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RATIONAL APPROXIMATIONS FOR THE HOLTSMARK DISTRIBUTION, ITS CUMULATIVE AND DERIVATIVE

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Abstract—The convergent series expansions of the Holtsmark distribution $P(\beta)$, its cumulative $Q(\beta)$, its derivative $R(\beta)$ and the semiconvergent asymptotic series for these functions are used to calculate rational approximations for P, Q, and R, which are valid for all positive β and have maximum errors of approximately 10^{-8}, 10^{-9} and 10^{-7}, respectively.

1. INTRODUCTION

Despite the great advances over the past three decades in the theory of plasma microfield distribution functions, which has been conveniently summarized by Iglesias, Hooper and DeWitt,[1] the distribution function derived by Holtsmark[2] is still of considerable value in dealing with certain properties of low-density, high-temperature plasmas in which particle correlations are unimportant. Recently, the need for the evaluation of the Holtsmark distribution, its cumulative and derivative has arisen in the calculation of atomic partition functions.[3] The technique of Padé approximants is used to calculate rational approximations for these three functions which are valid for all values of the reduced field strength β.

2. ANALYSIS

We consider the Holtsmark distribution function

$$P(\beta) = (2\beta/\pi)\int_0^\infty dt\ t \sin \beta t\ e^{-t^{3/2}} = (2/\beta\pi)\int_0^\infty dt\ t \sin t\ e^{-(t/\beta)^{3/2}}, \qquad (1)$$

its cumulative

$$Q(\beta) = \int_0^\beta d\beta'\ P(\beta') \qquad (2)$$

and its derivative, which we here call $R(\beta)$. Expanding $\sin \beta t$ in the first integral in Eq. (1) and integrating term by term, we obtain

$$P(\beta) = (4/3\pi)\,\beta^2 \sum_{n=0}^{\infty} a_n \beta^{2n}, \qquad a_n \equiv (-1)^n\, \Gamma\!\left(\frac{4}{3}n+2\right)/\Gamma(2n+2)\,. \qquad (3)$$

Integrating and differentiating this expression yields

$$Q(\beta) = (4/9\pi)\,\beta^3 \sum_{n=0}^{\infty} b_n \beta^{2n}, \qquad b_n \equiv 3\, a_n/(2n+3)\,, \qquad (4)$$

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and

$$R(\beta) = (8/3\pi)\,\beta \sum_{n=0}^{\infty} c_n \beta^{2n}, \qquad c_n \equiv (n+1)\,a_n\,. \tag{5}$$

The coefficients in Eqs. (3), (4) and (5) are normalized so that $a_0 = b_0 = c_0 = 1$; values for larger values of n are readily generated by recursion.

To obtain asymptotic expressions, we follow Holtsmark[2] by expanding $[\exp - (t/\beta)^{3/2}]$ in the second form of Eq. (1), then replacing $\sin t$ by $\mathrm{Im}\, e^{it}$. Defining a new variable of integration $u = -it$, we obtain

$$P(\beta) \sim (2/\pi\beta) \sum_{n=0} (-1)^{n+1} \sin(3n\pi/4)\, \Gamma\!\left(\frac{3n}{2} + 2\right) \beta^{-3n/2}/n!\,. \tag{6}$$

Terms with $n = 0,4,8,\ldots$ vanish. It is convenient to shift the index of summation by one so that the first term is nonzero:

$$P(\beta) \sim (15/8)\,\sqrt{2/\pi}\,\beta^{-5/2} \sum_{m=0} \tilde{a}_m \beta^{-3m/2}\,, \tag{7}$$

where

$$\tilde{a}_m \equiv (8/15)\,\sqrt{2/\pi}\,\sin(3(m+1)\pi/4)\,\Gamma\!\left(\frac{3m+7}{2}\right) \Big/ \Gamma(m+2)\,; \tag{8}$$

we have chosen the normalization so that $\tilde{a}_0 = 1$. Recursion relations for $\tilde{a}_m$ are easily derived from those for $\Gamma(n)$. Integrating and recalling that $P(\beta)$ is normalized to unity, we have

$$Q(\beta) = 1 - \int_\beta^\infty d\beta'\, P(\beta') \sim 1 - (5/4)\,\sqrt{2/\pi}\,\beta^{-3/2} \sum_{m=0} \tilde{b}_m \beta^{-3m/2},$$

$$\tilde{b}_m = \tilde{a}_m/m+1\,. \tag{9}$$

Differentiating Eq. (7) leads immediately to

$$R(\beta) \sim (-75/16)\,\sqrt{2/\pi}\,\beta^{-7/2} \sum_{m=0} \tilde{c}_m \beta^{-3m/2}, \qquad \tilde{c}_m = (1 + 3m/5)\tilde{a}_m\,. \tag{10}$$

3. RATIONAL APPROXIMATIONS

We now calculate rational approximations to $P(\beta)$, $Q(\beta)$ and $R(\beta)$ in the following form:

$$0 \le \beta \le \beta^*: \qquad P(\beta) = (4/3\pi)\,\beta^2\, u_1(x)/v_1(x)\,, \tag{11}$$

$$Q(\beta) = (4/9\pi)\,\beta^3\, u_2(x)/v_2(x)\,, \tag{12}$$

$$R(\beta) = (8/3\pi)\,\beta\, u_3(x)/v_3(x)\,; \tag{13}$$

$$\beta^* \le \beta < \infty: \qquad P(\beta) = (15/8)\,\sqrt{2/\pi}\,\beta^{-5/2}\, u_4(y)/v_4(y)\,, \tag{14}$$

Table 1. Coefficients for $P(\beta)$.

n	c_{1n}	d_{1n}	c_{4n}	d_{4n}
	$\beta^* = 5.35$, $L_1 = 12$, $M_1 = 13$, $L_4 = M_4 = 12$			
1	5.3213 58196 9315−2	5.1623 99953 7559−1	−5.8279 10992 3206+1	−6.3385 57111 2344+1
2	9.6792 05787 7047−3	1.2611 69195 1988−1	1.5802 12612 3226+3	1.8894 51071 1591+3
3	4.0835 19154 2469−4	1.9324 87786 9202−2	−2.5497 91301 9232+4	−3.4231 19239 9947+4
4	3.1725 79323 7450−5	2.0744 70827 3751−3	2.6383 10629 3289+5	4.1165 14954 9442+5
5	9.3347 92974 9771−7	1.6494 11125 0278−4	−1.7461 32102 3968+6	−3.3708 17266 3750+6
6	4.0297 73892 3818−8	1.0006 54492 1671−5	6.6626 75117 1900+6	1.8369 81594 1601+7
7	7.2634 59051 1209−10	4.6967 04829 4008−7	−7.3155 45947 2329+6	−5.9159 11859 7522+7
8	1.8177 22697 7773−11	1.7082 31490 7411−8	−5.2626 44361 9572+7	4.9132 01859 9912+7
9	1.5079 59904 3503−13	4.7633 89532 2910−10	2.4110 43181 9602+8	4.5437 54584 6985+8
10	2.0246 46471 2406−15	9.9179 96748 2997−12	−3.1988 13057 5441+8	−2.1012 34228 3505+9
11	1.4246 57599 3322−19	1.4634 21335 3989−13	−2.6036 18538 7058+7	3.8908 91660 4974+9
12	1.5130 90778 2746−20	1.3749 76403 5118−15	−3.7993 38432 7787+6	−2.7548 75377 6661+9
13		6.2306 99595 2226−18		

$$Q(\beta) = 1 - (5/4)\sqrt{2/\pi}\,\beta^{-3/2}\,u_5(y)/v_5(y)\,, \tag{15}$$

$$R(\beta) = (-75/16)\sqrt{2/\pi}\,\beta^{-7/2}\,u_6(y)/v_6(y)\,, \tag{16}$$

where

$$x \equiv \beta^2, \qquad y \equiv \beta^{-3/2}\,, \tag{17}$$

and

$$u_i(t) \equiv \sum_{n=0}^{L_i} c_{in} t^n, \qquad v_i(t) \equiv \sum_{n=0}^{M_i} d_{in} t^n, \qquad i = 1,2,\ldots,6\,. \tag{18}$$

The coefficients c_{in} and d_{in} are independent of β^* and are equal to unity for $n = 0$. The value of β^* may be different for each of the three functions, and is chosen after the coefficients are evaluated so as to minimize the error.

The coefficients c_{in} and d_{in} are determined by equating in turn each rational expression (11)–(16) to the appropriate series from Sec. 2, cross multiplying and formally equating like powers of x or y, as described, for example, by Baker;[4] this is the simplest form of the Padé approximation procedure. The resulting linear systems are described by matrices with elements that depend only on the *difference* of row and column indices, i.e. so-called Toeplitz matrices. We exploit this feature of the system matrix, using a subroutine written by G. B. Rybicki, in order to reduce computing time and to increase accuracy. The calculations have been carried out with 30 sf arithmetic for a number of choices of the polynomial orders L_i and M_i.

To select the optimum values of L_i, M_i and β^*, accurate values of $P(\beta)$ and $R(\beta)$ were evaluated by direct numerical integration using the form

$$P(\beta) = (2/\beta\pi)\int_0^{2\pi} \mathrm{d}t\,\sin t \sum_{n=0} (2n\pi + t)\exp\{-[(2n\pi + t)/\beta]^{3/2}\}\,, \tag{19}$$

Table 2. Coefficients for $Q(\beta)$.

n	c_{2n}	d_{2n}	c_{5n}	d_{5n}
	$\beta^* = 5.40$, $L_2 = M_2 = L_5 = M_5 = 12$			
1	1.8966 42013 8780-1	4.6748 [illegible]494 3157-[illegible]	-6.8144 04030 3307+1	-7.0697 27089 7826+1
2	2.5317 91426 7416-2	1.0267 05493 [illegible]802-[illegible]	2.1615 25739 9963+3	2.3372 19675 0053+3
3	2.1210 29801 3271-3	1.4013 74346 2989-2	-4.1627 60190 1822+4	-4.2254 83206 6084+4
4	1.3477 85085 8071-4	1.3261 [illegible] 9511-4	5.3595 42633 7602+5	6.4541 87020 5303+5
5	6.2081 05054 2043-6	9.1750 02904 [illegible]-5	-4.8015 92995 5805+6	-6.2259 53932 2157+6
6	2.1963 85329 1836-7	4.2662 42187 3130-6	3.0245 64108 [illegible]50+7	4.3151 58094 7636+7
7	5.6533 16102 3147-9	1.8740 12259 8294-7	-[illegible].3222 01586 4016+8	-[illegible].1455 27985 [illegible]+8
8	1.0682 77496 8143-10	5.5486 91202 3144-9	3.8632 16810 1547+8	[illegible].5088 82002 2297+8
9	1.3129 60298 0283-12	1.2082 08502 2752-10	-6.9545 [illegible]5225 8028+8	1.7750 62687 [illegible]+9
10	9.6643 53100 6326-15	1.8399 90125 8672-12	6.5789 91548 8388+8	[illegible].6229 75324 [illegible]+9
11	3.8572 26100 3340-17	1.2633 87494 2625-14	-1.9837 45649 8072+8	-[illegible].0238 49393 6619+9
12	-1.4989 85216 0908-20	8.0621 31538 2642-17	-2.0452 41410 7833+5	[illegible].8724 73984 6038+8

together with a similar expression for $R(\beta)$. Error curves were computed for each rational approximation for the interval $4.0 \leq \beta \leq 6.0$, in which the small-β and large-β forms were expected to have comparable errors. By choosing the small-β and large-β form with the smallest errors, plotting the error curves and locating the intersection point, we determined for $P(\beta)$ the optimum value $\beta^* = 5.35$, which corresponds to a maximum relative error $\epsilon_{max} = 8.9 \times 10^{-9}$ for the approximations with $L_1 = 12$, $M_1 = 13$, $L_4 = M_4 = 12$. For $R(\beta)$, we find $\beta^* = 5.15$, $\epsilon_{max} = 1.6 \times 10^{-7}$ for $L_3 = 13$, $M_3 = 14$, $L_6 = 14$, $M_6 = 13$. No check values were computed for $Q(\beta)$, but by comparing the small$-\beta$ and large$-\beta$ forms for $L_2 = M_2 = L_5 = M_5 = 12$ in the neighborhood of $\beta = 5.0$, we find $\beta^* = 5.40$ and $\epsilon_{max} \simeq 1 \times 10^{-9}$. The coefficients of the rational approximations

Table 3. Coefficients for $R(\beta)$.

n	c_{3n}	d_{3n}	c_{6n}	d_{6n}
	$\beta^* = 5.15$, $L_3 = 13$, $M_3 = 14$, $L_6 = 14$, $M_6 = 13$			
1	-3.7714 53756 2388-1	5.4890 74511 8867-1	-7.6330 65660 3984+1	-8.4500 99450 6606+1
2	2.8658 82706 1726-3	1.4339 80601 8754-1	2.7570 50239 6661+3	[illegible].4156 89437 8926+3
3	-2.4970 47597 8578-3	2.3655 23418 8604-2	-6.1205 28191 0748+4	-8.6428 65578 [illegible]322+4
4	-1.0908 00305 5922-5	2.7557 98165 8846-3	9.1767 67955 5383+5	1.5163 54314 3104+6
5	-5.9411 20278 1125-6	2.4012 66306 8210-4	-9.6479 93996 1065+6	-1.9369 29227 4955+7
6	-3.0835 58842 5598-8	1.6159 55789 5932-5	7.1555 23649 0707+7	1.8443 22194 1642+8
7	-6.0080 85045 0585-9	8.5441 37592 2706-7	-3.6602 83959 2271+8	-1.3194 58456 2566+9
8	-1.3513 40679 4647-11	3.5723 07268 9829-8	1.2099 42069 0429+9	7.0598 13144 3434+9
9	-2.4485 98156 7752-12	1.1773 49943 0426-9	-2.1701 95814 8170+9	-2.7760 30437 2412+10
10	4.5538 91752 5173-15	3.0167 15665 4715-11	7.8558 28079 8258+8	7.7500 66877 2928+10
11	-3.2230 60591 6261-16	5.8399 29379 7887-13	2.5883 18206 9229+9	-1.4429 55383 2890+11
12	1.5088 21629 8286-18	8.0938 93466 2632-15	-8.2367 64554 9389+7	1.5867 62018 3908+11
13	-7.4026 17976 4104-21	7.2087 55477 9050-17	-1.0598 79446 6865+8	-7.6147 09895 7657+10
14		3.1227 32286 7957-19	-3.8444 64866 9187+7	

Table 4. Values of Functions.

β	P(β)	Q(β)	R(β)
1.0	2.7122 08070-1	1.0860 77296-1	3.0773 83690-1
2.0	3.3693 87827-1	4.5176 18482-1	-1.3340 16902-1
3.0	1.7606 29272-1	7.0774 78512-1	-1.3848 70538-1
4.0	8.0673 54136-2	8.2946 55532-1	-5.9639 21261-2
5.0	4.1180 23725-2	8.8754 44652-1	-2.4943 81058-2
6.0	2.3822 08458-2	9.1896 59950-1	-1.1778 55295-2
7.0	1.5164 57557-2	9.3800 57335-1	-6.2662 27225-3
8.0	1.0349 76409-2	9.5054 78667-1	-3.6610 34529-3
9.0	7.4383 07487-3	9.5932 91180-1	-2.2977 05335-3
10.0	5.5613 46283-3	9.6576 48611-1	-1.5242 96364-3

defined by Eqs. (11)–(16) are given in Tables 1, 2 and 3 for $P(\beta)$, $Q(\beta)$ and $R(\beta)$, respectively. To enable users to check that these coefficients are correctly entered into their programs, we give in Table 4 ten-digit values of the functions for $\beta = 1(1)10$ generated by the approximations given here; not all of the digits are necessarily significant.

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