{\sigma_m^2}. \quad (\text{a41})$$

The tracking index is a function of the assumed target maneuverability variance  $\sigma_a^2$  (deviation from modeled behavior), radar measurement noise variance  $\sigma_m^2$ , and  $T$  is the update interval. The maneuverability is an unknown parameter in most cases because there is no direct means of determining it from system parameters, nor is it measurable. The following table contains parameters used to design equations used throughout the paper. The update interval,  $T$ , is .5 seconds,  $\sigma_\theta$  is equal to 1 milliradian and acceleration,  $a_0$ , is equal to 4 g's were predefined.

Range(n)	$\Gamma$	$\alpha_K$	$P_x^K(0)$	$P_v^K(0)$	$J_v^K(\tau, \Gamma)$
4	2.5	.88	1.21	.82	2.94
8	1.25	.78	.70	.46	1.61
16	.625	.67	.58	.17	.89
32	.3125	.54	.46	.06	.48
64	.15	.43	.35	.02	.26
128	.08	.33	.26	.007	.14

Table 2 - Filter Design Table

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