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Generation models for scale-free networks

Chavdar Dangalchev[1]

Pacific Northwest National Laboratory, 160 Van Giesen Str., ap. 212, Richland, WA 99352, USA

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Abstract

In the last few years it has been established that the connectivity distribution of the large real-world networks often follows the power-law, i.e., they are scale-free networks. In this article stochastic models leading to scale-free network are considered and a model close to them is proposed. Deterministic models for creating scale-free networks with given nodes (static model) are demonstrated. A characteristic of graphs, which could be used for determining the scale-free topology of networks, is suggested.
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1. Introduction

Networks are used extensively to study and describe topologically complex real-world systems: social networks, the World-Wide Web, the system of co-authorship in the scientific community, etc. The beginning of the study of topology of the complex networks started from the random graph theory of Erdos and Renyi ER in Ref. [1], followed by the small-world model of Watts–Strogatz [2]. Recent advances in the theory of complex networks by Barabasi (and his co-authors) in Refs. [3–8] led to the introduction of preference and growth as elements of creating scale-free networks. In Ref. [9] Dorogovtsev and Mendes have proved that aging (preference growth based on the age of nodes) can also lead to scale-free networks.

[1] Most of this work was done while the author was with Pacific Northwest National Laboratory and on leave from Institute of Mathematics, Bulgarian Academy of Sciences, Bulgaria.

E-mail address: dangalchev@hotmail.com (C. Dangalchev).

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In the Erdos ER network the nodes are connected randomly and all nodes have the same probability to be connected. Barabasi and Albert in Ref. [3] have suggested a preference attachment as a part of the network model (BA). At each step a new node is created and linked to the old nodes with probability proportional to the number of links each node already has. In this case we have a preference function of connection to node "i" $Pr^{(BA)}(i) = K_i$, where K_i is the number of connections to node i. For consistency we can assume that the Erdos ER network is created using a preference function which is a constant (all nodes have the same, given in advance, probability to be connected), i.e., the function using to link the nodes is $Pr^{(ER)}(i) = 1$. Dorogovtsev et al., in Ref. [10] have introduced another model DM with linear preference function $Pr^{(DM)}(i) = K_i + D$, where D is a constant ($D \geqslant 0$).

The goal of this paper is to present a network created using a different preference function, deterministic static models of scale-free networks and a property, which could be characteristic of scale-free networks.

2. Two-levels network model

We will describe the 2-levels network model. Let us consider the collaboration graph, i.e., the network presenting co-authorship (see Ref. [7]). It is true that each author prefers co-authors with more links (publications). But it is also true that the quality of the links matter. Authors with more respected co-authors will be preferred to authors with the same number of links but co-authoring with authors having less links. In Fig. 1 author 1 will be much more preferable than author 2 (in spite of the fact that they both have 5 publications).

That is why it is worth considering a model where the preference function is the sum of links of a node plus the sum of links of all nodes connected to it, e.g. to use the formula $Pr(i) = K_i + \sum K_j$. This preference function (we can call this model "2-levels" model) has advantages—it takes into consideration the quality of links. But it also has disadvantages—it values in the same way the publications of the author as publications of his co-authors.

In Fig. 2 authors 1 and 2 have the same preference function equal to 12. If we have to choose a co-author the obvious choice will be author 1 (with 6 publications)

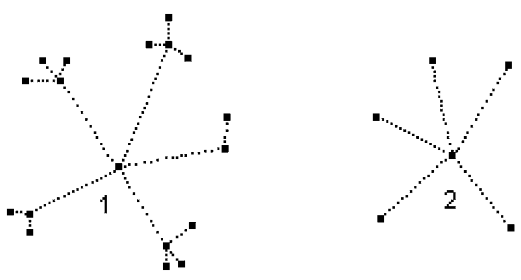

Fig. 1. Co-authorship preference. Author 1 is more preferable as a co-author than author 2.

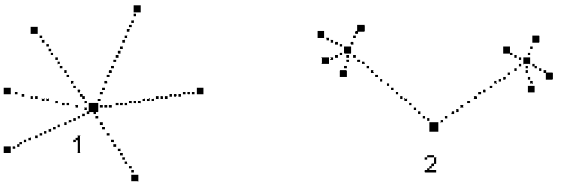

Fig. 2. Co-authorship preference. Author 1 is more preferable as a co-author than author 2.

compared to author 2 (with only 2 publications). To take this into account we change the preference function of the 2-levels model to $Pr^{(2L)}(i) = K_i + C.\sum K_j$. In the formula the sum is on all nodes j connected to node i and C is a coefficient: $C \in [0,1]$. When $C = 0$ the preference of the 2-levels model (2L) is equal to the scale-free BA preference function. When $C = 1$ we receive the first suggested formula. All the test results with model 2L, shown later, will use coefficient C equal to 0.5.

3. Preliminary tests

The natural question is what is the topology of 2-levels networks? Is the behavior of the 2L model similar to random ER networks or similar to scale-free networks? To answer these questions we have run experiments to study the probability $P(K)$ of a node having exactly K links.

Fig. 3 uses logarithmic scales to present the $P(K)$ as a function of K for a random ER network and networks generated by using BA and the 2L preference functions. The results show that the 2L model is closer to the BA model.

We used the model with linear preference function DM to find the topology of the 2L networks. We compared the hubs of the networks (the nodes with the maximal number $K_{\max}$ of links). The maximal connectivity for the random ER model is substantially less than those for other models ($K_{\max}^{(ER)} < 0.15\, K_{\max}^{(BA)}$) with the same size (nodes and links). Comparing statistically the maximal connectivity for the BA model, the 2L model with $C = 0.5$ and linear preference DM model with $D = 4$ (with the same number of nodes and links) we have received:

$$K_{\max}^{(DM)} < K_{\max}^{(2L)} < K_{\max}^{(BA)} . \qquad (1)$$

Networks with the same number of nodes (up to 6000) and 2 links for each new node are generated, using the 3 models. The maximal number of links for 2L is statistically between the BA model and the linear preference model. The linear regression of the numerical results gives $K_{\max}^{(DM)} \approx 0.55\, K_{\max}^{(2L)}$ and $K_{\max}^{(BA)} \approx 1.4\, K_{(\max)}^{(2L)}$.

Condition (1) is a clear indication that the 2L model is closer to BA models than to random models—the probability of having a node with n-links for the 2L model is between two power-law functions and it is substantially greater than exponential functions.

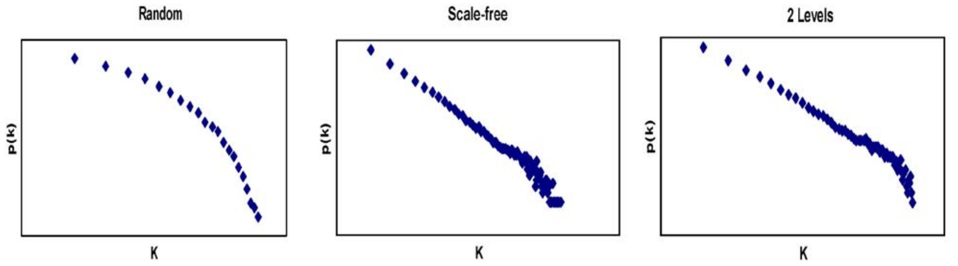

Fig. 3. Connectivity distribution for random, scale-free and 2-levels models. All three graphics use logarithmic scales.

4. Relationships between network models

To find more about the 2L topology we will compare the 2L network model with the well-known models. Let us start initially with m_0 nodes and m_0 links, which form a cycle. At each step we create a node, connecting it randomly to the "m" ($m \leqslant m_0$) old nodes (following the preference function). We use initially connected nodes because we want (for easier comparison) all the models to be in the same conditions. If we start models 2L and BA with unconnected nodes then the nodes, which after the first step are unconnected, remain unconnected—their probability for connection is 0. At any given moment "t" (after "t" steps) there are $(m_0 + mt)$ links (edges) connecting $(m_0 + t)$ existing nodes.

The random ER network model is a non-growing model. It starts with a fixed number of nodes ("n") and then they are connected with probability "p". We could assume (for the purpose of this section) that a given ER network with "n" nodes and "k" links (probability of connection $p = 2k/n(n-1)$) is received by a "growth" process with each new node connected to k/n nodes. This approximation for the ER model could work for larger size networks because of the independent probabilities for connecting the different nodes.

The probability of connecting to one particular node (for all randomly generated networks) is proportional to its preference function. To receive the probability functions we have to normalize the preference functions. For the approximation of the random ER model we have $Pr^{(\mathrm{ER})}(i) \cong 1$ and the probability function is

$$P^{(\mathrm{ER})}(i) = \frac{1}{\sum 1} = \frac{1}{m_0 + t}\ . \tag{2}$$

For BA model probability we receive

$$P^{(\mathrm{BA})}(i) = \frac{K_i}{\sum K_j} = \frac{K_i}{2m_0 + 2mt}\ . \tag{3}$$

For DM model the probability function is

$$P^{(\mathrm{DM})}(i) = \frac{K_i + D}{\sum (K_j + D)} = \frac{K_i + D}{\sum K_j + \sum D} = \frac{K_i + D}{2m_0 + 2mt + (m_0 + t)D}\ . \tag{4}$$

To calculate the probability for the 2L model we will use the following notations: V is the set of all nodes of the graph, E is the set of all edges of the graph, (i,j) is the edge connecting nodes i and j. Then the probability is proportional to the preference function $Pr^{(2L)}(i) = K_i + C\sum_{(i,j)\in E} K_j$. To calculate the probability function we have to calculate the sum:

$$\sum_{i\in V}\left(K_i + C\sum_{(i,j)\in E} K_j\right) = \sum_{i\in V} K_i + C\sum_{i\in V}\sum_{(i,j)\in E} K_j\ .$$

For the second part of the sum we have: each node j contributes to the sum its connectivity K_j exactly K_j times. This could be expressed by

$$\sum_{i\in V}\sum_{(i,j)\in E} K_j = \sum_{j\in V}\sum_{(i,j)\in E} K_j = \sum_{j\in V} K_j \sum_{(i,j)\in E} 1 = \sum_{j\in V} K_j K_j = \sum_{j\in V} K_j^2\ .$$

For the total sum we receive: $\sum_{i\in V} K_i + C\sum_{i\in V} K_j^2$. The probability function for the 2L model is

$$P^{(2L)}(i) = \frac{K_i + C\sum_{(i,j)\in E} K_j}{2m_0 + 2mt + C\sum_{j\in V} K_j^2}\ . \tag{5}$$

Let us assume that at moment "t" all models have generated the same networks (this is true at least at the beginning $t = 0$). How do the connection probabilities for the different models relate? At any moment "t" the average number of connections is $(2m_0 + 2mt)/(m_0 + t)$. Let us consider the most connected node "I_m" of the networks. For I_m: $K_{I_m} \geqslant (2m_0 + 2mt)/(m_0 + t)$ is true. From here (see Appendix A) we have

$$P^{(ER)}(I_m) \leqslant P^{(DM)}(I_m) \leqslant P^{(BA)}(I_m)\ . \tag{6}$$

Because of the difficulties calculating second-level connectivity instead of $\sum_{(I_m,j)\in E} K_j$ we will use the average connectivity for the rest of the nodes (without I_m): $(2m_0 + 2mt - K_{I_m})/(m_0 + t - 1)$. In Appendix B the following is proved

$$P^{(ER)}(I_m) \leqslant P^{(2L)}(I_m) \leqslant P^{(BA)}(I_m)\ . \tag{7}$$

Inequality (7) is true only statistically (or on average) because instead of the real second-level connectivity we have used its average value. There could be a moment t and a graph for which the above is not true.

Let us consider the preference functions of DM and 2L models ($Pr^{(DM)}(i) = K_i + D$ and $Pr^{(2L)}(i) = K_i + C.\sum K_j$). With constants D and C equal to zero both models transform into the BA model. With increase in the constants both preference functions move towards ER preference function. With the right proportion of C and D (for every D we could find a small enough C) we could write for the most connected nodes

$$P^{(DM)}(I_m) \leqslant P^{(2L)}(I_m) \leqslant P^{(BA)}(I_m)\ . \tag{8}$$

More precisely condition $P^{(\mathrm{DM})}(I_m) \leqslant P^{(2\mathrm{L})}(I_m)$ is equivalent statistically (or on average) to the following conditions:

$$A_1 D \geqslant A_2 C + A_3 CD, \quad \text{where coefficients } A_1,\ A_2,\ A_3 \text{ are}$$

$$A_1 = (m_0 + t - 1)\sum_{j \in V}(K_{I_m} - K_j) \geqslant 0 ,$$

$$A_2 = K_{I_m}\left(\sum_{j \in V} K_j(K_{I_m} - K_j) + (m_0 + t)\sum_{j \in V} K_j^2 - \sum_{j \in V} K_j \sum_{j \in V} K_j\right) \geqslant 0 ,$$

$$A_3 = \sum_{j \in V}(K_j - K_{I_m})((m_0 + t - 1)K_j - K_{I_m}) \leqslant 0 .$$

If $A_2 + A_3 D \leqslant 0$ then condition (8) is true for every $C \geqslant 0$. If $D < -A_2/A_3$ then $A_2 + A_3 D > 0$ and condition (8) is true if C satisfies

$$C \leqslant A_1 D/(A_2 + A_3 D) .$$

The exact conditions for coefficients C and D which are necessary for (8) to be true will be discussed in a future paper.

5. Characterization of the networks using dual graphs

We will introduce a characteristic of a graph, which could play a role in describing real networks. For a given graph G we create its dual (edge-dual) graph G' in the following way: for each edge (link) of G we create a vertex (node) of G'; two nodes of G' are connected if their corresponding links in G have a common node. For graph G' we calculate its average connectivity—twice the number of edges divided by the number of vertexes. This number (which could be called "dual connectivity") can characterize graph G.

Why do we think that dual connectivity can play a role in characterizing real networks? The real networks are characterized with the existence of hubs (nodes with very high connectivity).

Let us calculate the dual connectivity for some types of graphs:

- Chain (graph with n vertexes and $n-1$ edges, no vertex has more than 2 corresponding edges) with n nodes has a dual graph which is also a chain with $n-1$ nodes and the dual connectivity is $[2(n-2)]/(n-1) < 2$.
- Cycle (connected graph with n vertexes and n edges, each vertex has exactly 2 corresponding edges, there are no subcycles) with n nodes has as dual graph a cycle with n nodes and the dual connectivity is 2.
- Star (hub) with $n+1$ nodes (graph with $n+1$ vertexes and n edges, each vertex is connected to one central vertex) has a dual graph, which is a complete graph with n nodes and $[n(n-1)]/2$ links. The dual connectivity of the star is $(n-1)$.

Nodes	ER	BA	2L	DM
100	8.65	12.72	12.33	9.63
500	8.86	17.19	16.09	10.72
1000	8.93	21.52	17.12	11.18
2000	9.02	24.31	19.21	11.38

Fig. 4. Dual connectivity. Statistical results for different sizes and models of networks.

- Complete graph (graph with n nodes and $[n(n-1)]/2$ links) has a dual graph with each node connected to exactly $2(n-2)$ so the dual connectivity is $2(n-2)$.

It is clear that hubs and nodes with higher connection can ensure the increase of the dual connectivity with the increase of the size of the network. On the other hand lack of such structures will keep the dual connectivity constant.

To study the behavior of the dual connectivity we conduct a number of tests. We started with graphs with 3 nodes all connected and generated random networks using preference functions following the random ER, BA, 2L and the linear preference DM models. On each step we added a node with 2 random links. The average results are given in Fig. 4.

As we can see the dual connectivity is increasing for scale-free models and stays almost constant for the Erdos random network.

To support the statement that the dual connectivity is increasing in scale-free networks we will prove it for the deterministic model of Ref. [6]. At each step "k" the most connected hub of the model is connected to $2^{k+1}-2$ nodes, which means that the dual graph will have a complete subgraph with $2^{k+1}-2$ nodes and $(2^{k+1}-2)(2^{k+1}-3)/2$ edges. The total number of edges in the graph [6] at step "k" is $2.3^k - 2^{k+1}$, i.e., the dual graph will have the same number of nodes. Hence the total connectivity of the dual graph will be greater than

$$\frac{(2^{k+1}-2)(2^{k+1}-3)}{2.3^k - 2^{k+1}} = \frac{(1-\frac{1}{2^k})(1-\frac{3}{2^{k+1}})}{\frac{1}{2}\frac{3^k}{4^k} - \frac{1}{2}\frac{1}{2^k}} = 2\,\frac{(1-\frac{1}{2^k})(1-\frac{3}{2^{k+1}})}{(\frac{3}{4})^k - (\frac{1}{2})^k}\ .$$

With increasing of total steps k the denominator $(\frac{3}{4})^k - (\frac{1}{2})^k$ decreases and hence the dual connectivity increases.

6. Examples of deterministic models for a scale-free network with fixed nodes

The model we are going to describe is a deterministic static model, which could be constructed implementing the rule "each set of k nodes must be controlled by a node"—"k-control" network model. Let us start with n nodes. From these nodes we assign $n/(k+1)$ to be responsible for controlling the others (the division here is integer, we pick the biggest possible integer number not greater than the fraction). The rest $n - n/(k+1)$. nodes are said to be in 1-level. Each node from the controlling nodes is connected to k of the nodes from 1-level. The last controlling node could be connected to more than k nodes (but no more than $2k$: k nodes plus the remainder of the integer

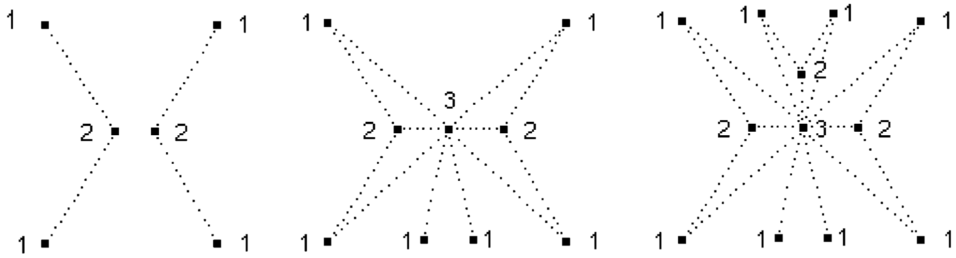

Fig. 5. “2-control” networks with 6, 9 and 12 nodes.

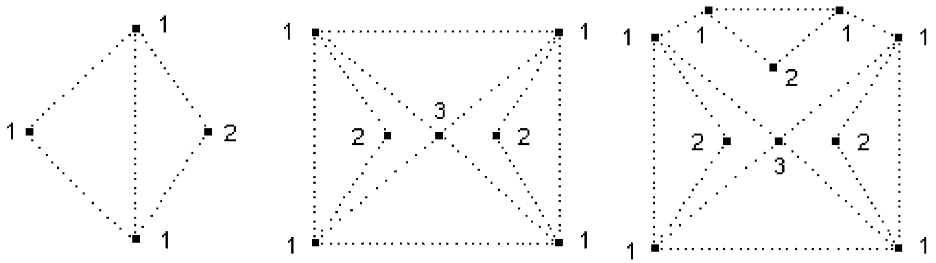

Fig. 6. “2-pyramid” networks with 4, 7 and 10 nodes.

division). From the nodes of 2-level we pick some nodes to control the others, i.e., we create 3-level nodes. In the 3-level we have $(n/(k+1))/(k+1)$ nodes. Each node of 3-level is connected to k subordinate nodes from 2-level (the last node could have more connections) and $k(k+1)$ (or more) corresponding subordinate nodes of 1-level. We continue this way until the upper level has less than $k+1$ nodes. Each node, from any level higher than 1, is connected to all of its subordinates—direct or remote.

In Fig. 5 “2-control” networks are shown (the numbers correspond to the levels of the nodes).

Let us consider another model, which could be called the “k-pyramid” model. Let us start with “n” nodes and we want to have hubs, connected to k^m nodes (n and k are given integer numbers). First we are going to describe a complete k-pyramid with “p” levels. (See Fig. 6).

The basis level (or 1-level) consists of k^{p-1} nodes. The second level (or 2-level) has k^{p-2} nodes and each node of 2-level is linked to k nodes from 1-level (no 1-level node is connected to 2 nodes of 2-level). On the 3-level we have k^{p-3} nodes and each of them is connected to k^2 nodes of 1-level (no 1-level node is connected to 2 nodes of 3-level). Actually, for each 3-level node, we select k nodes of 2-level and connect their corresponding nodes from 1-level to the node. We continue this way until we connect the only node of p-level to all nodes from 1-level. The k-pyramid with “p” levels has $1+k+k^2+\cdots+k^{p-1}=(k^p-1)/(k-1)$ nodes.

To create the graph with "n" nodes we follow the steps A–D:

(A) First we have to find "p" -the number of levels of k-pyramid we are going to implement. Let "p" be the biggest number satisfying the condition

$$1 + k + k^2 + \cdots + k^{p-1} \leqslant n \, .$$

(B) We will start building k-pyramids. We have enough material (nodes) to build one complete k-pyramid with "p" levels. If we have enough material to build more than one pyramid with "p" levels, i.e., the condition is satisfied, $s(1 + k + k^2 + \cdots + k^{p-1}) \leqslant n$, for $s > 1$ we build s k-pyramids (the maximal number of possible k-pyramids is k).

(C) If we do not have enough nodes to build the next pyramid with "p" levels and we still have nodes left, we build pyramids with "$p-1$" levels or "$p-2$" levels, etc. until all nodes are in pyramids with different levels.

(D) All nodes from 1-level we connect with a cycle with no subcycles—each node is connected to 2 other nodes from 1-level.

Now we can calculate the connectivity distribution probabilities. For easy calculation we will assume that $n = 1 + k + k^2 + \cdots + k^p$, i.e., we have built a complete $(p+1)$ levels pyramid. Let us consider $(s+1)$-level of the pyramid. Each of its nodes is connected to k^s nodes (from 1L) and there are exactly k^{p-s} nodes in this level. From here we can receive

$$P(k^s) = \frac{k^{p-s}}{n} = \frac{k^{p-s}}{1 + k + k^2 + \cdots + k^p} = \frac{k^{p-s}(k-1)}{k^{p+1} - 1} = \frac{k^p(k-1)}{k^{p+1} - 1}\,\frac{1}{k^s} \, .$$

As $[k^p(k-1)]/(k^{p+1} - 1)$ is a constant (not depending on "s") we have received: $P(k^s) \approx 1/k^s$, i.e., the connectivity distribution follows the power-law with degree 1.

We will calculate the dual connectivity using again complete pyramid with $n = 1 + k + k^2 + \cdots + k^p$. The pyramid has k^p links between 1-level nodes, $k.k^{p-1} = k^p$ links between 2L nodes and 1L nodes, $k^2.k^{p-2} = k^p$ links between 3L and 1L nodes,..., k^p links between $(p+1)$-level node and 1L nodes. The total number of links is $(p+1)k^p$, hence it is the number of nodes in the dual graph.

Each node from 1L has $(2+p)$ links, which means that each edge between 1L nodes has $2(p+1)$ adjacent edges, which means $2(p+1)$ links in the dual graph. Each edge connecting 2L node and 1L nodes has $(k-1)+(p+1)=k+p$ adjacent edges. Each edge connecting 3L node and 1L nodes has $(k^2-1)+(p+1)=k^2+p$ adjacent edges. Each edge connecting the $(p+1)$-level node and 1L nodes has $(k^p-1)+(p+1)=k^p+p$ adjacent edges. The total links in the dual graph is

$$\begin{aligned} T &= k^p(2p+2) + k^p(k+p) + k^p(k^2+p) + \cdots + k^p(k^p+p) \\ &= k^p\left(1 + p(p+2) + \frac{k^{p+1}-1}{k-1}\right). \end{aligned}$$

For the dual connectivity we receive

$$\frac{T}{(p+1)k^p} = (p+1) + \frac{k^{p+1}-1}{(k-1)(p+1)} \, ,$$

which means that the dual connectivity for the k-pyramid increases with the increase of the number of nodes.

Both versions of the deterministic models are with similar characteristics, which mean that the first version is also scale-free. The process of constructing the graph for the first version is without elements of growth, i.e., it is static (the second model could be interpreted as a result of the growth of k-pyramids and the increase of the number of the levels).

7. Conclusions

In this article we have proposed a stochastic model for generating networks—2L model. The 2L models could be found in the real world (we even think that the 2L model represents co-authorship networks better than the BA model). An idea similar to the 2L idea is used for Google's page-ranking procedure. Although we have proved that the 2L connectivity function is between two scale-free functions (for some coefficients C) the following questions remain: What is the topology of the 2L networks? How does the topology of the 2L models reacting to the changes of their coefficients?

Next we have suggested a property of graphs—dual connectivity. This characteristic of graphs increases when the network has hubs (or nodes with high connectivity). The dual connectivity (or "normalized dual connectivity"—dual connectivity divided by the connectivity of the original graph) could be used as indicator for BA topology of networks.

We have also created deterministic static models for scale-free networks. It extends earlier works where growth (preferential or aging) is an essential element of the scale-free networks creation. The interesting fact for the models is that they are created for a given number of nodes, which could be an indication that the scale-free topology is not necessarily related to the growth of the network.

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Appendix A. Proof of inequalities (6)

Starting from inequality of connections for the most connected node "I_m" of the networks, $K_{I_m} \geqslant \frac{2m_0+2mt}{m_0+t}$, we have

$$K_{I_m}(m_0+t) \geqslant 2m_0 + 2mt \ ,$$

$$K_{I_m}(m_0+t)D + (2m_0+2mt)K_{I_m} \geqslant 2m_0 D + 2mtD + (2m_0+2mt)K_{I_m} \ ,$$

$$K_{I_m}[2m_0 + 2mt + (m_0 + t)D] \geqslant (2m_0 + 2mt)(K_{I_m} + D) ,$$

$$P^{(\mathrm{BA})}(I_m) \geqslant P^{(\mathrm{DM})}(I_m) .$$

From the same inequality we get

$$K_{I_m}(m_0 + t) \geqslant 2m_0 + 2mt ,$$

$$K_{I_m}(m_0 + t) + D(m_0 + t) \geqslant 2m_0 + 2mt + D(m_0 + t) ,$$

$$(K_{I_m} + D)(m_0 + t) \geqslant 2m_0 + 2mt + D(m_0 + t) ,$$

$$P^{(\mathrm{DM})}(I_m) \geqslant P^{(\mathrm{ER})}(I_m)$$

which proves (6):

$$P^{(\mathrm{ER})}(I_m) \leqslant P^{(\mathrm{DM})}(I_m) \leqslant P^{(\mathrm{BA})}(I_m).$$

Appendix B. Proof of inequalities (7)

Instead of $\sum_{(I_m,j)\in E} K_j$ we will use the average connectivity for the rest of the nodes (without the most connected node "I_m"): $(2m_0 + 2mt - K_{I_m})/(m_0 + t - 1)$.

Using the following inequalities we obtain

$$K_{I_m}(m_0 + t) \geqslant 2m_0 + 2mt ,$$

$$(2m_0 + 2mt)(m_0 + t) - 2m_0 - 2mt \geqslant (2m_0 + 2mt)(m_0 + t) - K_{I_m}(m_0 + t) ,$$

$$(2m_0 + 2mt)(m_0 + t - 1) \geqslant (2m_0 + 2mt - K_{I_m})(m_0 + t) ,$$

$$\frac{2m_0 + 2mt - K_{I_m}}{m_0 + t - 1} \leqslant \frac{2m_0 + 2mt}{m_0 + t} ,$$

i.e., the average connections of the rest of the nodes (without I_m) are less than the average connections for the whole network. From here we get

$$\sum_{(I_m,j)\in E} K_j \leqslant \sum_{(I_m,j)\in E} \frac{2m_0 + 2mt}{m_0 + t} \leqslant K_{I_m} \frac{2m_0 + 2mt}{m_0 + t} . \tag{9}$$

The above inequalities are true only statistically (or on average).

Now we can estimate the probability function for the 2L model. Let us start from the inequality between arithmetic and quadratic averages and use (9)

$$\sum_{j\in V} K_j^2/(m_0 + t) \geqslant \left(\sum_{j\in V} K_j/(m_0 + t)\right)^2 ,$$

$$\sum_{j\in V} K_j^2 \geqslant \frac{(2m_0 + 2mt)^2}{m_0 + t} ,$$

$$K_{I_m}\sum_{j\in V}K_j^2 \geqslant K_{I_m}\frac{(2m_0+2mt)^2}{m_0+t} = (2m_0+2mt)\left(K_{I_m}\frac{2m_0+2mt}{m_0+t}\right)$$

$$\geqslant (2m_0+2mt)\sum_{(I_m,j)\in E}K_j\ .$$

Using $C \geqslant 0$ we get

$$K_{I_m}(2m_0+2mt)+C.K_{I_m}\sum_{j\in V}K_j^2 \geqslant (2m_0+2mt)C\sum_{(I_m,j)\in E}K_j + K_{I_m}(2m_0+2mt)\ ,$$

$$K_{I_m}\left((2m_0+2mt)+C\sum_{j\in V}K_j^2\right) \geqslant (2m_0+2mt)\left(K_{I_m}+C\sum_{(I_m,j)\in E}K_j\right)\ ,$$

$$P^{(\mathrm{BA})}(I_m) \geqslant P^{(\mathrm{2L})}(I_m)\ .$$

To prove the other part of inequality (7) we will start with the inequality

$$\sum_{j\in V}(K_{I_m}-K_j)((m_0+t-1)K_j-K_{I_m}) \geqslant 0\ ,$$

which is true because K_{I_m} is the most connected node and its connectivity is less than m_0+t-1. From here we get

$$\sum_{j\in V}(K_{I_m}-K_j)(m_0+t-1)K_j - \sum_{j\in V}(K_{I_m}-K_j)K_{I_m} \geqslant 0\ ,$$

$$(m_0+t-1)\sum_{j\in V}K_{I_m}K_j-(m_0+t-1)\sum_{j\in V}K_j^2+K_{I_m}\sum_{j\in V}K_j-(m_0+t)K_{I_m}^2 \geqslant 0\ ,$$

$$(m_0+t)\sum_{j\in V}K_{I_m}K_j-(m_0+t)K_{I_m}^2 \geqslant (m_0+t-1)\sum_{j\in V}K_j^2\ ,$$

$$(m_0+t)K_{I_m}\left(\sum_{j\in V}K_j-K_{I_m}\right) \geqslant (m_0+t-1)\sum_{j\in V}K_j^2\ ,$$

$$(m_0+t)K_{I_m}\frac{2m_0+2mt-K_{I_m}}{m_0+t-1} \geqslant \sum_{j\in V}K_j^2\ ,$$

$$(m_0+t)\sum_{(I_m,j)\in E}K_j \geqslant \sum_{j\in V}K_j^2\ . \tag{10}$$

Using (10) with $C \geqslant 0$ we get

$$C\left(\sum_{j\in V}K_j^2-(m_0+t)\sum_{(I_m,j)\in E}K_j\right) \leqslant 0 \leqslant \sum_{j\in V}(K_{I_m}-K_j)\ ,$$

$$\sum_{j\in V}K_j+C\sum_{j\in V}K_j^2 \leqslant (m_0+t)C\sum_{(I_m,j)\in E}K_j+(m_0+t)K_{I_m}\ ,$$

$$P^{(\mathrm{ER})}(I_m) \leqslant P^{(\mathrm{2L})}(I_m)\ .$$

Inequality (7)

$$P^{(\mathrm{ER})}(I_m) \leqslant P^{(\mathrm{2L})}(I_m) \leqslant P^{(\mathrm{BA})}(I_m)$$

is true only statistically (or on average) because instead of the real second-level connectivities we have used their average values.

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