Tables for addition and subtraction with better use of the place value system

Thomas Colignatus
http://thomascool.eu

April 2 and major update May 5 2018

Abstract

This notebook presents tables for addition and subtraction that have a better use of the place value system. The method is already used in Holland for addition in levels but this notebook extends for addition in differences and subtraction in levels and differences. The method is only intended for an intermediate stage in teaching before addition and subtraction are mentally fully automated. For example 99 + 21 can be added per digit position. To keep digits in the range [0, 9] we remove underflow or overflow. We work from right to left, since the numbers come from India and Arabia. Then we get 99 + 21 → {9, 9} + {2, 1} = {11, 10} = {11, 10} + {1, -10} = {12, 0} = {0, 12, 0} + {1, -10, 0} = {1, 2, 0} → 120. Compared to existing methods: (1) This method does not change the original sum. (2) The workflow is into a single direction. (3) Allowing positions to have values outside the [0 , 9] range focuses attention upon the place value. (4) There is a unity of approach to both addition and subtraction. The method fits within the US Common Core when we tell kids that a step with {1, -10} represents the subtraction 1 × 10 - 10 × 1 = 0. It would be more fundamental to adapt the curriculum for negative numbers though. Routines to create such tables for addition and subtraction are available in a package. Obviously pupils in elementary school must master the method by hand, but the package allows for examples and checking.

Keywords

Mathematics Education, Addition, Subtraction, Place Value System, Negative Numbers, Common Core, Mathematica, Wolfram language, Programming, Package

MSC2010

97M70 Mathematics education. Behavioral and social sciences

Cloud

This notebook with package is also available at:

(1) http://community.wolfram.com/groups/-/m/t/1313408 (though the latter displays the earlier version in html and a short community title)

(2) https://www.wolframcloud.com/objects/thomas-cool/MathEd/2018-04-02-AdditionTable-and-SubtractionTable.nb (same link but update notebook and package)

(3) https://zenodo.org/record/1241350 or DOI 10.5281/zenodo.1241350

Contents

Start (subsection for the initialisation packages)

---

1. Introduction

There is a more efficient way for tables of addition and subtraction, see Colignatus (2014) and (2015b), that takes better advantage of the place value system (decimal system). This would suit the Common Core State Standards (CCSS) (2018) objectives.

The method is already used in Holland for *addition* in *levels* but this notebook extends for (i) addition in *differences* and (ii) *subtraction* in levels and differences.

The method is only intended for an intermediate stage in teaching before addition and subtraction are mentally fully automated. For example, the addition table with differences would be a first encounter, then the table with levels would already rely on more work in memory, and then we would proceed with automated memory of underflow or overflow.

The method is now available in a package, and the method and routines are discussed here. Obviously pupils in elementary school must master the method by hand, but the package allows for examples and checking.

2. Discussion

2.1. Main advantages

To teach the place value system, it is advisable to add or subtract per position.

- This does not change the original sum.
- The workflow is into a single direction, not jumping around to the beginning and back.
- Allowing positions to have values outside the [0 , 9] range focuses attention upon the place value.
- There is a unity of approach to both addition and subtraction.

I have not seen this method elsewhere, except for addition in levels, so if you start using this, please refer to this present source, so that others can find the proper documentation. See the comparison with existing methods below (Section 8).

2.2. Place value system

The place value system tends to display with single digits. Adding or subtracting may come along with underflow and overflow.

```
ToSingleDigits[{11, 9, 14}]
```

1204

```
ToSingleDigitsTable[{11, 9, 14}]
```

	10^3	10^2	10^1	10^0
1	0	11	9	14
2			1	−10
3		1	−10	
4	1	−10		
5	+	+	+	+
6	1	2	0	4

Colignatus (2014) mentions the possible use of enclosing circles, or colours, or brackets for overflow or underflow positions, with numbers like 9 [11]7 = 1017 = [11][-9]7.

```
places = {9, 11, 7} - {11, -9, 7}
```

{−2, 20, 0}

```
places . {100, 10, 1}
```

0

2.3. US Common Core State Standards

The method fits within the US Common Core State Standards (2018) when we tell kids that a term or line {1, -10} represents the subtraction 1 × 10 - 10 × 1 = 0. It would be more fundamental to adapt the curriculum for negative numbers though. The latter is discussed by Colignatus (2018) and summarised in Section 8 below.

Wikipedia is a portal and no source, and compare: https://en.wikipedia.org/wiki/Subtraction#Subtraction_by_hand

3. With steps of differences

These tables use differences, which can be seen from the diagonal form.

The method of differences is to be preferred as a first phase, because it has less steps and shows them all.

Line 5 contains {1, -10} with a negative number, but we might tell kids that it is a subtraction, namely 1 × 10 - 10 × 1 = 0.

```
AdditionTable[99, 21]
```

	10^2	10^1	10^0
1	0	9	9
2	0	2	1
3	+	+	+
4	0	11	10
5		1	−10
6	1	−10	
7	+	+	+
8	1	2	0

Training on addition will eventually cause a different workflow, when addition becomes mentally fully automated. A higher level script for 99 + 21 would be: 9 + 1 = 10, write 0 and remember 1, 1 + 9 + 2 = 12, write 2 and remember 1, write 1, and check and finish. The addition table in levels (next Section) would be an intermediate phase between using differences here and the compact workflow with automation.

Practice makes perfect. More tables or bigger numbers.

```
AdditionTable[999, 777, 123]
```

	10^3	10^2	10^1	10^0
1	0	9	9	9
2	0	7	7	7
3	0	1	2	3
4	+	+	+	+
5	0	17	18	19
6			1	−10
7		1	−10	
8	1	−10		
9	+	+	+	+
10	1	8	9	9

Line 5 below contains {-1, 10, 0} with a negative number, but it we might tell kids that it is subtraction: - 1 × 100 + 10 × 10 + 0 × 1 = 0.

```
SubtractionTable[491, 753]
```

SubtractionTable: Order switched into 753 – 491

	10^2	10^1	10^0
1	7	5	3
2	4	9	1
3	–	–	–
4	3	–4	2
5	–1	10	
6	+	+	+
7	2	6	2

4. With steps in levels

That these tables use levels can be seen from the repeat of the place values.

For addition, in these levels, this method actually has been used widely in Holland. It however hides the subtraction, and it hasn't been developed further for a table for subtraction. This may suggest to kids that subtraction must be handled differently, while there is little need for that.

```
AdditionTable[99, 21, Differences → False]
```

	10^2	10^1	10^0
1	0	9	9
2	0	2	1
3	+	+	+
4	0	11	10
5	0	12	0
6	1	2	0

```
AdditionTable[999, 777, 123, Differences → False]
```

	10^3	10^2	10^1	10^0
1	0	9	9	9
2	0	7	7	7
3	0	1	2	3
4	+	+	+	+
5	0	17	18	19
6	0	17	19	9
7	0	18	9	9
8	1	8	9	9

```
SubtractionTable[753, 491, Differences → False]
```

	10^2	10^1	10^0
1	7	5	3
2	4	9	1
3	–	–	–
4	3	–4	2
5	2	6	2

5. Calculation strategy

Pupils must also develop some calculation strategies:

- Addition allows that more numbers are added at the same time, and the order doesn't matter.
- Simple subtraction has only two numbers, the largest minus the smallest. If one really wants to subtract a large number from a smaller one, change the order and multiply the outcome by -1. If there are more numbers to subtract, first add all numbers that must be subtracted.
- Involved subtraction uses more than two numbers: see the next section. In such case, there can be a negative outcome. One must be aware that a number like -123 will be displayed as {-1, -2, -3}.

6. Subtraction in given order or with more numbers

Here we must make a rough judgement whether the outcome will be positive or negative. Depending upon this judgement we add or subtract 10 in the various places.

```
SubtractionTable[{18, 30, 5}]
```

	10^1	10^0
1	1	8
2	3	0
3	0	5
4	–	–
5	−2	3
6	1	−10
7	+	+
8	−1	−7

```
SubtractionTable[{11, 99}]
```

	10^1	10^0
1	1	1
2	9	9
3	–	–
4	−8	−8
5	+	+
6	−8	−8

```
SubtractionTable[11, 99, 78]
```

	10^2	10^1	10^0
1	0	1	1
2	0	9	9
3	0	7	8
4	–	–	–
5	0	−15	−16
6		−1	10
7	−1	10	
8	+	+	+
9	−1	−6	−6

```
SubtractionTable[999, 11, 99, 78]
```

	10^2	10^1	10^0
1	9	9	9
2	0	1	1
3	0	9	9
4	0	7	8
5	–	–	–
6	9	−8	−9
7		−1	10
8	−1	10	
9	+	+	+
10	8	1	1

7. When the routines are overly complex

When we add or subtract single digit numbers, then the routines are overly complex, since there is no real need for more columns. The routines now give a warning on this. The routines still show the general algorithm, for this might have some didactic value.

```
AdditionTable[8, 3, 6]
```

AdditionTable: Numbers all less than 10: routine is overly complex

	10^1	10^0
1	0	8
2	0	3
3	0	6
4	+	+
5	0	17
6	1	−10
7	+	+
8	1	7

```
SubtractionTable[8, 3, 6, 9]
```

SubtractionTable: Numbers all less than 10: routine is overly complex

	10^1	10^0
1	0	8
2	0	3
3	0	6
4	0	9
5	–	–
6	0	−10
7	−1	10
8	+	+
9	−1	0

8. Comparison with Common Core and existing methods

8.1. Position in the US Common Core

8.1.1. It can be used already

The US Common Core State Standards (2018) has subtraction from kindergarten. The method thus can be used when we tell kids that {1, -10} codifies a subtraction, namely 1 × 10 - 10 × 1 = 0.

Keep in mind that this method is only intended for an intermediate stage in teaching before addition and subtraction are mentally fully automated.

8.1.2. Option to adapt the curriculum too

We rather would want to speak about negative numbers. The method can be applied with open use of negative numbers when the order in the curriculum of *fractions* and *negative numbers* is interchanged. See Colignatus (2018) for an involved discussion.

- Currently negative numbers are introduced in Grade 6, but they better be mastered in Grade 2 and 3. First the whole numbers and later the fractions.
- Grade 2 now works up to 1000 but it would better to have -10 first.
- The major objective is to get to understand the place value system and the properties of addition and subtraction. Negative numbers appear to be relevant for this.
- Now fractions are discussed in Grade 3-5 but they better be discussed in Grade 4-6.

8.2. Use in Holland for addition in levels only

There is a widely used application in Holland already, but only for the *addition in levels* (Section 4 above). Working in levels obscures the internal subtraction that is being done. Thus, the present news is the development of (i) differences for addition and (ii) tables for subtraction (either in levels or differences).

8.3. Methods of partial sums and partial differences

The proposed method compares a bit to the *partial sums* and *partial differences* methods in “realistic mathematics education” at the University of Utrecht, also adopted by *Everyday Mathematics* (2018abc) at the University of Chicago. An (here) irrelevant difference is that the latter work from left to right instead from right to left. The relevant distinction is that the latter create new big numbers, which departs from the notion of place value again. There can also be awkward combinations of addition and subtraction. Everyday Mathematics (2018c) ends up with the input of this sum, and its first three lines, but it obscures the work that still has to be done in the second part of the table.

```
MixedAdditionTable[500, -40, -2]
```

	10^2	10^1	10^0
1	5	0	0
2	0	−4	0
3	0	0	−2
4	+	+	+
5	5	−4	−2
6		−1	10
7	−1	10	
8	+	+	+
9	4	5	8

PM. The *partial differences method* also has "subtraction of numbers" but this can better be interpreted as the implied use of negative numbers. It would be better to have the whole number line already available at the end of Grade 3, which repeats above argument.

9. Some additional features

9.1. Mixed addition table

The following allows a mixture of positive and negative integers. It might not be so handy to mix such cases, but perhaps it is useful for exercises. It was used above to look at the method of partial differences.

```
MixedAdditionTable[10, -15, -6, 23]
```

	10^1	10^0
1	1	0
2	−1	−5
3	0	−6
4	2	3
5	+	+
6	2	−8
7	−1	10
8	+	+
9	1	2

9.2. For the step from counting to addition

Key didactics is to properly memorise the “*Table of Basic Addition*”, since this is the step from counting to addition. Pupils who get stuck at counting - still using their fingers - will not have memorised this table. Its “version in sounds” - also before reading and writing - is part of memorisation. I thank pedagogue and teacher drs. A.I. Roessingh for these insights.

PM. (i) The entries of the table better have the orientation of the system of co-ordinates. (ii) An element within the table also shows the possible additions and subtractions w.r.t. the entries.

```
TableOfBasicAddition[]
```

	0	1	2	3	4	5	6	7	8	9	10
10	10										
9	9	10									
8	8	9	10								
7	7	8	9	10							
6	6	7	8	9	10						
5	5	6	7	8	9	10					
4	4	5	6	7	8	9	10				
3	3	4	5	6	7	8	9	10			
2	2	3	4	5	6	7	8	9	10		
1	1	2	3	4	5	6	7	8	9	10	
0	0	1	2	3	4	5	6	7	8	9	10

9.3. For the step from positive to negative

For some negative numbers it looks like this.

```
TableOfBasicAddition[-5, 5]
```

	−5	−4	−3	−2	−1	0	1	2	3	4	5
5	0	1	2	3	4	5					
4	−1	0	1	2	3	4	5				
3	−2	−1	0	1	2	3	4	5			
2	−3	−2	−1	0	1	2	3	4	5		
1	−4	−3	−2	−1	0	1	2	3	4	5	
0	−5	−4	−3	−2	−1	0	1	2	3	4	5
−1		−5	−4	−3	−2	−1	0	1	2	3	4
−2			−5	−4	−3	−2	−1	0	1	2	3
−3				−5	−4	−3	−2	−1	0	1	2
−4					−5	−4	−3	−2	−1	0	1
−5						−5	−4	−3	−2	−1	0

10. The routines

```
? Cool`MathEd`AdditionTable`*
```

▼ Cool`MathEd`AdditionTable`

AdditionTable	SubtractionTable	ToSingleDigitsTable
MixedAdditionTable	TableOfBasicAddition	
SquareTableOfAddition	ToSingleDigits	

`? AdditionTable`

AdditionTable[n__Integer] gives a table that adds the integers,
that must be positive, using the place value system. The integers are added
per digit position, and each overflow generates a new line in the table
AdditionTable[{n__Integer}] is another input format
Options are passed on to ToSingleDigitsTable.
Default there is Differences –> True (default); otherwise levels
When all integers are in [0, 9] then there is a warning that the routine is overly complex for such a case

`? SubtractionTable`

SubtractionTable[n, m] gives a table for n – m, for n > m > 0, using the place value system. The integers
are subtracted per digit position, and each underflow generates a new line in the table. If
n < m then the routine switches the order. When subtracting a block of numbers, perhaps
first add the numbers that must be subtracted, so that subtraction only applies to n – m
SubtractionTable[n__Integer] (for more than 2) or SubtractionTable[{n__Integer}]
cannot switch the order though. Integers must be positive
Options are passed on to ToSingleDigitsTable. Default there is
Differences –> True (default); otherwise levels
When all integers are in [0, 9] then there is a warning that the routine is overly complex for such a case

`? MixedAdditionTable`

MixedAdditionTable[n__Integer] (for more than 2) or MixedAdditionTable[{n__Integer}]
adds positive or negative integers. Options are passed on to ToSingleDigitsTable.
Default there is Differences –> True (default); otherwise levels
When all integers are in [0, 9] then there is a warning that the routine is overly complex for such a case

`? ToSingleDigitsTable`

ToSingleDigitsTable[vec] decomposes the place value vector.
ToSingleDigitsTable["Tested", vec, table] complements the table. The test is
on the proper length of vec for the table. The routine is called by AdditionTable,
SubtractionTable and MixedAdditionTable for completion, and its option setting
controls their output. Default option is Differences –> True; otherwise levels

`? TableOfBasicAddition`

TableOfBasicAddition[n:0, m:10] gives i + j in the range [n, m]
for values Abs[i + j] <= Max[Abs[n], Abs[m]], and n and m may be negative.
Elements x within the table show j = x – i. Options are passed on to TableForm

11. Conclusions

It is obviously advantageous when tables of addition and subtraction use the place value system in a better manner.

The place value system by its very definition supports this way of working. It is rather amazing that this method has not been developed before, except for the addition in levels. Apparently overflow was recognised but underflow not, and this probably relates to the negative numbers.

The method fits in the current Common Core when we tell kids that a line with {1, -10} represents the subtraction 1 × 10 - 10 × 1 = 0, or with other appropriate values depending upon the positions.

Remarkably, the US Common Core has negative numbers only in Grade 6. See Colignatus (2018) for a fundamental discussion that it could be well better to have the negative integers in Grade 2 & 3, and only then start with fractions in Grade 4-6. First the whole numbers and place value system, and then the fractions.

12. Literature

Thomas Colignatus is the scientific name of Thomas Cool, econometrician and teacher of mathematics, Scheveningen, Holland.

Colignatus, Th. (2014), *Taking a loss*, https://boycottholland.wordpress.com/2014/08/30/taking-a-loss/

Colignatus, Th. (2009, 2015a), *Elegance with Substance,*
(1) website: http://thomascool.eu/Papers/Math/Index.html,
(2) PDF on Zenodo: https://zenodo.org/record/291974,
(3) Publisher for a print: https://www.mijnbestseller.nl/shop/index.php/catalog/product/view/id/115148/s/elegance-with-substance-78023-www-mijnbestseller-nl/

Colignatus, Th. (2015b), *A child wants nice and no mean numbers*,
(1) website: http://thomascool.eu/Papers/NiceNumbers/Index.html
(2) PDF on Zenodo: https://zenodo.org/record/291979
(3) Publisher for a print: https://www.mijnbestseller.nl/shop/index.php/catalog/product/view/id/118082/s/a-child-wants-nice-and-no-mean-numbers-79074-www-mijnbestseller-nl/

Colignatus, Th. (2018), *Arithmetic with H = -1: subtraction, negative numbers, division, rationals and mixed numbers*, update May 5, https://www.wolframcloud.com/objects/thomas-cool/Math-Ed/2018-04-02-Arithmetic-with-H.nb and http://community.wolfram.com/groups/-/m/t/1313408 (same links with updated files) or https://zenodo.org/record/1241383

Common Core Standards (2018), *Number & Operations in Base Ten,* http://www.corestandards.org/Math/Content/NBT/ with also *Understand place value*

Everyday Mathematics (2018a), *Alternative algorithms*,
http://everydaymath.uchicago.edu/parents/understanding-em/alternative-algorithms/

Everyday Mathematics (2018b), *Addition - Partial Sums*, http://everydaymath.uchicago.edu/teaching-topics/computation/add-partial-sums/

Everyday Mathematics (2018c), *Partial Differences*,
http://everydaymath.uchicago.edu/teaching-topics/computation/sub-part-diff.html