

## H-infinity optimal filters for a class of nonlinear models

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### Abstract

The design of an H-infinity filter or observer based on a nonlinear model involves the solution of the Hamilton-Jacobi inequality. Alternatively, this paper uses the expansion of the nonlinear model as a series of linear ones and provides a sufficient condition, in the form of a set of bilinear matrix inequalities, for the solution of the H-infinity state estimation problem.

### 1 Introduction

One approach to design observers or filters for general nonlinear systems is via a transformation of coordinates that converts the original problem to one whose solution has been charted; see [1] and the references therein. Another approach is to tackle the problem directly by asserting that an observer or filter whose model is a copy of the plant's can, with the assistance of feedback, estimate the state or output of interest within certain bounds of error [2].

As with control design, the assessment of performance in filtering follows several paths. The ideas on how to measure performance of nonlinear filters stem from linear design. For example, the design for a specified gain, as measured by the Euclidean norm, from the disturbance to the output error is an extension of the

$H_\infty$  norm criterion to nonlinear systems. The theory for the design of nonlinear filters in accordance with such criterion has been developed but involves solving an associated Hamilton-Jacobi equation [3-6].

In this article, we follow an approach to nonlinear filter design similar to solving the linear filtering problem—cast in the  $H_\infty$  or  $H_2$  framework—using Linear Matrix Inequalities [7]. The success of the proposed approach lies in the modeling of the nonlinear plant as an aggregation of linear models, hereafter, referred to as a piecewise-linear model. The ability of such piecewise-linear model to capture the dynamics of a large class of nonlinear plants has been investigated extensively in the literature; see [8] and the references therein. Using the piecewise-linear model, this article shows that the design of filters with guaranteed or optimal gain is possible by solving a feasibility or optimization problem, respectively, subject to Bilinear Matrix Inequalities (BMIs). Problems associated with BMIs have received some attention in the literature; see [9]. Such approach has similarities with the treatment of the filtering problem for uncertain systems with jumping parameters [10].

The paper is organized as follows. A complete problem formulation is offered in Section 2, followed by our main result on filter design for certain nonlinear plants

in Section 3. The simulation in Section 4 verifies our theoretical result and Section 5 concludes the paper.

## 2 Problem Statement

Consider the following plant:

$$\dot{x} = f(x) + B(x)w \quad (1)$$

$$z = E(x)x \quad (2)$$

where  $w$  is some disturbance and  $z$  the output of interest. The measured output is as follows

$$y = C(x)x$$

To estimate the output of interest, we propose the following filter:

$$\dot{\hat{x}} = A(\hat{x})\hat{x} + M(C(\hat{x})\hat{x} - y) \quad (3)$$

$$\hat{z} = E(\hat{x})\hat{x} \quad (4)$$

The error in the estimation of the output of interest is

$$e = \hat{z} - z$$

In this paper, we design the filter by finding  $M$  so that

$$\int_0^t e^T e \, d\tau \leq \gamma^2 \int_0^t w^T w \, d\tau \quad (5)$$

where  $\gamma$  is the  $\mathcal{L}_2$  gain of the composite plant-and-filter system. Furthermore, we seek for the matrix  $M$  that minimizes the  $\mathcal{L}_2$  gain from the disturbance,  $w$ , to the output error,  $e$ .

## 3 Main Result

Note that the system (1) with  $f(0) = 0$  and  $\nabla f$  continuously differentiable is a particular of the following system [11]:

$$\dot{x} = A(x)x + B(x)w \quad (6)$$

Indeed, given the drift dynamics  $f$ ,  $A$  is a continuously differentiable matrix-valued function as follows:

$$A(x) \triangleq \int_0^1 \nabla f(\nu x) d\nu$$

Hereafter, the matrices  $0$  and  $I$  denote a matrix of zeros and the identity matrix of appropriate size, respectively.

**Theorem 1** Consider the plant (6)-(2) and the filter (3)-(4). Suppose there exist  $X > 0$ ,  $Y > 0$ , and  $M$  that satisfy the following BMIs for all  $i$ :

$$\begin{bmatrix} A_i^T X + X A_i & X M C_i & 0 & 0 \\ * & Y \bar{A}_i + \bar{A}_i^T Y & E_i^T & -Y B_i \\ * & * & -I & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0$$

where  $\bar{A}_i = A_i + M C_i$ . Then, the  $\mathcal{L}_2$  gain of the plant-and-filter system from the disturbance,  $w$ , to the output error,  $e$ , is less or equal to  $\gamma$ .

**Proof:** We develop the solution to the filtering design problem using the following expansion [8]:

$$A(x) = \sum_{i=1}^N \alpha_i(x) A_i \quad (7)$$

where the interpolation functions,  $\alpha_i$ , satisfy the constraints below

$$\alpha_i(x) \geq 0 \text{ and } \sum_{i=1}^N \alpha_i(x) = 1 \quad (8)$$

Using the notation  $\text{Co}A_i$  for the convex hull of the matrices  $\{A_1, \dots, A_N\}$ , expansion (7) implies that

$$A(x) \in \text{Co}A_i$$

and, treating  $B(x)$ ,  $C(x)$ , and  $E(x)$  similarly,

$$B(x) \in \text{Co}B_i$$

$$C(x) \in \text{Co}C_i$$

$$E(x) \in \text{Co}E_i$$

Subtracting (6) from (3), we get

$$\dot{\hat{x}} - \dot{x} = A(\hat{x})\hat{x} - A(x)x + M(C(\hat{x})\hat{x} - C(x)x) - B(x)w \quad (9)$$

It can be shown that [8]

$$A(\hat{x})\hat{x} - A(x)x \in \text{Co}A_i \cdot (\hat{x} - x)$$

Similarly, we have

$$C(\hat{x})\hat{x} - C(x)x \in \text{Co}C_i \cdot (\hat{x} - x)$$

With this in mind, we write the equations of the composite system as follows

$$\begin{aligned} \dot{\tilde{x}} &= \sum_{i=1}^N \alpha_i(\tilde{x}) (\tilde{A}_i \tilde{x} + \tilde{B}_i w) = \tilde{A}(\tilde{x}) \tilde{x} + \tilde{B}(\tilde{x}) w \\ e &= \sum_{i=1}^N \alpha_i(\tilde{x}) \tilde{E}_i \tilde{x} = \tilde{E}(\tilde{x}) \tilde{x} \end{aligned} \quad (10)$$

where the  $\alpha_i$ 's are interpolation functions satisfying the properties (8). The state vector and the matrices of the composite system have as follows:

$$\begin{aligned} \tilde{x} &= \begin{bmatrix} \hat{x} \\ \hat{x} - x \end{bmatrix}, \quad \tilde{A}_i = \begin{bmatrix} A_i & MC_i \\ 0 & A_i + MC_i \end{bmatrix}, \\ \tilde{B}_i &= \begin{bmatrix} 0 \\ -B_i \end{bmatrix}, \quad \tilde{E}_i = \begin{bmatrix} 0 & E_i \end{bmatrix} \end{aligned}$$

Suppose there exists a smooth solution,  $V \geq 0$ , of the dissipation inequality

$$\frac{\partial V}{\partial \tilde{x}} (\tilde{A}(\tilde{x}) \tilde{x} + \tilde{B}(\tilde{x}) w) \leq \gamma^2 w^T w - e^T e \quad (11)$$

Then, the  $\mathcal{L}_2$  gain condition (5) holds [12]. This result implies that the value of  $V$  at an equilibrium of the composite system is zero. Let us consider the following candidate for a solution of (11)

$$V(\tilde{x}) = \tilde{x}^T \tilde{P} \tilde{x} \quad (12)$$

where  $\tilde{P} \geq 0$ . Thus, a sufficient condition for the dissipation inequality (11) is

$$\begin{bmatrix} \tilde{A}_i^T \tilde{P} + \tilde{P} \tilde{A}_i & \tilde{E}_i^T & \tilde{P} \tilde{B}_i \\ * & -I & 0 \\ * & * & -\gamma^2 I \end{bmatrix} < 0 \quad (13)$$

From this point on, one will proceed by assuming a particular structure for the matrix  $\tilde{P}$ . Let us consider a matrix  $\tilde{P}$  of block-diagonal structure as follows:

$$\tilde{P} = \begin{bmatrix} X & \\ & Y \end{bmatrix}$$

Manipulation of the matrix inequality (13) leads to the sufficient condition (7). ■

Also, we note that minimizing  $\gamma$  subject to (7) solves the optimal filtering problem.

#### 4 Simulation

We simulated a second-order nonlinear system, subject to a sinusoidal disturbance, along with its optimal  $\mathcal{L}_2$  gain filter. Figure 1 shows that the ratio of the filter's accumulated error,  $\int_0^T e^2(t)dt$ , stemming from the energy of the disturbance,  $\int_0^T w^2(t)dt$ , is less than  $\gamma^2 \int_0^T w^2(t)dt$ .

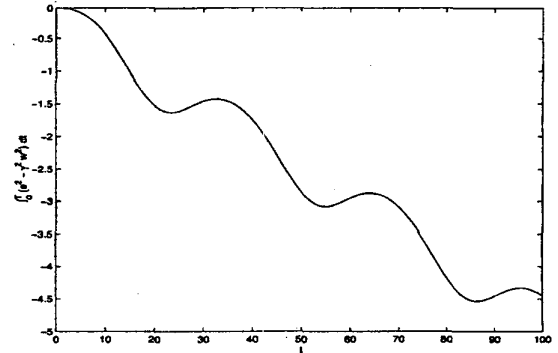


Figure 1: Verification of the filter's  $\mathcal{L}_2$  gain

#### 5 Conclusion

In this paper, we have shown that the filtering problem for a large class of nonlinear systems can be formulated as a feasibility problem with BMI constraints. For the

minimum gain filter, the proposed approach leads to an optimization problem instead.

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