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COMPUTATION OF THE GAMMA, DIGAMMA, AND TRIGAMMA FUNCTIONS*

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Abstract. This paper gives an approximation for the gamma function that, while different, has the same form as one by Lanczos [*SIAM J. Numer. Anal.*, B1 (1964), pp. 86–96]. Both approximations correct Stirling's approximation with contributions from the gamma function's poles, and both require $O(-\log\epsilon)$ time independent of z to calculate $z!$ with a relative error ϵ. At comparable accuracies, this paper's approximation requires slightly more computation than Lanczos' but is clearly superior in two ways: its coefficients are given by simple formulas and the error estimates apply not only to the gamma function, but also to its derivatives. Thus approximations for the digamma and trigamma functions are also given. Let $F_{a,1/2}(z) = z!(z+a)^{-z-(1/2)}e^{z+a}(2\pi)^{-(1/2)}$ and $f_a(z) = \ln F_{a,1/2}(z)$, with z complex and a real. Several lemmas in the paper require "Stirling inequalities," i.e., bounds on $|F_{a,1/2}(z)|$. To this end, the classical Binet integral for $f_0(z)$ is generalized from $a = 0$ to all real a. Since the generalized Binet integral is a Laplace transform, the requisite Stirling inequalities follow from the complete monotonicity of such transforms. The proof of the main theorem uses the calculus of residues and presents a lemma generalizing Plana's theorem, which, like the Euler–Maclaurin sum formula, evaluates the difference between a sum and the corresponding integral. This paper also presents a factorial approximation that is both simpler and more accurate than Stirling's approximation.

Key words. Stirling's approximation, calculus of residues, Binet integral, complete monotonicity inequalities

AMS subject classifications. 40-04, 41A20, 30E10

1. Introduction and statement of results.

This paper gives simple and computationally efficient formulas for the gamma, digamma, and trigamma functions. Section 1.1 provides a historical introduction. The new results then fall into three distinct classes corresponding to §§1.2, 1.3, and 1.4: Stirling inequalities, formulas for the gamma function and its derivatives, and formulas for the digamma and trigamma functions.

Section 1 attempts a general, heuristic overview of the results and defers detailed proofs to §2. (No heuristic in §1 is intended as a completed proof.) Theorems and lemmas are therefore numbered by subsection, so that the reader can always find the proofs for §1 in the corresponding subsection of §2 (e.g., the proof for Theorem 1.3.4, which is the fourth theorem in §1.3, is found in §2.3).

1.1. General background.

The gamma function extends the factorial function from the natural numbers to the complex plane according to the normalization

$$z! = \Gamma(z+1). \tag{1}$$

To avoid the gamma function's cumbrous normalization, the factorial function can be defined directly for all complex z with a Weierstrass product

$$\frac{1}{z!} = e^{\gamma z}\prod_{k=1}^{\infty}\left[\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}}\right], \tag{2}$$

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where Euler's constant $\gamma = \lim_{n\to\infty}\left(\sum_1^n k^{-1} - \ln n\right) = 0.57721\ldots$ [1, p. 255]. In the complex plane, the factorial is an analytic function whose only singularities are simple poles at the negative integers. The fundamental recursion $z! = z\cdot(z-1)!$ shows that for $k = 1, 2, 3, \ldots$ the residue at $z = -k$ is $(-1)^{k-1}[(k-1)!]^{-1}$. Because of the reflection theorem

$$(-z)!z! = \frac{\pi z}{\sin \pi z}, \tag{3}$$

methods for computing $z!$ can assume Re $z \geq 0$ without loss of generality.

$\Psi(z+1) = (d/dz)\ln z!$ is often called the digamma function, while $\Psi'(z)$ is called the trigamma function [1, p. 258]. The Weierstrass product equation (2) shows

$$\Psi(z+1) = -\gamma + \sum_{k=1}^{\infty}\left[k^{-1} - (z+k)^{-1}\right], \tag{4}$$

which implies the fundamental recursion $\Psi^{(n-1)}(z+1) = \Psi^{(n-1)}(z) + (-1)^{n-1}(n-1)!z^{-n}$. Consequently, tables of $\Psi(z)$ and its derivatives once helped to compute sums of the form $\sum(z+k)^{-n}$. Currently the digamma and trigamma functions probably arise most frequently as definite integrals, e.g., the Gauss integral

$$\Psi(z+1) = \int_0^{\infty}\left(\frac{e^{-t}}{t} - \frac{e^{-tz}}{e^t - 1}\right)dz \tag{5}$$

for Re $z > 0$, which can be derived from (4) by noting that $w^{-1} = \int_0^{\infty} e^{-tw}dt$ for Re $w > 0$, and knowing that $-\gamma = \int_0^{\infty}[e^{-t}t^{-1} - (e^t-1)^{-1}]dt$.

In this paper, a will denote an arbitrary real number unless stated otherwise. Define $w^z = e^{z\,\mathrm{Ln}\,w}$, where $\mathrm{Ln} z = \ln|z| + i$ Arg z, $|\mathrm{Arg}\ z| < \pi$, is the principal branch of the natural logarithm. The theory of the factorial often uses functions of the form

$$F_{a,\theta}(z) = z!(z+a)^{-z-\theta}e^{z+a}(2\pi)^{-\frac{1}{2}}. \tag{6}$$

For example, let Δ be any real number, $0 < \Delta < \pi$. Stirling's approximation $F_{0,1/2}(z) \to 1$ holds as $|z| \to \infty$, uniformly in any sector $|\mathrm{Arg}\ z| \leq \pi - \Delta < \pi$ [10, p. 278]. Consequently, $F_{a,1/2}(z) \to 1$ as $|z+a| \to \infty$, uniformly in any sector $|\mathrm{Arg}\ (z+a)| \leq \pi - \Delta < \pi$.

Press, Flannery, Teukolsky, and Vetterling [8, p. 167] state without qualification that no method for calculating the factorial is "quite as neat as the [following] approximation derived by Lanczos," which uses contributions from the factorial function's poles to correct Stirling's approximation [5]:

$$z! = (z+a)^{z+\frac{1}{2}}e^{-(z+a)}(2\pi)^{\frac{1}{2}}\left[c_0 + \sum_{k=1}^{N} c_k(z+k)^{-1} + \epsilon(z)\right], \tag{7}$$

where the optimal value of N depends on a. Lanczos' formulas for the $\{c_k\}$ take about a page to write out and are too complicated to reproduce here. Anyway, since a computer program should avoid unnecessary recalculations by storing the $\{c_k\}$ to required decimal accuracy, Lanczos tabulated some numerical values. The smallest error bound in his table, $|\epsilon(z)| \leq 2\cdot 10^{-10}$ for Re $z \geq 0$ and $a = 5.5$, requires only $N = 6$ terms in the sum.

Equation 7 can be rewritten as

$$F_{a,\frac{1}{2}}(z) = c_0 + \sum_{k=1}^{N} c_k(z+k)^{-1} + \epsilon(z). \tag{8}$$

Neglecting $\epsilon(z)$ in (8) incurs a relative error $|\epsilon(z)[F_{a,1/2}(z)]^{-1}|$, which propagates to (7) when computing $z!$. The following Stirling inequality assures that this relative error is at most $\left(\frac{\pi}{e}\right)^{1/2} = 1.07504\ldots$ times $\epsilon(z)$.

LEMMA 1.1.1. *$|F_{a,1/2}(z)| \geq (\frac{e}{\pi})^{1/2}$ whenever $a \geq 0$ and* Re $z \geq 0$. *Moreover, the bound is sharp since $F_{1/2,1/2}(0) = (\frac{e}{\pi})^{1/2}$.*

Neither Lanczos [5] nor his references appear to give this crucial link between $\epsilon(z)$ and the relative error in $z!$, and despite other authors' impressions [8, p. 168], Lanczos explicitly links $\epsilon(z)$ and the relative error only when $N = 1$ in equation (7). This apparent lacuna motivates a digression on Stirling inequalities.

1.2. Results on Stirling inequalities. Define

$$f_a(z) = \operatorname{Ln} F_{a,\frac{1}{2}}(z). \tag{9}$$

(The subscript $\theta = \frac{1}{2}$ will be fixed in $f_a(z)$ and can remain implicit.) Stirling inequalities are bounds on $|F_{a,1/2}(z)|$, or equivalently, on Re $f_a(z)$.

The Binet integral

$$f_0(z) = \int_0^\infty e^{-tz}\left(\frac{t}{e^t-1} - 1 + \frac{1}{2}t\right)t^{-2}dt \tag{10}$$

for Re $z > 0$ [10, p. 248] can be demonstrated by differentiating and comparing the result to the Gauss integral in (5). The Binet integral has an easy generalization.

LEMMA 1.2.1. *Let a be any real number. Then*

$$f_a(z) = \int_0^\infty e^{-tz}\left\{\frac{t}{e^t-1} - [1+(a-\tfrac{1}{2})t]e^{-at}\right\}t^{-2}dt \tag{11}$$

for Re $z > \max(-1,-a)$.

Because Binet integrals are Laplace transforms, simple inequalities on their integrands yield the following lemma.

LEMMA 1.2.2. *Let $z = x + iy$. $f_a(z)$ has the following properties whenever $x > \max(-1,-a)$.*

(1) *Fix z, and consider $f_a(z)$ as a function of a.* Re $f_a(z)$ *decreases for $a < \frac{1}{2}$ and increases for $\frac{1}{2} < a$.*

(2) *Fix $a \leq \frac{1}{2}$.* Re $f_a(x+iy) \geq f_a(x)$, *and $f_a(x)$ strictly increases to $0 = \lim_{x\to\infty} f_a(x)$.*

(3) *Fix $a \geq 1$.* Re $f_a(x+iy) \leq f_a(x)$, *and $f_a(x)$ strictly decreases to $0 = \lim_{x\to\infty} f_a(x)$.*

The proof of Lemma 1.1.1 in §2.1 is based on the hierarchy of Stirling inequalities in Lemma 1.2.2.

1.3. Results on the gamma function. The following theorem is the most practical distillate of this paper's results. Let the ceiling $\lceil a \rceil$ denote the unique integer satisfying $\lceil a \rceil - 1 < a \leq \lceil a \rceil$.

THEOREM 1.3.1. *Neglect* $\epsilon(z)$ *in equation* (7), *and set* $N = \lceil a \rceil - 1$, $c_0 = 1$, *and*

$$c_k = (2\pi)^{-\frac{1}{2}} \frac{(-1)^{k-1}}{(k-1)!} (-k+a)^{k-\frac{1}{2}} e^{-k+a} \tag{12}$$

for $k = 1, 2, \ldots, \lceil a \rceil - 1$. *For* $a \geq 3$ *and* Re $(z+a) > 0$, *the relative error when computing* $z!$ *is less than* $a^{1/2}(2\pi)^{-(a+1/2)}[\mathrm{Re}(z+a)]^{-1}$. *Thus for* Re $z \geq 0$, *the relative error is less than* $a^{-1/2}(2\pi)^{-(a+1/2)}$.

The next theorem is a curiosity.

THEOREM 1.3.2. *The relative error in the approximation*

$$z! \approx \left(\frac{z+\frac{1}{2}}{e}\right)^{z+\frac{1}{2}} (2\pi)^{\frac{1}{2}} \tag{13}$$

is less than $(2\pi e)^{-1/2}\pi^{-1}(\ln 2)[\mathrm{Re}(z+\frac{1}{2})]^{-1}$.

Since $(2\pi e)^{-1/2}\pi^{-1}(\ln 2) = 0.053\ldots$, equation (13) is not only simpler than Stirling's approximation, but nearly twice as accurate for any $z > 0$.

Both Theorems 1.3.1 and 1.3.2 follow from a much more general theorem, which applies not only to $F_{a,1/2}(z)$ but also to $F_{a,\theta}(z) = F_{a,1/2}(z)(z+a)^{-(\theta-(1/2))}$, where $a \geq \theta \geq \frac{1}{2}$. To motivate the generalization, consider once again the $\{c_k\}$ in Theorem 1.3.1. By Stirling's approximation, $c_0 = 1$ is the value of $F_{a,1/2}(z)$ at $z = \infty$, while other c_k are then residues of $F_{a,\frac{1}{2}}(z)$ at $z = -1, -2, \ldots, -(\lceil a \rceil - 1)$.

Define $F_\theta = 1$ if $\theta = \frac{1}{2}$, and $F_\theta = 0$ otherwise. Let

$$G_{a,\theta}(w) = \frac{(-w-a)^{-w-\theta} e^{w+a} (2\pi)^{\frac{1}{2}}}{(-w-1)!}, \tag{14}$$

so Stirling's approximation implies $G_{a,\theta}(w) \to F_\theta$ as $|-w-a| \to \infty$, uniformly in any sector $|\mathrm{Arg}(-w-a)| \leq \pi - \Delta < \pi$. Also, let $\epsilon_{a,\theta}(z)$ satisfy

$$\begin{aligned} F_{a,\theta}(z) = F_\theta + (2\pi)^{-\frac{1}{2}} \sum_{k=1}^{\lceil a \rceil - 1} \frac{(-1)^{k-1}}{(k-1)!} (-k+a)^{k-\theta} e^{-k+a} (z+k)^{-1} \\ + \frac{\cos \pi\theta}{\pi} \int_{-\infty}^{-a} \frac{G_{a,\theta}(w)}{w-z} dw + \epsilon_{a,\theta}(z), \end{aligned} \tag{15}$$

where any empty sum ($a \leq 1$) is to be interpreted as 0. Since $F_{a,\theta}(z) \to F_\theta$ as $|z+a| \to \infty$, uniformly in any sector $|\mathrm{Arg}(z+a)| \leq \pi - \Delta < \pi$, F_θ is the value of $F_{a,\theta}(z)$ at $z = \infty$, while the terms of the sum are the residues of $F_{a,\theta}(z)$ at $z = -1, -2, \ldots, -(\lceil a \rceil - 1)$, just as in Theorems 1.3.1 and 1.3.2.

Since the branch cut for $(-w-a)^{-w-\theta}$ is $|\mathrm{Arg}(-w-a)| = \pi$, or equivalently $-w < a$, the integrand $G_{a,\theta}(w)(w-z)^{-1}$ is well-defined for $w \leq -a$ when $a \geq \theta$. For $\theta > \frac{1}{2}$, the integral converges at $w = -\infty$ by Stirling's approximation and ratio comparison with $(-w-a)^{-\theta+(1/2)}(w-z)^{-1}$ [9, p. 638]. For $\theta = \frac{1}{2}$, $\cos \pi\theta = 0$ and the integral term is to be interpreted as 0.

THEOREM 1.3.3. *Let* $a \geq \theta \geq \frac{1}{2}$. *The error term in* (15) *is*

$$\begin{aligned} \epsilon_{a,\theta}(z) = \frac{e^{\pi i\theta}}{2\pi} \int_0^\infty \frac{G_{a,\theta}(-a-iv)}{e^{2\pi(v-ia)}-1} \frac{dv}{-a+iv-z} \\ - \frac{e^{-\pi i\theta}}{2\pi} \int_0^\infty \frac{G_{a,\theta}(-a+iv)}{e^{2\pi(v+ia)}-1} \frac{dv}{-a+iv-z}. \end{aligned} \tag{16}$$

The two sides of (15) are matched for their behavior at $z = \infty$ and their poles at $z = -1, -2, \ldots, -(\lceil a \rceil - 1)$. Taking the difference of the two sides suggests a loose analogy with Liouville's theorem: if $f(z)$ is a bounded entire function, then $f(z)$ is constant. Standard proofs of Liouville's theorem differentiate the Cauchy integral for $f(z)$, then use an arbitrarily large circular contour to show that $f'(z) \equiv 0$ (e.g., [6, p. 137]). The proof of Theorem 1.3.3 is similar in spirit.

By Cauchy's theorem

$$F_{a,\theta}(z) = \frac{1}{2\pi i} \int_C \frac{F_{a,\theta}(w)}{w - z} dw, \tag{17}$$

where C is a sufficiently small closed contour encircling z counterclockwise. Since $F_{a,\theta}(w)(w-z)^{-1}$ is analytic in the domain between C and contour C_b in Fig. 1, equation (17) remains true when C in it is replaced by C_b [6, p. 248]. The Cauchy integral around C_b is conveniently partitioned into contributions from the four subcontours specified in Fig. 1: (1) the large circular arc ${}_bC_{2\pi-2\Delta}$ centered at $w = -a$; (2) the two vertical segments ${}_bC_{\pm\Delta}$ subtending a total angle 2Δ from $w = -a$; (3) the small closed circles C_{-k} around the poles $z = -1, -2, \ldots, -(\lceil a \rceil - 1)$ of $w!$; and (4) the upper and lower indented contours $\pm C^{\pm}_{[b,-a]}$ along the branch cut $(-\infty, -a]$ of $F_{a,\theta}(w)$.

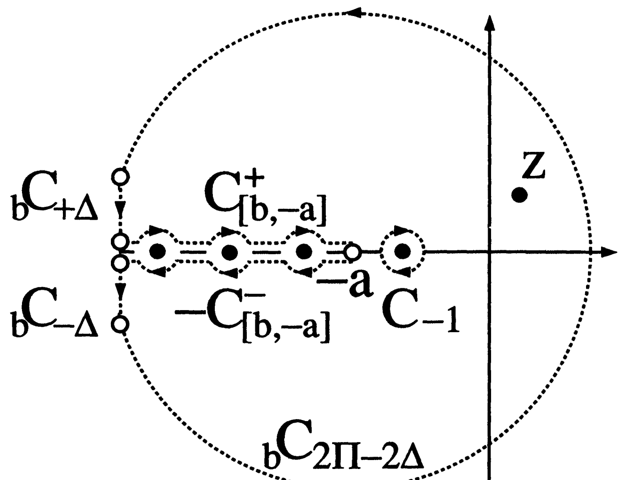

FIG. 1. *The oriented contour C_b in the w-plane is shown in dotted line. Solid points are marked at z and $-1, -2, -3$, and -4. A branch cut extends from $-\infty$ to $-a = -1.5$. Hollow points are marked at $-a$, at the point $b = -4.5$ above and below the branch cut, and at the points $b \pm i(-b-a)\tan\Delta$ subtending a total angle 2Δ from $-a$. Also labeled are: (1) the large circular* arc ${}_bC_{2\pi-2\Delta}$ *comprising points w satisfying both $|w + a| = -(b + a)$ and $|\mathrm{Arg}(w + a)| \leq \pi - \Delta$; (2) the two vertical segments ${}_bC_{\pm\Delta}$ comprising points w satisfying both* $\mathrm{Re}\, w = b$ *and* $\pi - \Delta \leq |\mathrm{Arg}(w+a)| < \pi$; *(3) the small closed circle C_{-1} around the pole -1 of $z!$; and (4) the subcontours $\pm C^{\pm}_{[b,-a]}$ above and below $(-b, -a]$. The minus sign preceding the subcontour $C^{-}_{[b,-a]}$ indicates that the contour's normal orientation from b to $-a$ is reversed. $C^{\pm}_{[b,a]}$ are indented around the poles $-2, -3$, and -4 of $z!$.*

As $b \to -\infty$ for fixed Δ, followed by $\Delta \to 0$, the subcontours make the following contributions to $F_{a,\theta}(w)$ in equation (15): (1) the subcontour ${}_bC_{2\pi-2\Delta}$ contributes F_θ; (2) the subcontours ${}_bC_{\pm\Delta}$ contribute nothing; (3) the subcontours C_{-k} contribute the residues at $z = -1, -2, \ldots, -(\lceil a \rceil - 1)$; and (4) the subcontours $\pm C^{\pm}_{[b,-a]}$ contribute the integral and error terms. The generalization of Plana's theorem [10, p. 145] in Lemma 2.3.1 exposes the branch-cut contribution from $\pm C^{\pm}_{[b,-a]}$ as the difference between a sum and the corresponding integral, much like the difference in the Euler–Maclaurin sum formula [10, p. 127].

When (15) is differentiated term by term n times ($n = 0, 1, 2, \ldots$) with respect to z, the derivatives of the integral in (15) [9, p. 669] can be obtained by differentiating the respective integrands. Since the same is true for the two integrals in (16) [6, p. 370], the following bound can be obtained.

THEOREM 1.3.4. *The error term's nth derivative is bounded by*

$$\left|\epsilon^{(n)}_{a,\theta}(z)\right| \le C_{a,\theta} n! [\mathrm{Re}(z+a)]^{-(n+1)}, \tag{18}$$

where

$$C_{a,\theta} = \frac{1}{(a-1)!}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{v^{a-\theta}}{\left|e^{2\pi v} - e^{2\pi i a}\right|} dv. \tag{19}$$

When $\theta = \frac{1}{2}$, the absolute error bound for $\epsilon_{a,\frac{1}{2}}(z)$ in Theorem 1.3.4 *and the lower bound for $F_{a,\frac{1}{2}}(z)$ in Lemma* 1.1.1 *combine to give the relative errors in Theorems* 1.3.1 *and* 1.3.2 *for computing $z!$.*

The following theorem is another curiosity.

THEOREM 1.3.5. *Equation* (15) *has a limiting form valid for $\theta \ge \frac{1}{2}$ and all values of z:*

$$z! = \lim_{a\to\infty} (z+a)^{z+\theta} \sum_{k=1}^{\lceil a \rceil - 1} \frac{(-1)^{k-1}}{(k-1)!} (-k+a)^{k-\theta} e^{-(z+k)} (z+k)^{-1}. \tag{20}$$

Multiplying (20) *by* $\lim_{a\to\infty} a^{z+\theta}(z+a)^{-(z+\theta)} = 1$ *gives a form similar to the final equation in Lanczos* [5].

1.4. Results on the digamma and trigamma functions. Because Theorem 1.3.4 bounds errors for derivatives of the factorial function, it effectively bounds errors for the digamma and trigamma functions, the logarithmic derivatives of the factorial function. The notation that follows facilitates the statement of the theorems.

Let $Q(z)$ be any quantity expressed in terms of the $\{F^{(n)}_{a,1/2}(z)\}$. Equation (8) and its derivatives give expressions for $\{F^{(n)}_{a,1/2}(z)\}$. If the error terms $\{\epsilon^{(n)}_{a,1/2}(z)\}$ are omitted when substituting these expressions into $Q(z)$, an error is induced. The error's absolute value will be denoted by $\epsilon[Q(z)]$.

Because of (9), the derivatives of $f_a(z)$ can be expressed in terms of the $\{F^{(n)}_{a,1/2}(z)\}$. From (6), the digamma function can be expressed as

$$\Psi(z+1) = \ln(z+a) - (a - \tfrac{1}{2})(z+a)^{-1} + f'_a(z); \tag{21}$$

and the trigamma function as

$$\Psi'(z+1) = (z+a)^{-1} + (a - \tfrac{1}{2})(z+a)^{-2} + f''_a(z). \tag{22}$$

Straightforward but tedious inequalities lead to absolute error bounds for the digamma and trigamma functions.

Let $D = [1 - (\frac{2}{3})^{1/2}(2\pi)^{-2}]^{-1} = 1.02111\ldots$.

THEOREM 1.4.1. *For* Re $z \geq 0$ *and* $a \geq \frac{3}{2}$, *the absolute error induced in* $\Psi(z+1)$ *satisfies*

$$\begin{aligned} \epsilon\,[f'_a(z)] &< D(\ln 2a)a^{\frac{1}{2}}(2\pi)^{-(a+\frac{1}{2})}[\mathrm{Re}(z+a)]^{-1} \\ &\leq D(\ln 2a)a^{-\frac{1}{2}}(2\pi)^{-(a+\frac{1}{2})}. \end{aligned} \tag{23}$$

Bounds on the corresponding relative error can be derived from the Gauss integral in (5). If $z = x+iy$, then $|\Psi(z+1)| \geq \Psi(x+1)$, and if $x > 0$, then $\Psi(x+1)$ increases. $\Psi(x_0+1) = 0$ for $x_0 = 0.462\ldots$, so for Re $z \geq x_1 > x_0$, the relative error in $\Psi(z+1)$ is less than the absolute error in (23) divided by $\Psi(x_1+1)$.

For the reader's reference, $\Psi(2) = 0.42278\ldots$ for $x_1 = 1$ and $\Psi(x) \sim \ln x$.

THEOREM 1.4.2. *For* Re $z \geq 0$ *and* $a \geq \frac{3}{2}$, *the absolute error induced in* $\Psi'(z+1)$ *satisfies*

$$\epsilon\,[f''_a(z)] < D(3\ln^2 2a - \ln 2a + 2)a^{\frac{1}{2}}(2\pi)^{-(a+\frac{1}{2})}[\mathrm{Re}(z+a)]^{-1}. \tag{24}$$

Differentiating the Gauss integral shows that $|\Psi'(z+1)| \leq \Psi'(x+1)$, so the following bound on the relative error applies only to real numbers: for $x \geq 0$, the relative error in $\Psi'(x+1)$ is less than $D(3\ln^2 2a - \ln 2a + 2)a^{1/2}(2\pi)^{-(a+1/2)}$.

For the reader's reference $\Psi'(x) \sim x^{-1}$ as $x \to \infty$.

Typically, the relative errors in Theorems 1.3.1, 1.4.1, and 1.4.2 are not very different, although the digamma and trigamma functions theoretically require a few more terms in (8) than the factorial function to obtain comparable accuracy. Numerical experiments for $3 \leq a \leq 11$ indicate, however, that for $x \geq 1$ the maximum relative errors for $\Psi(x+1)$ and $\Psi'(x+1)$ never exceed the one for $x!$. Hence in practice, even when computing the digamma and trigamma functions, the number of terms in (7) can be set by Theorem 1.3.1 for the factorial.

2. Proofs of results. Each result in §1 is proved in the corresponding part of §2, e.g., the proof for Theorem 1.3.4 is found in §2.3.

2.1. Proof of Lemma 1.1.1. Let $a \geq 0$ and Re $z \geq 0$. Lemma 1.2.2, proved in §2.2, justifies all inequalities in the following. $\mathrm{Ln}|F_{a,1/2}(z)| = \mathrm{Re\,Ln}\,F_{a,1/2}(z) = \mathrm{Re}\,f_a(z) \geq \mathrm{Re}\,f_{1/2}(z) \geq \mathrm{Re}\,f_{1/2}(\mathrm{Re}\,z) \geq \mathrm{Re}\,f_{1/2}(0) = \frac{1}{2}\ln(\frac{e}{\pi})$. Exponentiation yields Lemma 1.1.1.

2.2. Proofs of results on Stirling inequalities. The order of proof is Lemma 1.2.1 and then Lemma 1.2.2.

Proof of Lemma 1.2.1. Fix $a = a_0$, where a_0 is an arbitrary real number. Both sides of (11) are analytic in the simply connected domain $D = \{z : \mathrm{Re}\,z > \max(-1, -a_0)\}$ [10, p. 92]. Hence, proving (11) for a subdomain D' of D proves it for D [6, p. 147].

Restrict z to the subdomain $D' = \{z : \mathrm{Re}\,z > \max(0, -a_0) + 1\}$. For $a = 0$, equation (11) is just Binet's integral (10) at z. When a is between 0 and a_0 inclusive, differentiation with respect to a under the integral is justified in D' [10, p. 92]. Both sides of (11) have the derivative $(a - \frac{1}{2})(z+a)^{-1}$, so (11) holds in D' for arbitrary $a = a_0$, which proves the lemma.

Proof of property 1 *in Lemma* 1.2.2. From the proof of Lemma 1.2.1, $\frac{\partial}{\partial a}\mathrm{Re}\,f_a(z) = \mathrm{Re}\left[\left(a - \frac{1}{2}\right)(z+a)^{-1}\right] = \left(a - \frac{1}{2}\right)(x+a)\left[(x+a)^2 + y^2\right]^{-1}$, which is negative for $a < \frac{1}{2}$ and positive for $\frac{1}{2} < a$.

Proof of property 2 *in Lemma* 1.2.2. Equation (11) defines $f_a(z)$ as a Laplace transform. In general, if $\phi(t)$ is continuous for $t \geq 0$, and the Laplace transform $\Phi(z) = \int_0^\infty e^{-tz}\phi(t)dt$ converges for Re $z > x_0$, then $\Phi^{(n)}(z) = \int_0^\infty e^{-tz}(-t)^n\phi(t)\,dt$ for Re $z > x_0$, $n = 0, 1, \ldots$ and $\lim_{x\to\infty} \Phi^{(n)}(x) = 0$ [11, pp. 440–451]. Also, if $\phi(t) > 0$ for all $t > 0$, then $\Phi(z)$ has the following "complete monotonicity properties" [2, p. 439] for $x > x_0$:

(1) $(-1)^{n+1}\Phi^{(n+1)}(x) > 0$ so $(-1)^n\Phi^{(n)}(x)$ strictly decreases to 0 as $x \to \infty$.

(2) Re $\Phi^{(n)}(x+iy) \leq |\Phi^{(n)}(x+iy)| \leq (-1)^n\Phi^{(n)}(x)$.

The integrand in (11) suggests the inequality

$$\frac{t}{e^t - 1} = e^{-\frac{1}{2}t}\frac{\frac{1}{2}t}{\sinh\frac{1}{2}t} < e^{-\frac{1}{2}t} \tag{25}$$

for $t > 0$. Hence for all a,

$$\begin{aligned}\phi_a(t) &= \left\{\frac{t}{e^t-1} - [1 + (a - \frac{1}{2})t]e^{-at}\right\}t^{-2} \\ &< \left\{e^{-\frac{1}{2}t} - e^{-at} - (a - \tfrac{1}{2})te^{-at}\right\}t^{-2} = \phi_{a0}(t),\end{aligned} \tag{26}$$

where the equalities define $\phi_a(t)$ and $\phi_{a0}(t)$. When $a \leq \frac{1}{2}$, $-\phi_a(t) > -\phi_{a0}(t) \geq 0$, so property 2 follows from the complete monotonicity properties with $\phi \equiv -\phi_a$.

Proof of property 3 *in Lemma* 1.2.2. Property 3 is best proved in conjunction with the next lemma, which is needed later.

LEMMA 2.2.1. *For* $a \geq 1$, $x > -\frac{1}{2}$ *and* $n = 1, 2, \ldots$,

$$\begin{aligned}\left|f^{(n)}_{a,\frac{1}{2}}(x+iy)\right| &\leq (-1)^n f^{(n)}_{a,\frac{1}{2}}(x) \\ &< (-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}\left[\ln\frac{x+a}{x+\frac{1}{2}} - (a - \tfrac{1}{2})(x+a)^{-1}\right].\end{aligned} \tag{27}$$

If $\phi_a(t) > 0$ held for all $t > 0$ and $a \geq 1$, the complete monotonocity properties would imply both property 3 in Lemma 1.2.2 and the first inequality in (27). But when $t > 0$, $e^{at}(e^t - 1) > 0$, so $\phi_a(t)$ and $\phi_a(t)e^{at}(e^t - 1)$ have the same sign. When $t > 0$ and $a \geq 1$, $\phi_a(t)e^{at}(e^t - 1)$ equals a power series in t with nonnegative coefficients, so $\phi_a(t)e^{at}(e^t - 1) > 0$, implying $\phi_a(t) > 0$.

The second inequality on the right side of (27) follows from the complete monotonicity property 1 with $\phi \equiv (\phi_{a0} - \phi_a)t$, because differentiating the quantity inside the square brackets in (27) with respect to x shows that it is the Laplace transform of $\phi_{a0}(t)t$.

2.3. Proofs of results on the gamma function.

The order of proofs is Theorems 1.3.3, 1.3.4, 1.3.1, 1.3.2, and finally 1.3.5.

Proof of Theorem 1.3.3. Fix $\Delta > 0$, and consider a sequence of contours C_b like the one shown in Fig. 1, but with $b = -\lceil a\rceil - k - \frac{1}{2}$, $k = 0, 1, 2, \ldots$. The contributions from the four subcontours will be estimated as $b \to -\infty$ for fixed Δ, followed by $\Delta \to 0$.

(1) *The subcontour* ${}_bC_{2\pi-2\Delta}$: Since $F_{a,\theta}(z) \to F_\theta$ uniformly on ${}_bC_{2\pi-2\Delta}$ as $b \to -\infty$, ${}_bC_{2\pi-2\Delta}$'s contribution to the Cauchy integral approaches $F_\theta(1 - \Delta\pi^{-1})$. The corresponding limit as $\Delta \to 0$ is F_θ. Hence ${}_bC_{2\pi-2\Delta}$ contributes the first term in (15).

(2) *The subcontours* ${}_bC_{\pm\Delta}$: Restricting w^z to its principal branch mandates that $(-w)^z = w^z e^{-\pi i z(\text{sgn Im } w)}$, where sgn x denotes the sign of a real number x. Because sgn Im $(w+a)$ = sgn Im w and because of the reflection theorem in (3),

$$F_{a,\theta}(w) = G_{a,\theta}(w)\frac{-ie^{-\pi i(w+\theta)\text{ sgn Im } w}}{e^{\pi i w} - e^{-\pi i w}}. \tag{28}$$

Stirling's approximation implies $G_{a,\theta}(w) \to F_\theta$ as $b \to -\infty$, uniformly for any w on ${}_bC_{\pm\Delta}$. Also Re $w = b = -\lceil a \rceil - k - \frac{1}{2}$, so $e^{2\pi i \text{Re } w} = -1$. Hence

$$\left|\frac{-ie^{-\pi i(w+\theta)\text{sgn Im } w}}{e^{\pi i w} - e^{-\pi i w}}\right| = \frac{1}{e^{-2\pi|\text{Im } w|} + 1} \leq 1. \tag{29}$$

So $F_{a,\theta}(w) = O(F_\theta) = O(1)$ as $b \to -\infty$, uniformly on ${}_bC_{\pm\Delta}$. ${}_bC_{\pm\Delta}$'s contribution to the Cauchy integral in (17) is bounded by $(2\pi)^{-1}$ times the length of ${}_bC_{\pm\Delta}$ times the maximum magnitude of the integrand on ${}_bC_{\pm\Delta}$. When $b \to -\infty$ for fixed Δ, the contribution is of order

$$(2\pi)^{-1}(2|b+a|\tan\Delta)|b+a|^{-1} = \pi^{-1}(\tan\Delta). \tag{30}$$

Thus, when $\Delta \to 0$, ${}_bC_{\pm\Delta}$ makes no contribution to (15).

(3) *The subcontours* C_{-k}: C_{-k}'s contribution to the Cauchy integral in (17) is the residue of $-F_{a,\theta}(w)(w-z)^{-1}$ at $w = -k$. Because the residue of $w!$ at $w = -k$ is $(-1)^{k-1}[(k-1)!]^{-1}$, the residue of $-F_{a,\theta}(w)(w-z)^{-1}$ is the kth term of the sum in (15).

(4) *The subcontours* $\pm C^{\pm}_{[b,-a]}$: From (28), along the upper side of the branch cut, $F_{a,\theta}(w)$ equals

$$F^+_{a,\theta}(w) = G_{a,\theta}(w)\frac{-ie^{-\pi i\theta}}{e^{2\pi i w} - 1}, \tag{31}$$

while on the lower side, it equals

$$F^-_{a,\theta}(w) = G_{a,\theta}(w)\frac{ie^{\pi i\theta}}{e^{-2\pi i w} - 1}. \tag{32}$$

A series of lemmas will calculate how much the branch contours $\pm C^{\pm}_{[b,-a]}$ contribute to the Cauchy integral in (17).

LEMMA 2.3.1. *Let b and c be real numbers, $b < c$. Let $G(w)$ be any function bounded and analytic in the strip $b \leq$ Re $w \leq c$. A branch point at $w = b$ or $w = c$ is permitted if $G(w)$ is finite there. Then*

$$\begin{aligned}\sum_{k=\lceil b\rceil}^{\lfloor c\rfloor} & G(k)\cos\pi\theta - P\int_b^c G(w)\frac{\sin\pi(w-\theta)}{\sin\pi w}dw \\ &= \int_{C^+_{[b,c]}} G(w)\frac{e^{-\pi i\theta}}{e^{-2\pi i w} - 1}dw + \int_{C^-_{[b,c]}} G(w)\frac{e^{\pi i\theta}}{e^{2\pi i w} - 1}dw \\ &= i\int_0^\infty \left[e^{-\pi i\theta}\frac{G(u+iv)}{e^{2\pi(v-iu)} - 1} - e^{\pi i\theta}\frac{G(u-iv)}{e^{2\pi(v+iu)} - 1}\right]_{u=b}^{u=c} dv,\end{aligned} \tag{33}$$

where the floor $\lfloor c\rfloor$ is the unique integer satisfying $\lfloor c\rfloor \leq c < \lfloor c\rfloor + 1$. "P" indicates the Cauchy principal value of the integral [6, *p.* 203] *and* $[g(u)]_{u=b}^{u=c} := g(c) - g(b)$. *On* *the left side, if either b or c is an integer, the corresponding term in the sum should be halved. In the second expression, $C^{\pm}_{[b,c]}$ (analogous to $C^{\pm}_{[b,-a]}$ in Fig. 1) are upper and lower indented contours oriented along the real axis from b to c.*

The left side is incidental to the rest of the paper, but interesting. When $\theta = 0$, the expression is the difference between a sum and the corresponding integral. Equation (33) then parallels a standard proof of Plana's theorem [10, p. 145], which like the Euler–Maclaurin sum formula [10, p. 127] involves the difference between a sum and the corresponding integral. When $\theta = \frac{1}{2}$, however, (33) becomes the principal value of a cotangent integral [3, p. 339], and intermediate values $0 < \theta < \frac{1}{2}$ give linear combinations of the two extreme results.

Proof of Lemma 2.3.1. In the second expression, the indentations of $C^{\pm}_{[b,c]}$ around poles of $(e^{\mp 2\pi i w} - 1)^{-1}$ contribute partial residues adding up to the sum in the first expression [6, p. 206]. The rest of $C^{\pm}_{[b,c]}$ contributes the principal value because of the identity

$$\frac{e^{-\pi i\theta}}{e^{-2\pi i w} - 1} + \frac{e^{\pi i \theta}}{e^{2\pi i w} - 1} = -\frac{\sin \pi(w-\theta)}{\sin \pi w}. \tag{34}$$

The second equality in (33) is proved by integrating $G(w)(e^{-2\pi i w} - 1)^{-1}$ around an indented rectangle with corners b, c, $c + i\infty$, and $b + i\infty$, equating the result to 0, and then multiplying by $e^{-\pi i \theta}$ to yield

$$\int_{C^{+}_{[b,c]}} G(w) \frac{e^{-\pi i \theta}}{e^{-2\pi i w} - 1} dw = i \left[\int_0^{\infty} e^{-\pi i \theta} \frac{G(u+iv)}{e^{2\pi(v-iu)} - 1} dv \right]_{u=b}^{u=c}. \tag{35}$$

Adding this equation to a similar one with $-i$ replacing i gives the second equality.

LEMMA 2.3.2. *Under the conditions of Lemma* 2.3.1,

$$\begin{aligned} &\int_{C^{+}_{[b,c]}} G(w) \frac{e^{-\pi i \theta}}{e^{2\pi i w} - 1} dw + \int_{C^{-}_{[b,c]}} G(w) \frac{e^{\pi i \theta}}{e^{-2\pi i w} - 1} dw \\ &\quad = -2 \cos \pi\theta \int_b^c G(w) dw \\ &\quad -i \int_0^{\infty} \left[e^{-\pi i \theta} \frac{G(u+iv)}{e^{2\pi(v-iu)} - 1} - e^{\pi i \theta} \frac{G(u-iv)}{e^{2\pi(v+iu)} - 1} \right]_{u=b}^{u=c} dv. \end{aligned} \tag{36}$$

Proof of Lemma 2.3.2. The second term on the right side of (36) is the common expression in (33) times (-1). Substituting (34) with $\theta = 0$ into the integrands on the left of (36) then proves Lemma 2.3.2.

Because of (31) and (32), the contribution $\pm C^{\pm}_{[b,-a]}$ makes to the Cauchy integral in (17) is the same as the common value in (36) when $c = -a$, $G(w) = G_{a,\theta}(w)(w - z)^{-1}(-i)$, and the integrals are premultiplied by $(2\pi i)^{-1}$. The restriction $a \geq \theta$ in Theorem 1.3.3 derives from the requirement that $G(w)$ be analytic in the strip $b \leq \operatorname{Re} w \leq c$.

After substitution and premultiplication in (36), as $b \to -\infty$ the first term on the right side approaches the integral in (15). When $u = b$ in the second term and $b \to -\infty$, Stirling's approximation implies $G_{a,\theta}(b \pm iv) \to 1$ uniformly for any v. Because $b = -\lceil a \rceil - k - \frac{1}{2}$, $e^{\mp 2\pi i b} = -1$. Thus the magnitude of the second term evaluated as $u = b \to -\infty$ is bounded by

$$\frac{1}{\pi} \int_0^{\infty} \frac{1}{e^{2\pi v} + 1} \frac{1}{\operatorname{Re}(z - b)} dv \to 0. \tag{37}$$

Because all other terms in (15) have been accounted for, the second term on the right side of (36) evaluated at $u = c = -a$ must be the error term $\epsilon_{a,\theta}(z)$. This proves Theorem 1.3.3.

Proof of Theorem 1.3.4. The factors in the nth derivative of the integrand of (16) must be bounded for $v \geq 0$. The following equations bound factors in $G_{a,\theta}(-a \pm iv)$:

$$\begin{aligned} \ln\left|\frac{(a-1)!}{(a \mp iv - 1)}\right| & \\ &= \frac{1}{2}\sum_{k=0}^{\infty} \ln\left[1 + v^2(a+k)^{-2}\right] \leq \frac{1}{2}\int_{a-1}^{\infty} \ln(1+v^2x^{-2})dx \\ &= \frac{1}{2}v\left[x\ln(1+x^{-2}) + 2\arctan x\right]_{x=(a-1)v^{-1}}^{x=\infty} \leq \frac{1}{2}\pi v. \end{aligned} \tag{38}$$

The first equality in (38) follows from the factorial function's Weierstrass representation in (2). Hence $(a-1)! \leq |(a \mp iv - 1)!| e^{(1/2)\pi v}$. Also, because the principal branch of the logarithm is used in complex exponentiation,

$$\begin{aligned} \left|(\mp iv)^{a-\theta\mp iv}\right| &= \left|e^{(a-\theta\mp iv)(\ln v \mp \frac{1}{2}\pi i)}\right| \\ &= e^{(a-\theta)\ln v - \frac{1}{2}\pi v} \\ &= v^{a-\theta}e^{-\frac{1}{2}\pi v}. \end{aligned} \tag{39}$$

Combining (38) and (39) with $|e^{\pm\pi i\theta}| = |e^{\pm iv}| = 1$ and other obvious estimates yields (18) and (19), which proves Theorem 1.3.4.

Proof of Theorem 1.3.1. $|e^{2\pi v} - e^{2\pi i a}|^{-1} \leq (e^{2\pi v} - 1)^{-1} = \sum_{k=1}^{\infty} e^{-2\pi k v}$, so integrating term by term in (19) when $a > \theta$ gives $C_{a,\theta} \leq C^*_{a,\theta}$, where

$$C^*_{a,\theta} = \frac{(a-\theta)!}{(a-1)!}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \zeta(a-\theta+1)(2\pi)^{-(a-\theta+1)}, \tag{40}$$

and $\zeta(z) = \sum_{k=1}^{\infty} k^{-z}$ is the Riemann zeta function [10, p. 265]. Set $\theta = \frac{1}{2}$. The next lemma bounds $C^*_{a,\frac{1}{2}}$.

LEMMA 2.3.3. *For $a > 0$,*

$$\frac{(a-\frac{1}{2})!}{(a-1)!} < a^{\frac{1}{2}}. \tag{41}$$

Proof of Lemma 2.3.3. Let $a > 0$. Lemma 1.2.2 implies that $f_{\frac{1}{2}}(a - \frac{1}{2}) = \ln F_{\frac{1}{2},\frac{1}{2}}(a-\frac{1}{2}) < 0$ and that $f_1(a-1) = \ln F_{1,\frac{1}{2}}(a-1) > 0$. Hence

$$\frac{(a-\frac{1}{2})!}{(a-1)!}a^{-\frac{1}{2}} = \frac{F_{\frac{1}{2},\frac{1}{2}}(a-\frac{1}{2})}{F_{1,\frac{1}{2}}(a-1)} < 1. \tag{42}$$

Thus $C^*_{a,\frac{1}{2}} < (\frac{e}{\pi})^{1/2}a^{1/2}(2\pi)^{-(a+1/2)}$ for $a \geq 3$, because $\zeta(a+\frac{1}{2}) \leq \zeta(3.5) = 1.12\ldots < 1.16\ldots = (\frac{e}{2})^{1/2}$. Because of Lemma 1.1.1, multiplying a bound for $\epsilon_{a,\frac{1}{2}}(z)$ by $(\frac{\pi}{e})^{\frac{1}{2}}$ bounds the relative error in $z!$, so Theorem 1.3.1 follows.

Proof of Theorem 1.3.2. Theorem 1.3.4 with $a = \theta = \frac{1}{2}$ and $n = 0$ implies $F_{1/2,1/2}(z) = 1 + \epsilon_{1/2,1/2}(z)$, where $|\epsilon_{1/2,1/2}(z)| \leq C_{1/2,1/2}[\mathrm{Re}(z+\frac{1}{2})]^{-1}$. Since

$(e^{2\pi v}+1)^{-1} = \sum_{k=1}^{\infty}(-1)^{k-1}e^{-2\pi kv}$, integrating term by term in (19) gives $C_{1/2,1/2} = 2^{-1/2}\pi^{-2}\ln 2$. Multiplying $C_{1/2,1/2}$ by the factor $(\frac{\pi}{e})^{1/2}$ from Lemma 1.1.1 gives the stated relative error.

Proof of Theorem 1.3.5. Fix z and θ. Multiply both sides of (15) by $(z+a)^{z+\theta}e^{-(z+a)}(2\pi)^{1/2}$, which has limit 0 as $a \to \infty$. Thus any term on the right of (15) that is bounded for sufficiently large a can be omitted in the limit as $a \to \infty$. Because of Theorem 1.3.4 and (40), both 1 and $\epsilon_{a,\theta}(z)$ can be omitted. For $\theta = \frac{1}{2}$, this proves the theorem because the integral term in (15) is 0 then, whereas for $\theta > \frac{1}{2}$ the next lemma bounds the integral term, thus showing that it can be neglected.

LEMMA 2.3.4. $|G_{a,1/2}(w)| \le (\frac{8}{e^2})^{1/2}$ *for* $w \le -a-1$ *and* $a \ge \frac{1}{2}$.

Proof of Lemma 2.3.4. Let $w \le -a-1$, where $a \ge \frac{1}{2}$. Lemma 1.2.2 justifies all the inequalities in the following: $\ln G_{a,\frac{1}{2}}(w) = -\ln F_{1-a,\frac{1}{2}}(-w-1) = -f_{1-a}(-w-1) \le -f_{\frac{1}{2}}(-w-1) \le -f_{\frac{1}{2}}(\frac{1}{2}) = \frac{1}{2}\ln(\frac{8}{e^2})$. Lemma 2.3.4 follows by exponentiating.

Hence $\left|\int_{-\infty}^{-a-1} G_{a,\theta}(w)(w-z)^{-1}dw\right| \le \left(\frac{8}{e^2}\right)^{1/2}\int_{-\infty}^{-a-1}(-w-a)^{-(1/2)-\theta}dw$. Also, $\left|\int_{-a-1}^{-a} G_{a,\theta}(w)(w-z)^{-1}dw\right| \le (2\pi)^{1/2}[(a-1)!]^{-1}[\mathrm{Re}(z+a)]^{-1}$, so $\left|\int_{-\infty}^{-a} G_{a,\theta}(w) \times (w-z)^{-1}dw\right| \le \left(\frac{8}{e^2}\right)^{1/2}(\theta-\frac{1}{2})^{-1}+(2\pi)^{1/2}(a!)^{-1}$. Thus the integral term is bounded and can be neglected.

2.4. Proof of results on the digamma and trigamma functions. In this section $\theta = \frac{1}{2}$ always. For brevity, the dependencies on θ, a and z are sometimes suppressed. The order of proof is Theorem 1.4.1 and then Theorem 1.4.2.

Proof of Theorem 1.4.1. Rearranging the nth derivative of the equation $F_{a,\frac{1}{2}}(z) = \exp[f_a(z)]$ gives

$$f^{(n)} = \frac{F^{(n)}}{F} - P_n\left(f', \ldots, f^{(n-1)}\right), \tag{43}$$

where P_n is a multivariate polynomial given explicitly by the Ivanoff [4] and Faa di Bruno formulas [7, p. 214]. All the coefficients in P_n are nonnegative.

Recall the ϵ-notation defined in the Introduction. The recursion (43) for $f^{(n)}$ has one ancillary quantity $F^{(n)}/F$ whose error is bounded by

$$\begin{aligned}\epsilon\left[\frac{F^{(n)}}{F}\right] &\le \frac{|F^{(n)}|+\epsilon\left[F^{(n)}\right]}{|F|-\epsilon[F]} - \left|\frac{F^{(n)}}{F}\right| \\ &= \left(\epsilon\left[F^{(n)}\right] - \left|\frac{F^{(n)}}{F}\right|\epsilon[F]\right)(|F|-\epsilon[F])^{-1}.\end{aligned} \tag{44}$$

The final factor $(|F_{a,1/2}| - \epsilon[F_{a,1/2}])^{-1}$ can be bounded because Lemma 1.1.1 shows that $|F_{a,1/2}| \ge (\frac{e}{\pi})^{1/2}$, while Theorem 1.3.4 with $n = 0$ shows that $\epsilon[F_{a,1/2}] \le a^{-1}C_{a,1/2}$. The next lemma shows $a^{-1}C_{a,1/2} < (\frac{e}{\pi})^{1/2}(\frac{2}{3})^{1/2}(2\pi)^{-2}$ for $a \ge \frac{3}{2}$. The restriction $a \ge \frac{3}{2}$ in the theorems simplifies inequalities while maintaining reasonable stringency.

LEMMA 2.4.1. $C_{a,1/2} < (\frac{e}{\pi})^{1/2}(\frac{3}{2})^{1/2}(2\pi)^{-2}$ *for* $a \ge \frac{3}{2}$.

Proof of Lemma 2.4.1. The proof of Theorem 1.3.1 in §2.3 contains the inequalities

$$C_{a,\frac{1}{2}} \le C^*_{a,\frac{1}{2}} < \left(\frac{e}{\pi}\right)^{\frac{1}{2}} a^{\frac{1}{2}}(2\pi)^{-(a+\frac{1}{2})}. \tag{45}$$

Since the right side decreases for $a \geq \frac{3}{2}$, Lemma 2.4.1 follows by setting $a = \frac{3}{2}$ on the right.

All quantities on the right side of (44) can be bounded. First, $|F_{a,1/2}| - \epsilon[F_{a,1/2}] > (\frac{e}{\pi})^{1/2}[1 - (\frac{2}{3})^{1/2}(2\pi)^{-2}] = (\frac{e}{\pi})^{1/2}D^{-1}$. Second, Theorem 1.3.4 bounds $\epsilon[F^{(n)}]$ and $\epsilon[F]$. Finally, $|F^{(n)}_{a,1/2}/F_{a,1/2}|$ can be bounded by solving (43) for $F^{(n)}/F$, and bounding $|f_a^{(n)}|$ with Lemma 2.2.1.

Equation (44) with $n = 1$ requires $\epsilon[F] \leq C_{a,1/2}[\mathrm{Re}(z + a)]^{-1}$ and $\epsilon[F'] \leq C_{a,1/2}[\mathrm{Re}(z+a)]^{-2} \leq a^{-1}C_{a,1/2}[\mathrm{Re}(z+a)]^{-1}$ from Theorem 1.3.4. Lemma 2.2.1 gives $|F'_{a,1/2}/F_{a,1/2}| = |f'_a| \leq -f'_a(0) < \ln 2a - 1 + \frac{1}{2}a^{-1}$. Hence because $-1+\frac{1}{2}a^{-1}+a^{-1} \leq 0$, for $a \geq \frac{3}{2}$ and $\mathrm{Re}\, z \geq 0$, (44) with $n = 1$ yields

$$\epsilon\left[\frac{F'_{a,\frac{1}{2}}}{F_{a,\frac{1}{2}}}\right] < D(\ln 2a)\left(\frac{\pi}{e}\right)^{\frac{1}{2}} C_{a,\frac{1}{2}}[\mathrm{Re}(z+a)]^{-1}. \tag{46}$$

The bound for $C_{a,\frac{1}{2}}$ in (45) gives (23) in Theorem 1.4.1.

Proof of Theorem 1.4.2. Equation (43) with $n = 2$ gives $P_2(f') = (f')^2$, so a bound for $\epsilon[P_2] \leq \epsilon[f'](2|f'| + \epsilon[f'])$ is required. Equation (46) bounds $\epsilon[f']$, and the paragraph preceding, $|f'| = |f'_a|$. The bound for $(2|f'| + \epsilon[f'])$ can be simplified because for $a \geq \frac{3}{2}$, $\epsilon[f'] \leq D(\ln 2a)a^{-1/2}(2\pi)^{-(a+1/2)} \leq 1 - a^{-1}$. The first inequality is (23), while the second is elementary. Hence

$$\epsilon[P_2] \leq D(2\ln^2 2a - \ln 2a)\left(\frac{\pi}{e}\right)^{\frac{1}{2}} C_{a,\frac{1}{2}}[\mathrm{Re}(z+a)]^{-1}. \tag{47}$$

Equation (44) with $n = 2$ requires $\epsilon[F] \leq C_{a,1/2}[\mathrm{Re}(z+a)]^{-1}$ and $\epsilon[F''] \leq 2C_{a,1/2}[\mathrm{Re}(z + a)]^{-3} \leq 2a^{-2}C_{a,1/2}[\mathrm{Re}(z + a)]^{-1}$ from Theorem 1.3.4. Equation (43) and Lemma 2.2.1 give $|F''_{a,1/2}/F_{a,1/2}| = |f''_a + (f'_a)^2| \leq f''_a(0) + |f'_a(0)|^2 \leq \ln^2 2a + 2 - 2a^{-2}$, the last inequality holding for $a \geq \frac{3}{2}$. Hence for $a \geq \frac{3}{2}$ and $\mathrm{Re}\, z \geq 0$, equation (44) with $n = 2$ yields

$$\epsilon\left[\frac{F''_{a,\frac{1}{2}}}{F_{a,\frac{1}{2}}}\right] \leq D(\ln^2 2a + 2)\left(\frac{\pi}{e}\right)^{\frac{1}{2}} C_{a,\frac{1}{2}}[\mathrm{Re}(z+a)]^{-1}. \tag{48}$$

Adding (47) and (48) and substituting the bound for $C_{a,1/2}$ from (45) gives (24) in Theorem 1.4.2.

3. Discussion. The calculation of $z!$ in Theorem 1.3.1 is unusually efficient, since a computation with relative error ϵ requires $O(-\log \epsilon)$ time independent of z. The only numerical drawback to Theorem 1.3.1, which is not serious, is that for large a its coefficients are large and their signs alternate, causing numerical cancellation in (7).

The coefficients computed from Theorem 1.3.1 agree remarkably with Lanczos' coefficients for equation (7) [5], despite the dissimilarity of the generating formulas. For example, for $a = 2$ and $N = 1$, Lanczos gave $c_0 = 0.999\underline{\mathbf{779}}\ldots$ and $c_1 = 1.084\underline{\mathbf{635}}\ldots$, and an error bound $|\epsilon(z)| \leq 2.4 \cdot 10^{-4}$ for $\mathrm{Re}\, z \geq 0$. For $a = 2$, Theorem 1.3.1 gives

$$z! = (z+2)^{z+\frac{1}{2}}e^{-(z+2)}(2\pi)^{\frac{1}{2}}\left[1 + (2\pi)^{-\frac{1}{2}}\frac{e}{z+1} + \epsilon_{2,\frac{1}{2}}(z)\right]. \tag{49}$$

Numerically, the coefficients are $c_0 = 1$ and $c_1 = 1.084\underline{\mathbf{437}}\ldots$.

The error bound in Theorem 1.3.1 is reasonably tight. Numerical experiments for $3 \leq a \leq 11$ indicate that the bound is too generous by a factor of about $(2\pi)^2$ to $(2\pi)^3$. Given a desired error $|\epsilon(z)|$, the bound therefore overestimates the number of terms $N = \lceil a \rceil - 1$ required in (7) by only 2 or 3.

Numerical experiments show Lanczos' coefficients $\{c_k\}$ for (7) to be slightly superior to the coefficients in Theorem 1.3.1. For example, the smallest error Lanczos [5] tabulated was $|\epsilon(z)| \leq 2 \cdot 10^{-10}$ for Re $z \geq 0$, requiring $N = 6$ terms in the sum, and numerical experiments do indeed confirm that $N = 6$ terms suffice. By comparison, Theorem 1.3.1 gives a relative error of $2 \cdot 10^{-10}$ for 10 terms in the sum (with $a = 11$), while numerical experiments indicate that 7 terms (with $a = 8$) actually suffice.

A few extra terms in (7) do not prolong computations significantly, however. (7) contains the exponential operation, which requires more computing time than elementary arithmetic operations. Thus, when the accuracy required exceeds available tabulations of Lanczos' coefficients, calculating $\{c_k\}$ from Theorem 1.3.1 is certainly simpler than using Lanczos' complicated formulas, and the resulting computation is not significantly slower.

The series (15) for $F_{a,\theta}(z)$ computes $z!$ more easily and accurately with $\theta = \frac{1}{2}$ than with $\theta > \frac{1}{2}$. First, the unattractive integral term persists except when $\cos \pi\theta = 0$. More importantly, even when $\cos \pi\theta = 0$, $F_{a,\theta}(z) = F_{a,\frac{1}{2}}(z)(z+a)^{-(\theta-\frac{1}{2})} \to 0$ as $|z+a| \to \infty$, which inflates the relative error when computing $z!$ at large $|z+a|$. (15) with $\theta = \frac{3}{2}$ does, however, compute $z!$ remarkably efficiently.

Theorem 1.3.4 also provides error bounds for the factorial's derivatives that have no analogue in Lanczos' paper [5]. In Theorems 1.4.1 and 1.4.2 these bounds yield approximations for the factorial's logarithmic derivatives, the digamma and trigamma functions. The computations given for $\Psi(z)$ and $\Psi'(z)$ are quite efficient, since a relative error ϵ again requires only $O(-\log \epsilon)$ time independent of z.

REFERENCES

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1965.

[2] W. Feller, *An Introduction to Probability Theory and its Applications*, vol. 2, John Wiley and Sons, New York, 1968.

[3] G. H. Hardy, *Divergent Series*, Oxford University Press, Oxford, 1973.

[4] V. F. Ivanoff, *Problem 4782*, Amer. Math. Monthly, 65 (1958), p. 212.

[5] C. Lanczos, *A precision approximation of the gamma function*, SIAM J. Numer. Anal., B1 (1964), pp. 86–96.

[6] N. Levinson and R. M. Redheffer, *Complex Variables*, Holden-Day, San Francisco, CA, 1970.

[7] Z. A. Melzak, *Companion to Concrete Mathematics*, John Wiley and Sons, New York 1973.

[8] W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes in C*, Cambridge University Press, Cambridge, UK, 1988.

[9] A. E. Taylor, *Advanced Calculus*, Ginn, New York, 1955.

[10] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge University Press, Cambridge, UK, 1978.

[11] D. V. Widder, *Advanced Calculus*, Prentice-Hall, Englewood Cliffs, NJ, 1961.