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AN EXACT SOLUTION OF THE LONG ROD PENETRATION EQUATIONS

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Summary—An exact solution is presented for the long rod penetration equations first formulated by Alekseevski in 1966 and independently by Tate in 1967. This analytical solution allows a faster and easier solution of the penetration equations, since stability considerations associated with any numerically integrated solutions are avoided. Additionally, an analytical solution provides greater insight into the penetration mechanism than a comparable numerically integrated solution.

NOTATION

$A = \Sigma/(2K\sqrt{\gamma})$

$B = \dfrac{1}{8K\sqrt{\gamma}(\sqrt{\gamma}+1)}$

$C = \dfrac{\Sigma^2(1-\gamma)(\sqrt{\gamma}+1)}{8K\sqrt{\gamma}}$

E = a constant in the W integral

$E_1 = 2/(A+1)$

$E_2 = 2/(A-1)$

$F = 4\dot{u}_0 M\sqrt{\gamma}\exp\left[N - \dfrac{\Sigma}{4K}\right]$

$G = N + \ln M + \ln \dot{u}_0$

$H = \dfrac{4\sigma}{\rho U_0^2}$

$K = Y/(\rho_r U_0^2)$

L = instantaneous penetrator length

$M = \{\sqrt{\gamma} + \sqrt{\gamma + \Sigma(1-\gamma)}\}^{\Sigma/(2K\sqrt{\gamma})}$

$N = \dfrac{\sqrt{\gamma+\Sigma(1-\gamma)}}{2K(1-\gamma)} - \dfrac{\gamma}{2K(1-\gamma)}$

P = depth of penetration

R = target resistance

t = time after impact

U = speed of rear of penetrator

u = normalized speed of rear of penetrator (U/U_0)

V = penetration velocity

v = normalized penetration velocity (V/U_0)

W = an integral function of five parameters, $W(B, C, E, y_1, y_2) = E\displaystyle\int_{y1}^{y2} \exp(By^E - Cy^{-E})\,dy$

Y = penetrator yield stress

y = dummy integration variable

y_1 = dummy integration lower limit

y_2 = dummy integration upper limit

z = transformation variable, $\sqrt{z} = u\sqrt{\gamma} + \sqrt{\gamma u^2 + \Sigma(1-\gamma)}$

z_x = terminal value of z

$\gamma = \rho_t/\rho_r$

θ = transformation variable

ρ = density of rod and target under special case of $\rho_t = \rho_r$

ρ_r = rod density

ρ_t = target density

$\Sigma = 2(R-Y)/(\rho_r U_0^2)$

σ = resistive stress of rod and target under special case of $R = Y$
ϕ = tranformation varbiable
A subscript '0' denotes initial value
A dotted quantity represents the time derivative d/dt
A double dotted quantity represents the second time derivative $\mathrm{d}^2/\mathrm{d}t^2$

INTRODUCTION

The impact of a long, slender, eroding rod at high speed on a thick semi-infinite target was initially formulated by Alekseevski [1] and Tate [2,3]. The governing equations, using the notation of Wright and Frank [4], are:

$$\dot{L} = V - U, \tag{1}$$

$$L\dot{U} = -Y/\rho_r, \tag{2}$$

$$\tfrac{1}{2}\rho_r(U - V)^2 + Y = \tfrac{1}{2}\rho_t V^2 + R, \tag{3}$$

and

$$\dot{P} = V, \tag{4}$$

where U is the speed of the rear of the penetrator, L is the instantaneous penetrator length, V is the penetration velocity, P is the depth of penetration, ρ_r is the penetrator density, Y is the penetrator yield stress, ρ_t is the target density, and R is the target resistance. In the equations, a dotted quantity represents the time derivative, d/dt.

Wright and Frank [4] and Frank and Zook [5] discuss these equations in detail, including the assumptions made in the derivation and approximate solutions. Our intent is to analyse the mathematical, not the physical, aspects of Eqns (1)–(4). Basically, L, U, V and P are the unknown, dependent variables, t (the time) is the independent variable, and ρ_r, Y, ρ_t and R are known constants. These equations are commonly used to solve a multitude of rod/jet impact problems.

THE SOLUTION

An exact analytical solution of Eqns (1)–(4) for L, U, V and P is now obtained. From Eqn (3),

$$v = \frac{u - \sqrt{\gamma u^2 + \Sigma(1 - \gamma)}}{1 - \gamma}, \tag{5}$$

where, if U_0 is the initial (known) penetrator velocity, $v = V/U_0$, $u = U/U_0$, $\gamma = \rho_t/\rho_r$ and $\Sigma = 2(R - Y)/(\rho_r U_0^2)$. The minus sign is chosen for the radical in the solution of the quadratic Eqn (3), to guarantee that V and U remain real, with $V < U$. Note that for the case $R > Y$ ($\Sigma > 0$), the minimum admissable value of v is zero, corresponding to the moment that penetration ceases, which occurs at $u = \sqrt{\Sigma}$. For the case of $R < Y$ ($\Sigma < 0$), the minimal admissable value of u is $\sqrt{-\Sigma/\gamma}$, in order to keep the root real in (5). In this case $v = u$, corresponding to the situation where rod erosion ceases, and rigid body penetration commences.

Next, from Eqn (2), with $K = Y/(\rho_r U_0^2)$,

$$L\dot{u} = -KU_0. \tag{6}$$

Differentiation gives

$$L\ddot{u} + \dot{u}\dot{L} = 0,$$

and the solution for dL/dt, eliminating L, is

$$\dot{L} = \frac{KU_0\ddot{u}}{\dot{u}^2}.$$

Alternately, from Eqns (1) and (5),

$$\dot{L} = \frac{U_0}{(1-\gamma)}\{u\gamma - \sqrt{\gamma u^2 + \Sigma(1-\gamma)}\}.$$

Combining these two expressions to eliminate $\mathrm{d}L/\mathrm{d}t$ gives

$$\frac{\ddot{u}}{\dot{u}} = \frac{\gamma u \dot{u}}{K(1-\gamma)} - \frac{\dot{u}\sqrt{\gamma u^2 + \Sigma(1-\gamma)}}{K(1-\gamma)}. \tag{7}$$

Straightforward integration yields

$$\ln(\dot{u}\{u\sqrt{\gamma} + \sqrt{\gamma u^2 + \Sigma(1-\gamma)}\}^{\Sigma/2K\sqrt{\gamma}}) = \frac{\gamma u^2}{2K(1-\gamma)} - \frac{u}{2K(1-\gamma)}\sqrt{\gamma u^2 + \Sigma(1-\gamma)} + G, \tag{8}$$

where G is a constant of integration which results from evaluation of the integral at the onset of penetration, when $u = 1$ and $\dot{u} = \dot{u}_0$. Note that $\dot{u}_0$, which equals $(1/U_0)\,\mathrm{d}U/\mathrm{d}t|_0$, and has dimensions of $[1/t]$, can be evaluated from Eqn (6) as $\dot{u}_0 = -KU_0/L_0$. The constant G may be expressed as

$$G = N + \ln M + \ln \dot{u}_0,$$

where:

$$M = \{\sqrt{\gamma} + \sqrt{\gamma + \Sigma(1-\gamma)}\}^{\Sigma/(2K\sqrt{\gamma})}$$

$$N = \frac{\sqrt{\gamma + \Sigma(1-\gamma)}}{2K(1-\gamma)} - \frac{\gamma}{2K(1-\gamma)}.$$

By substituting the constant A for the exponent, $\Sigma/(2K\sqrt{\gamma})$, Eqn (8) may be expressed in the following length nondimensionalized form:

$$\ln\left[\frac{\dot{u}}{\dot{u}_0}\frac{\{u\sqrt{\gamma} + \sqrt{\gamma u^2 + \Sigma(1-\gamma)}\}^A}{M}\right] = \frac{\gamma u^2}{2K(1-\gamma)} - \frac{u}{2K(1-\gamma)}\sqrt{\gamma u^2 + \Sigma(1-\gamma)} + N. \tag{9}$$

By introducing the following transformation variable z,

$$\sqrt{z} = u\sqrt{\gamma} + \sqrt{\gamma u^2 + \Sigma(1-\gamma)}, \tag{10}$$

Eqn (9), under the transformation (10), yields

$$(z^{(A-1)/2} + \Sigma(1-\gamma)z^{(A-3)/2})\exp(Bz - C/z)\,\mathrm{d}z = F\,\mathrm{d}t, \tag{11}$$

where

$$B = \frac{1}{8K\sqrt{\gamma}(\sqrt{\gamma}+1)},$$

$$C = \frac{\Sigma^2(1-\gamma)(\sqrt{\gamma}+1)}{8K\sqrt{\gamma}},$$

and

$$F = 4\dot{u}_0 M\sqrt{\gamma}\exp\left[N - \frac{\Sigma}{4K}\right].$$

The use of the transformation (10) produces a differential equation (11), in which the variables z and t are now separable. If integrated from time 0 to some finite time t in the penetration process, the limits on z will vary from its initial value, when $u = 1$, of

$$z_0 = (\sqrt{\gamma} + \sqrt{\gamma + \Sigma(1-\gamma)})^2,$$

to some intermediate value z. For the case of $R > Y$, the terminal value of time at which the governing equations are applicable occurs when penetration ceases at $v = 0$, in which case $u = \sqrt{\Sigma}$ and the terminal value of z, expressed as z_x, is given by

$$z_x = \Sigma(\sqrt{\gamma} + 1)^2.$$

For the case of $R < Y$, the long rod penetration equations are only valid (without modification) to the time at which the penetrator begins rigid body penetration, in which case $u = v = \sqrt{-\Sigma/\gamma}$, and the terminal value of z becomes

$$z_x = (-\Sigma)(\sqrt{\gamma} + 1)^2.$$

Note that z is always positive, since Σ is positive for $R > Y$, and $(-\Sigma)$ is also positive for $R < Y$.

Equation (11) may be further simplified by letting

$$\phi = z^{(A+1)/2}$$

in the first integral over z and

$$\theta = z^{(A-1)/2}$$

in the second integral over z. Under these transformations, and letting $E_1 = 2/(A+1)$ and $E_2 = 2/(A-1)$, the integration of Eqn (11) reduces to

$$E_1 \int_{\phi_0}^{\phi} \exp(B\phi^{E_1} - C\phi^{-E_1})\,\mathrm{d}\phi + \Sigma(1-\gamma)E_2 \int_{\theta_0}^{\theta} \exp(B\theta^{E_2} - C\theta^{-E_2})\,\mathrm{d}\theta = F \int_0^t \mathrm{d}t. \quad (12)$$

The solution is now reduced to a straightforward integration, though it requires evaluation of an exponential integral which, in theory, is a tabulated function of five input parameters. Defining the function W as

$$W(B, C, E, y_1, y_2) = E \int_{y_1}^{y_2} \exp(By^E - Cy^{-E})\,\mathrm{d}y \quad (13)$$

the solution for t becomes simply

$$t = [W(B, C, E_1, \phi_0, \phi) + \Sigma(1-\gamma)W(B, C, E_2, \theta_0, \theta)]/F.$$

In practice, our function is evaluated by expanding the exponential function in Eqn (13) in a power series and integrating term by term to the desired degree of precision. The result is z as an implicit function of time variable t.

The number of power series terms required for convergence of our W function varies a great deal with the input conditions to the problem. In particular, the evaluation of our W function by way of power series can be exacerbated for problems where the penetrator velocity overwhelms the strengths of the rod and target materials. Fortunately, problems in this velocity range are generally beyond the range of interest for typical long rod penetrator impacts.

Because B is always positive, the first part of the exponential term grows with z. As B is made parabolically larger by increasing the penetrator striking velocity U_0, more terms are required to make the power series converge. Because it is a binomial that needs to be exponentiated, use of n terms in the exponential expansion requires that $n(n-1)/2$ monomials be evaluated. Similarly, the coefficient for each of the n highest order monomials requires $(2n)$ operations to evaluate. Thus, the computational effort required to evaluate our W function varies greatly with initial conditions to the problem.

Typical penetration problems involving significant, but not total, penetrator erosion require that 10 to 20 exponential terms be evaluated in order to keep the relative error of the time variable in the fifth decimal place. Such calculations require mere seconds of computation on a PC. As hypervelocity conditions are approached, the number of exponential terms required for the same convergence epsilon may exceed 200 (recall that 200 exponential terms implies $200 \times 199/2 = 19\,900$ monomials), requiring several minutes on a PC. Fortunately, this solution technique need not be pursued for problems in hypervelocity since, under these conditions, the long rod penetration equations approach the standard Bernoulli flow conditions, which may be readily solved by hand.

Having t as a function of z, the normalized rod speed u follows from Eqn (10) as

$$u = \frac{z - \Sigma(1-\gamma)}{2\sqrt{\gamma}\sqrt{z}}. \tag{14}$$

Equation (5) may then be employed to obtain the normalized penetration velocity v. The rate of rod erosion comes from Eqn (1) in the form

$$\dot{L} = U_0(v-u).$$

The penetrator length L may be obtained in the following fashion. From Eqns (1) and (2), one obtains

$$\frac{\dot{L}}{L} = \frac{(u-v)\dot{u}}{K}.$$

Substituting for v and u gives the following:

$$-\frac{\dot{L}}{L} = \frac{\gamma u\dot{u}}{K(1-\gamma)} - \frac{\dot{u}\sqrt{\gamma u^2 + \Sigma(1-\gamma)}}{K(1-\gamma)}.$$

Note the identical form of this relation and Eqn (7). As a result, the solution looks nearly identical to Eqn (9), including the definition of constants A, M and N:

$$-\ln(L/L_0) = -\ln\left[\frac{\{u\sqrt{\gamma} + \sqrt{\gamma u^2 + \Sigma(1-\gamma)}\}^A}{M}\right] + \frac{\gamma u^2}{2K(1-\gamma)} - \frac{u}{2K(1-\gamma)}\sqrt{\gamma u^2 + \Sigma(1-\gamma)} + N.$$

Finally, the penetration P is obtained as follows:

$$P = \int_0^t V\,\mathrm{d}t = U_0\int_0^t v(\mathrm{d}t/\mathrm{d}z)\,\mathrm{d}z. \tag{15}$$

The quantity $\mathrm{d}t/\mathrm{d}z$ has been previously obtained in Eqn (11) as

$$\frac{\mathrm{d}t}{\mathrm{d}z} = \frac{(z^{(A-1)/2} + \Sigma(1-\gamma)z^{(A-3)/2})\exp(Bz - C/z)}{F}.$$

We may express v in terms of z, using Eqns (5) and (14), as follows:

$$v = \frac{1}{2\sqrt{\gamma}}\left[\frac{(1-\sqrt{\gamma})\sqrt{z}}{(1-\gamma)} - \frac{\Sigma(1+\sqrt{\gamma})}{\sqrt{z}}\right].$$

These substitutions into (15) produce an expression for P in terms of the W function defined in Eqn (13), and the solution for penetration P proceeds in a fashion analogous to Eqn (12).

SPECIAL CASE SOLUTIONS

Two special cases are considered:

(a) $\rho_r = \rho_t = \rho$, and $Y = R = \sigma$
(b) $\rho_r = \rho_t = \rho$.

Special case

Equation (3) reduces to

$$(u - v)^2 = v^2,$$

with the non-trivial solution $u = 2v$. Differentiating Eqn (2), and combining the result with Eqn (1), as before, yields:

$$\frac{\ddot{u}}{\dot{u}} = \frac{2u\dot{u}}{-H},$$

where

$$H = \frac{4\sigma}{\rho U_0^2}.$$

Integrating this equation, and evaluating the constant of integration at time equal 0, where $\dot{u} = \dot{u}_0$, which has the value $\dot{u}_0 = (-HU_0)/(4L_0)$, results in

$$\dot{u} = \dot{u}_0 \exp\left[\frac{1 - u^2}{H}\right].$$

Integrating again for u gives the result in terms of our W function as

$$t = \frac{\exp(-1/H)}{2\dot{u}_0} W(1/H, 0, 2, 1, u),$$

which gives u implicitly as a function of time t. As mentioned above, $u = 2v$ applies, and thus determines penetration rate v. Also

$$\dot{L} = \frac{-U_0 u}{2},$$

and

$$L = \frac{-HU_0}{4\dot{u}}$$

follow directly. To determine penetration P, employ the tactic of Eqn (15), namely

$$P = \int_0^t V\,\mathrm{d}t = U_0 \int_1^u v(\mathrm{d}t/\mathrm{d}u)\,\mathrm{d}u.$$

Penetration rate v is known directly in terms of u, equal to $(u/2)$, and $\mathrm{d}t/\mathrm{d}u$ is simply $1/\dot{u}$, given by

$$\frac{\mathrm{d}t}{\mathrm{d}u} = \frac{1}{\dot{u}_0} \exp\left[\frac{u^2 - 1}{H}\right].$$

Thus, making use of the term $\dot{u}_0 = (-HU_0)/(4L_0)$, the penetration may be computed as

$$P = L_0\left\{1 - \exp\left[\frac{u^2 - 1}{H}\right]\right\}.$$

Special case

Equation (3), which is no longer quadratic, as in the general case, yields

$$v = \frac{1}{2}(u - \Sigma/u).$$

Differentiating Eqn (2), and combining the result with Eqn (1), and the expression for v

above, yields

$$K\frac{\ddot{u}}{\dot{u}} = -\frac{u\dot{u}}{2} - \frac{\Sigma\dot{u}}{2u}.$$

Direct integration results in

$$\dot{u} = \dot{u}_0 u^{-A} \exp\left[\frac{1-u^2}{4K}\right],$$

where the exponent A has the value, analogous to the general case derivation, of $\Sigma/(2K)$. The variables are separable, and

$$\int_1^u u^A \exp\left[\frac{u^2-1}{4K}\right] \mathrm{d}u = \dot{u}_0 \int_0^t \mathrm{d}t.$$

This integral is evaluated using the same procedure used for Eqn (11). By letting

$$\phi = u^{(A+1)},$$

the integral reduces to

$$\frac{1}{(A+1)} \int_1^\phi \exp\left[\frac{\phi^{2/(A+1)}-1}{4K}\right] \mathrm{d}\phi = \dot{u}_0 t,$$

the evaluation of which may be expressed in terms of our W function as

$$t = \frac{1}{2\dot{u}_0} \exp\left[\frac{-1}{4K}\right] W[(4K)^{-1}, 0, 2/(A+1), 1, u^{(A+1)}].$$

The evaluation of the remaining variables v, $\mathrm{d}L/\mathrm{d}t$, L and P follows an analogous approach as in the general case.

CONCLUSIONS

An analytical solution to the long rod penetration equations for long rod penetration is offered. The general case is solved, as well as two special cases in which some of the target and penetrator parameters (e.g. density and/or strength) are equal. This analytical solution allows a faster and easier solution of the penetration equations, since stability considerations associated with any numerically integrated solution are avoided.

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