The Reduced Basis Method Applied to Aerothermal Simulations

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The Reduced Basis Method

Motivations

- Modeling : multi-physics non-linear models, complex geometries, genericity
- Uncertainty management / Risk analysis
- Optimization in early design, certification or operating phases

Objective 1: Fast

- Complex geometries
- ⟶ Large number of dofs
- Uncertainty quantification
- ⟶ Large number of runs

Objective 2: Reliable

- Field quality
- Design optimization
- ⟶ Certified bounds
- ⟶ Reach material limits

Main Idea

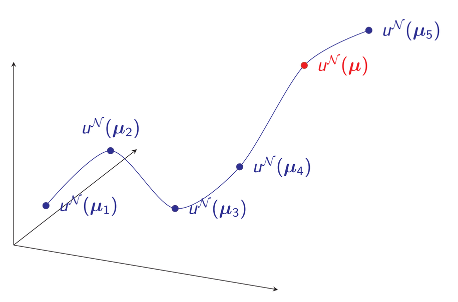
Weak formulation of the model : $a(u(\mu), v; \mu) = f(v; \mu)$

FEM Approximation:

$$\underbrace{X^{\mathcal{N}} = span\{\phi_1, \dots, \phi_{\mathcal{N}}\}}_{\text{FEM approximation space}} \longrightarrow \underbrace{u^{\mathcal{N}}(\mu) = \sum_{i=1}^{\mathcal{N}} u_i^{\mathcal{N}} \phi_i}_{\mathcal{N} \approx 10^6} \longrightarrow \underbrace{A^{\mathcal{N}}(\mu) u^{\mathcal{N}}(\mu) = F^{\mathcal{N}}(\mu)}_{\mathcal{N} \times \mathcal{N} \text{ system expensive to solve}}$$

RB Approximation: $u^N(\mu) \approx u^{\mathcal{N}}(\mu)$: linear combination of FEM solution

$$\underbrace{W^N = span\{u^{\mathcal{N}}(\mu_1), \dots, u^{\mathcal{N}}(\mu_N)\}}_{\text{RB approximation space}} \to \underbrace{u^N(\mu) = \sum_{i=1}^{N} u_i^N(\mu) u^{\mathcal{N}}(\mu_i)}_{10 \leqslant N \leqslant 100} \to \underbrace{A^N(\mu) u^N(\mu) = F^N(\mu)}_{N \times N \text{ system cheaper to solve}}$$



Ingredients

- Set of parameters : $\mathcal{D}^\mu$
- FEM "truth" approximation
  $X^{\mathcal{N}}$: finite element approximation space of dimension $\mathcal{N} >> 1$
  $u^{\mathcal{N}}(\mu) \in X^{\mathcal{N}}$ is solution of $a(u^{\mathcal{N}}(\mu), v; \mu) = f(v; \mu) \quad \forall v \in X^{\mathcal{N}}$
- RB approximation
  Sample : $S_N = \{\mu_1 \in \mathcal{D}^\mu, \dots, \mu_N \in \mathcal{D}^\mu\}$
  Approximation space : $W_N = span\{u^{\mathcal{N}}(\mu_1), \dots, u^{\mathcal{N}}(\mu_N)\}$ with $N << \mathcal{N}$
  Galerkin projection on W_N to determine RB coefficients

Efficient offline-online strategy

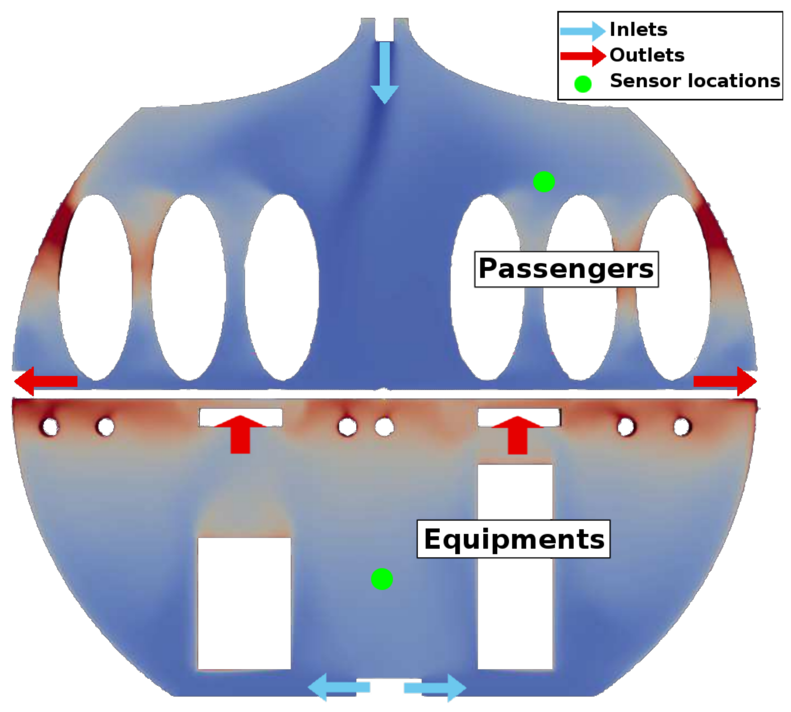
$$u^N(\mu) = \sum_{i=1}^{N} u_i^N(\mu) u^{\mathcal{N}}(\mu_i)$$

$N \times N$ system to solve : $\sum_{i=1}^{N} a(u^{\mathcal{N}}(\mu_i), v_k; \mu) u_i^N(\mu) = f(v_k; \mu), \ 1 \leqslant k \leqslant N$ If the parameter dependance can be expressed as an affine decomposition :

$$a(u, v; \mu) = \sum_{q}^{Q_a} \theta_q^a(\mu)\, a_q(u, v) \quad \text{and} \quad f(v; \mu) = \sum_{q}^{Q_f} \theta_q^f(\mu)\, f_q(v)$$

$$\Rightarrow \sum_{i=1}^{N} \left[\sum_{q=1}^{Q_a} \theta_a^q(\mu) \underbrace{a^q(u^{\mathcal{N}}(\mu_i), u^{\mathcal{N}}(\mu_j))}_{\text{precomputed}} \right] u_i^N(\mu) = \sum_{q=1}^{Q_f} \theta_f^q(\mu) \underbrace{f^q(u^{\mathcal{N}}(\mu_j))}_{\text{precomputed}}$$

Airbus Use-Case



Propose in the context of the ANR Project CHORUS

Objective : Apply the Certified Reduced Basis Methods on an aerothermal simulation in an avionic bay

Model :

- Steady Navier-Stokes/Heat transfer
- Incompressible Newtonian Fluid
- Boussinesq Approximation
- Turbulent Flow

Mathematical Model

Governing Equations

$$\begin{cases} \rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \mu \Delta \mathbf{u} = & \rho \beta (T - T_0) g, & \text{in } \Omega \times [0, T_f], \\ \nabla \cdot \mathbf{u} = & 0, & \text{in } \Omega \times [0, T_f], \\ \mathbf{u} \cdot \nabla T - \kappa \Delta T = & 0, & \text{in } \Omega \times [0, T_f], \\ +\text{Boundary Conditions.} \end{cases} \quad (1)$$

with

- ρ : fluid density $(kg.m^{-3})$,
- $\kappa \approx 10^{-5}\ m^2.s^{-1}$: thermal diffusivity $(m^2.s^{-1})$,
- $\mu \approx 10^{-5}\ m^2.s^{-1}$: dynamic viscosity $(kg.m^{-1}.s^{-1})$,
- β : thermal expansion coefficient (K^{-1}),
- Length Scale $L \approx 1\ m$,
- $\Rightarrow Re \approx 10^5, \quad Pr \approx 1, \quad Pe \approx 10^5$.

Solving Strategy

- Finite Element Discretization
- Newton Algorithm with transient continuation
- Parallel implementation using Feel++ library: `http://www.feelpp.org/`

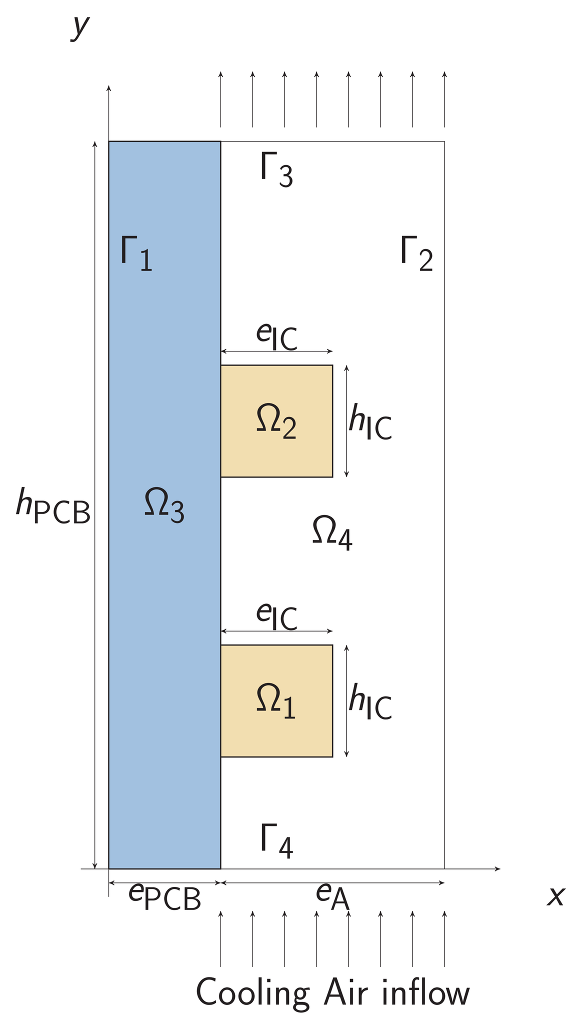
Challenge and Difficulties

- **Multi-physic coupled model**: simultaneous construction of the different reduced spaces
- **High Reynolds flow**: use of stabilization methods (SUPG/GLS) in the FEM and the RB model
- **Non-Linearity**: Newton algorithm with an affine decomposition of the Jacobian/Residual
- **Non-affine terms**: use of Empirical Interpolation Method (EIM) for discrete operators
- **Non-Linearity**: Use of Simultaneous EIM and RB (SER) algorithm to generate an affine approximation of the non-linear terms (stabilization terms)
- **Complex Formulation**: Due to geometric parameters, use of EIM to automatically recover the affine decomposition.

Perspectives

- Development of Efficient Error Estimators for the Reduced Model
- Reduction of the Coupled Turbulence Model

Numerical Results: Cooling of an Printed Circuit Board, Reduced Model



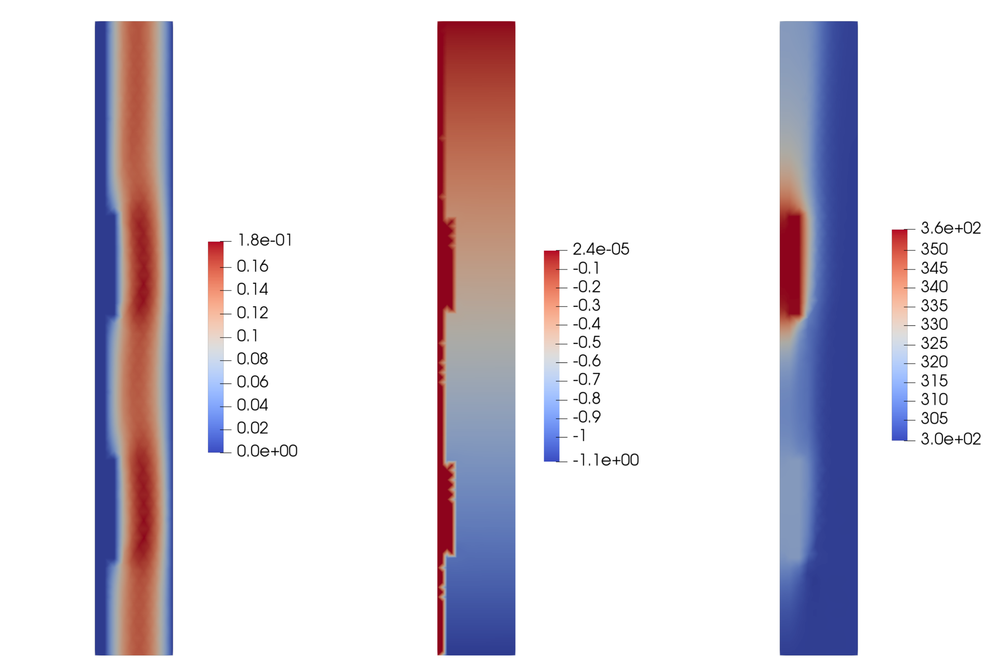
2D model representative of the neighboring of an electronic component submitted to a cooling air flow.

Physical Model

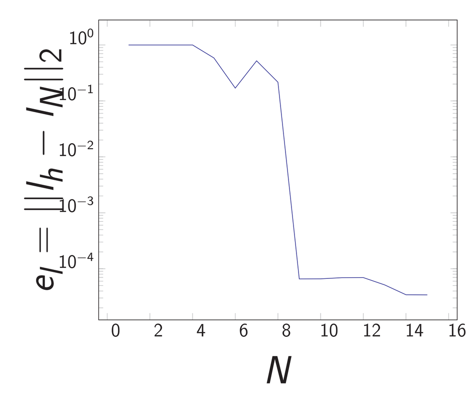
- Air thermal diffusivity: $\kappa_a = 2.7 \cdot 10^{-5}$
- Air kinematic viscosity: $\mu_a = 1.9 \cdot 10^{-5}$

Parameters

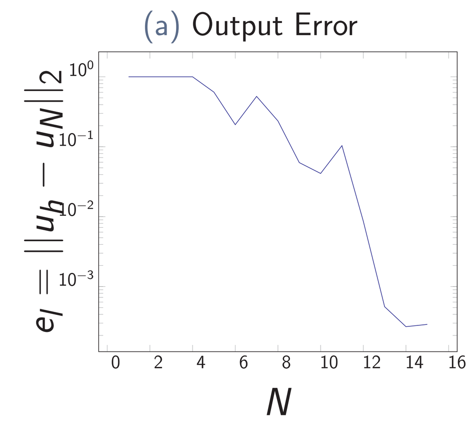
- Q_1 and Q_2: Heat sources from the two Integrated Circuits: $[0, 10^6]$
- κ_1 and κ_2:Thermal conductivity of the two Integrated Circuits: $[0.2, 150]$
- D: The inflow rate: $[5 \cdot 10^{-4}, 10^{-2}]$



(a) Velocity Profile (b) Pressure Profile (c) Temperature Profile



(a) Output Error



(b) Field Error ($\mathcal{L}^2$)

Convergence with respect to the size of the basis. Maximum of the error, evaluated on 100 RB approximations compared with the FEM approximations

Sponsor

MSO4SC:
Mathematical Modeling, Simulation and Optimization for Societal Challenges with Scientific Computing

The main objective of this project is to construct an e-infrastructure that provides, in a user-driven, integrative way, tailored access to the necessary services, resources and even tools for the fast prototyping, providing the service producers with the mathematical frameworks as well.

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