

A STREAMLINED APPROACH FOR TEACHING ROOT LOCUS COMPENSATOR DESIGN

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ABSTRACT

In this paper, classical root locus compensator design methods are streamlined with the objective of simplifying the computational procedures. The proposed methods place less emphasis on compensator synthesis and allow for more emphasis on compensator selection. The design procedures for four standard compensators are developed from a design procedure for the PD compensator. A numerical example is included.

1. INTRODUCTION

Design procedures for the standard compensators are well established in the classical linear control systems literature [1,2,3]. In a typical undergraduate linear control systems course, design procedures are taught using root locus and frequency domain techniques. However, despite their longevity, some of these design methods can yield poor closed-loop time-domain response [4]. In general, the design procedure for each compensator is different and a significant portion of the lecture time must be spent on teaching the individual methods. As a result, there is less time to discuss the issues involved in choosing a compensator for a given problem.

In this paper, classical root locus compensator design methods are streamlined with the objective of simplifying the computational procedures. The proposed methods place less emphasis on compensator synthesis and allow for more emphasis on compensator selection. Established classical control concepts are presented in a logical progression that facilitates comprehension for students in a first course in classical control. The design procedures for four compensators: lead, PI, PID, and PI-lead (practical PID) are developed from a design procedure for the PD compensator. It should be noted that the design procedures for the PD and lead compensators are similar to those found in standard linear control texts [1,2,3]. Procedures for rate feedback and lag compensators have been developed using the PD design procedure and can be found in [5]. The proposed design procedures require few computations. This allows students to quickly complete a single iteration of the design by hand or write a short MATLAB program to iterate the design process.

The remainder of the paper is organized as follows. A review of root locus design and the general compensator design problem is presented in Section 2. A PD compensator design procedure is presented in Section 3. In Section 4, the lead compensator design

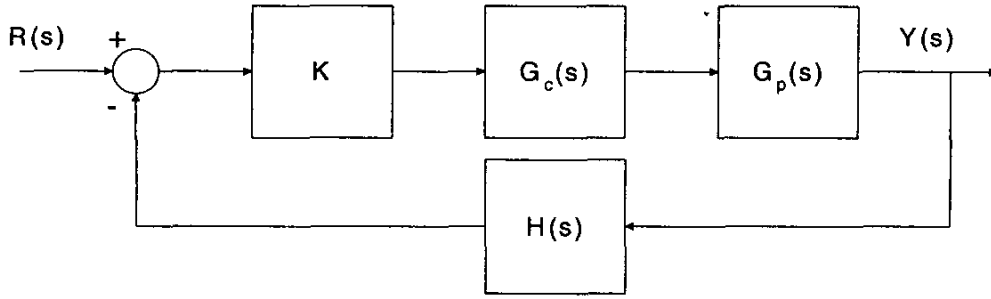


Figure 1: Compensated system

procedure is presented and the limits of its performance are evaluating using a comparable PD compensator design. A design procedure for the PI compensator is presented in Section 5 and design procedures for the PID and PI-lead (practical PID) are presented in Section 6. An example of a PI-lead design is presented in Section 7. Concluding remarks are presented in Section 8.

2. COMPENSATOR DESIGN USING ROOT LOCUS

The block diagram of a feedback control system is shown in Figure 1 where $G_p(s)$ represents the plant and actuator, $G_c(s)$ represents the compensator, and $H(s)$ represents the sensor. If a compensator is required to achieve the design point

$s_0 = -\sigma + j\omega$, the open loop transfer function

must satisfy the angle and magnitude criteria [1].

Equivalently, the compensator and gain must satisfy

$$\angle G_c(s_0) = 180^\circ - \angle G_p(s_0) - \angle H(s_0) \quad (1)$$

$$K = \frac{1}{|G_c(s_0)G_p(s_0)H(s_0)|} \quad (2)$$

Given the particular structure of the chosen compensator, the relationships in (1) and (2) are used to compute the compensator parameters and the control gain.

3. PD COMPENSATOR

3.1 Synthesis

The PD compensator represents the most basic compensator for improving the transient response beyond the capabilities of a proportional controller (uncompensated design). Although the PD compensator may suffer from actuator saturation or noise amplification, the PD design procedure has direct application in rate feedback compensator design and will provide the basis for the other compensator design procedures. In Section 4, the PD compensator is used to establish the limits of performance for the lead compensator.

The PD compensator has a transfer function

$$G_c(s) = s + z \quad \text{with the angle of the compensator}$$

at the design point s_0 given by

$$\angle G_c(s_0) = \angle(s_0 + z) = \tan^{-1}\left(\frac{\omega}{z - \sigma}\right) \quad (3)$$

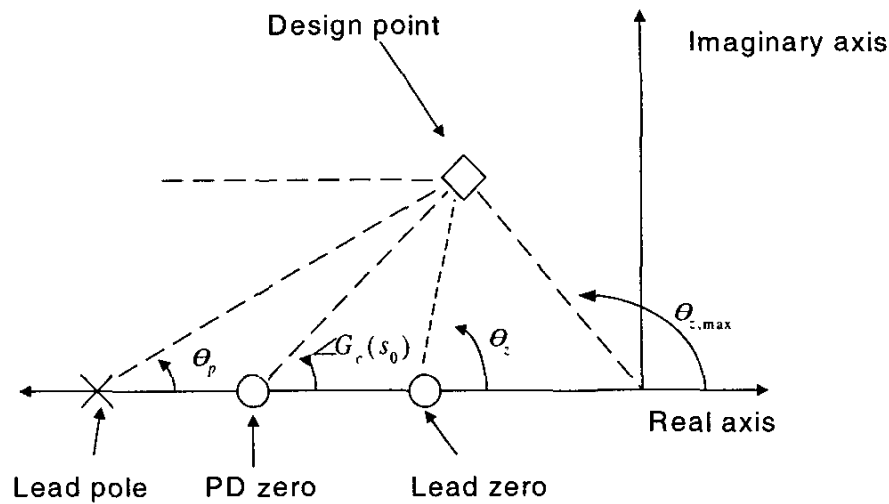


Figure 2: Relationship between the PD and lead compensators

Given the desired compensator angle from (1), the compensator zero z is computed from

$$z = \sigma + \frac{\omega}{\tan(\angle G_c(s_0))} \quad (4)$$

After $G_c(s)$ is determined, the magnitude criterion

(2) is used to compute the control gain K .

3.2 Feasibility of the PD design

There is a limit to the improvement that a PD compensator can achieve. In general, the compensator zero should not be placed in the right half plane because this design can lead to poor performance and/or instability. Under this assumption, the maximum compensator angle $\theta_{z,max}$ is achieved by placing the compensator zero at the origin or, equivalently, a pure derivative compensator. The angle $\theta_{z,max}$ can be computed from the design point from

$$\theta_{z,max} = 180^\circ - \tan^{-1}(\omega/\sigma) \quad (5)$$

It follows that the design point can be achieved by a single PD compensator if and only if

$$\angle G_c(s_0) < \theta_{z,max}.$$

4. LEAD COMPENSATOR

4.1 Synthesis

The lead compensator improves the transient response as compared to the uncompensated system without the practical limitations caused by the derivative action in the PD compensator. The lead compensator has a

transfer function $G_c(s) = \frac{s+z}{s+p}$ where $z < p$.

The compensator angle is given by

$$\begin{aligned} \angle G_c(s_0) &= \tan^{-1}\left(\frac{\omega}{z-\sigma}\right) - \tan^{-1}\left(\frac{\omega}{p-\sigma}\right) \\ &= \theta_z - \theta_p \end{aligned} \quad (6)$$

For a given desired compensator angle $\angle G_c(s_0)$,

the angle contributed by the lead compensator zero θ_z must be greater than that for a PD compensator due to the angle contributed by the lead compensator pole θ_p . Figure 2 provides a graphical comparison of the location of the PD compensator zero and the location of the lead compensator zero and pole. From (6), it follows that the PD compensator angle (3) is obtained if $p \rightarrow \infty$. In this sense, the PD compensator design establishes limits on the lead compensator design.

While there are numerous methods for calculating the compensator pole and zero, the graphical relationship in Figure 2 shows that the lead compensator zero must be to the right the PD zero location for a given design point. This result provides some guidance in the calculation of the lead compensator. One possible design procedure is as follows:

1. Given the desired compensator angle $\angle G_c(s_0)$, compute the PD zero location.
2. Choose a lead zero location
 - (a) to the right of the PD zero location (required)
 - (b) on or to the left of the pole, p_{2z} , of $G_p(s)H(s)$ that is to the left of one more pole than zero (suggested). If $G_p(s)H(s)$ has no zeros, p_{2z} is the 2nd most dominant pole.
3. Compute the lead pole from

$$p = \sigma + \frac{\omega}{\tan(\theta_p)} \quad (7)$$

where

$$\theta_p = \tan^{-1}\left(\frac{\omega}{z - \sigma}\right) - \angle G_c(s_0) \quad (8)$$

4. After $G_c(s)$ is determined, compute the control gain K using (2).

Placing the lead zero to the left of the pole p_{2z} helps to ensure that the transient response is dominated by the closed-loop modes associated with the design point (and its complex conjugate).

4.2 Limits of the lead compensator

The diagram in Figure 2 shows that the location of the PD zero provides an upper limit on the lead zero location. It follows from the feasibility discussion in Section 3.2 that a design point is feasible for a lead compensator if and only if it is feasible for a PD compensator. As a result, the PD compensator design provides a limit on the improvement in transient response that can be achieved with a lead compensator. Furthermore, it can be shown that the compensator error constant for the lead compensator cannot exceed the error constant for the PD compensator.

5. PI COMPENSATOR

Without loss of generality, assume the system in Figure 1 has unity feedback. The PI compensator has a

transfer function $G_c(s) = \frac{s+z}{s}$ and the design

process is simplified by transforming the PI design problem into a PD compensator design problem. The compensator pole (the integrator) is absorbed in the

plant to yield a transformed plant

$$\tilde{G}_p(s) = G_p(s) \cdot \left(\frac{1}{s} \right) \quad (9)$$

and a transformed compensator $\tilde{G}_c(s) = s + z$. It

follows that the compensator zero can be calculated using the PD compensator design procedure on the

transformed plant $\tilde{G}_p(s)$.

6. PID AND PI-LEAD COMPENSATOR (PRACTICAL PID)

6.1 PID compensator

The PID compensator has a transfer function

$$G_c(s) = \frac{(s + z_1)(s + z_2)}{s} \quad . \text{ If the integrator is}$$

absorbed into the plant, the PID-compensated system can be represented by the transformed plant $\tilde{G}_p(s)$

given by (9) and the transformed compensator

$$\tilde{G}_c(s) = (s + z_1)(s + z_2) \quad . \text{ If the compensator}$$

zeros are set equal ($z = z_1 = z_2$), the design process

is greatly simplified.

The desired compensator angle for the transformed plant $\tilde{G}_p(s)$ is given by

$$\angle \tilde{G}_c(s_0) = 180 - \angle \tilde{G}_p(s_0) \quad (10)$$

assuming unity feedback. Therefore, the compensator zeros are given by

$$z = \sigma + \frac{\omega}{\tan\left(\frac{\angle \tilde{G}_c(s_0)}{2}\right)} \quad (11)$$

The magnitude criterion is used to compute the control gain K as in (2). While the assumption that $z_1 = z_2$ is mildly restrictive, the resulting design procedure is a straightforward extension of the PD design procedure.

6.2 PI-lead compensator (practical PID)

The PI-lead compensator has a transfer function

$$G_c(s) = \frac{(s + z_1)(s + z_2)}{s(s + p)} \quad . \text{ The design procedure}$$

for the PI-lead compensator is very similar to the design procedure for the PID compensator. The PI-lead-compensated system can also be represented by

the transformed plant $\tilde{G}_p(s)$ is defined as in (9) and

the transformed compensator is defined as

$$\tilde{G}_c(s) = \frac{(s + z_1)(s + z_2)}{s + p} \quad (12)$$

If the compensator zeros are set equal ($z = z_1 = z_2$),

the design can be completed with a slight modification of the lead compensator design procedure. The angle of the transformed compensator is

$$\begin{aligned} \angle \tilde{G}_c(s_0) &= 2 \tan^{-1}\left(\frac{\omega}{z - \sigma}\right) - \tan^{-1}\left(\frac{\omega}{p - \sigma}\right) \\ &= 2\theta_z - \theta_p \end{aligned} \quad (13)$$

Given the desired angle for the transformed compensator (10), the compensator zeros z must be

placed to the right of the PID compensator zeros. Following the lead compensator procedure in Section 4.1, the recommended region for the PI-lead compensator zeros is between the PID compensator zeros and the pole p_{2z} of $\tilde{G}_p(s)$. Given the location of the PI-lead zeros, the PI-lead pole is given by (7) where

$$\theta_p = 2 \tan^{-1} \left(\frac{\omega}{z - \sigma} \right) - \left(\angle \tilde{G}_c(s_0) \right) \quad (14)$$

The magnitude criterion is used to compute the control gain K as in (2).

7. PI-LEAD EXAMPLE

Consider example 9.5 in [1]. This problem is equivalent to the feedback control system in Figure 1

$$\text{with } G_p(s) = \frac{s+8}{(s+3)(s+6)(s+10)} \text{ and}$$

$H(s) = 1$. The specifications for the PI-lead compensator design are a percent overshoot of 20% and a time-to-peak of 0.2 seconds in the unit step response. These specifications yield a design point of $s_0 = -8.13 + j15.9$. Using the procedure in Section 6.2, the PI-lead zeros compensator should satisfy $3 < z_{PI-lead} < 14.6$. After some iteration, a PI-lead

$$\text{compensator } G_c(s) = \frac{(s+4.70)^2}{s(s+14.6)} \text{ is computed}$$

and the resulting gain is $K = 309$. The closed loop unit step response is computed using MATLAB and it has 20% overshoot and a time-to-peak of 0.2 seconds.

8. CONCLUSIONS

Classical root locus compensator design methods have been streamlined with the objective of simplifying the computational procedures. The design procedures for four compensators: lead, PI, PID, and PI-lead (practical PID) were developed from a design procedure for the PD compensator. Completely analogous design methods for frequency domain compensator design will be presented in a future paper.

ACKNOWLEDGMENT

The authors would like to thank their colleague Professor Thomas Bechert of the Systems Engineering Department at The United States Naval Academy for the derivation of the relationship between the error constants for the lead and PD compensators discussed in Section 4.2.

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