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One reason why Brownian motion and Johnson noise are difficult subjects to teach is that their mathematical requirements transcend the capabilities of ordinary differential calculus. Presented here is an exposition of the needed generalization of calculus, namely continuous Markov process theory, in a form that should be accessible to advanced physics undergraduates. It is shown how this mathematical framework enables one to give clear, concise derivations of all the principal results of Brownian motion and Johnson noise, including fluctuation–dissipation formulas, auto-covariance transport formulas, spectral density formulas, Nyquist's formula, the notions of white and $1/f^2$ noise, and an accurate numerical simulation algorithm. An added benefit of this exposition is a clearer view of the mathematical connection between the two very different approaches to Brownian motion taken by Einstein and Langevin in their pioneering papers of 1905 and 1908. © *1996 American Association of Physics Teachers.*

I. INTRODUCTION

Two significant accomplishments of physics in the first half of this century were the theoretical elucidations of Brownian motion[1-3] and Johnson noise.[4,5] Despite the extensive theoretical advancements in our understanding made since those classic works,[6-11] the presentation of these topics to students has always posed a challenge. Typical expositions invoke some combination of a nonintuitive and complicated partial differential equation named "Fokker–Planck," and a more plausible but disquietingly not-so-ordinary differential equation named "Langevin." In the case of the latter, for example, students are usually presented with a benign looking equation of the form

$$\frac{dX(t)}{dt} = -aX(t) + f(t),$$

then told to assume, for no compelling reason, that the driving function $f(t)$ on the right is "delta-correlated and Gaussian," and finally informed much later (if the teacher can muster the courage) that the derivative on the left side does *not*, after all, really exist.

The purpose of this paper is to suggest a more palatable and convincing way of presenting Brownian motion and Johnson noise to students of physics. This approach avoids the Fokker–Planck equation entirely and relies solely on the Langevin equation, but not quite in the way just described. The key is a careful and well-motivated exposition of the underlying mathematics, namely, the theory of *continuous Markov processes*. Section II gives such an exposition for students who have only the usual background in ordinary calculus. The scope of the exposition is limited to developing only those results that will actually be required. This mathematical machinery is then applied to Brownian motion in Sec. III and Johnson noise in Sec. IV, with all the standard results being deduced in an efficient and transparent manner. Among those results are fluctuation–dissipation formulas, auto-covariance transport formulas, spectral density formulas, Nyquist's formula, the notions of white noise and $1/f^2$ noise, and a numerical simulation algorithm that is simple and accurate. A byproduct of this exposition is a clearer perspective of the relation between the two approaches to Brownian motion originally advanced by Einstein[1] and

Langevin.[3] It was their papers that gave birth to the "stochastic generalization" of ordinary calculus that we know today as continuous Markov process theory.

II. MATHEMATICAL FOUNDATIONS

A. The processes of ordinary calculus

Much of science and engineering is based on the presumption that some physical quantity X evolves with time t according to a differential equation of the form

$$\frac{dX(t)}{dt} = A(X(t), t), \qquad (2.1)$$

where A is some smooth function of its two arguments. An equivalent way of expressing this dynamical behavior is to say that

$$X(t+dt) = X(t) + A(X(t), t)dt, \qquad (2.2)$$

where dt is a *non-negative infinitesimal*, i.e., a real variable that is confined to some interval $[0, \epsilon]$, with ϵ taken so small that any higher order dt terms on the right side of Eq. (2.2) are rendered negligible. The nonnegativity of dt reflects the fact that time is something we perceive to be always increasing. We generally refer to any function X of time t as a *process*.

Although Eq. (2.2) might be held by some to be mathematically "less respectable" than Eq. (2.1), it evidently plays the conceptually useful role of an *update formula* for the process X: If we know the value of X at the current time t, then Eq. (2.2) allows us to compute the value of X at any infinitesimally later time $t+dt$. The fact that this update formula gives $X(t+dt)$ unequivocally prompts us to call the process "deterministic." The fact that the update formula implies that $X(t+dt) \rightarrow X(t)$ as $dt \rightarrow 0$ prompts us to call the process "continuous." And the fact that the update formula does not require for the prediction of $X(t+dt)$ the value of X at any time *before* t prompts us to call the process "memoryless;" i.e., the process does not have to "remember" any of its earlier values explicitly in order to advance itself from time t to time $t+dt$. So Eq. (2.2), and hence also Eq. (2.1), define a *continuous memoryless deterministic* process $X(t)$. Such processes are the traditional objects of study of ordinary differential calculus.

We shall be concerned here with *generalizing* Eqs. (2.1) and (2.2) so that they describe a process X that is *stochastic* rather than deterministic. A *continuous memoryless stochastic* process is called a *continuous Markov* process. For such a process, we can assign definite probabilities to all possible values of $X(t+dt)$ knowing the values of t, dt and $X(t)$, but those probabilities cannot be "sharpened" by taking cognizance of any values of X *before* time t.

We shall find that the evolutionary character of the variable t gives rise to some surprisingly strict limitations on how Eqs. (2.1) and (2.2) can be stochastically generalized. Such limitations are in fact already implicit in those formulas. For instance, the formula

$$X(t+dt)=X(t)+A(X(t),t)(dt)^{1/2} \qquad (2.3a)$$

might appear to define a continuous memoryless deterministic process X quite as ably as does Eq. (2.2) (although certainly not the same process), but that is actually not so. The problem with Eq. (2.3a) is *not* that there is some law against taking the square root of an infinitesimal variable (because there is not), *nor* that the process so defined would have no derivative (for we need not insist *a priori* that all continuous memoryless deterministic processes be differentiable). The problem with Eq. (2.3a) is rather that, as an "update" formula, it is *not self-consistent*: Observe that Eq. (2.3a) would update the process from time t to time $t+\frac{1}{2}dt$ as

$$X(t+\tfrac{1}{2}dt)=X(t)+A(X(t),t)(\tfrac{1}{2}dt)^{1/2}, \qquad (2.3b)$$

and from time $t+\frac{1}{2}dt$ to time $t+dt$ as

$$X(t+dt)=X(t+\tfrac{1}{2}dt)+A(X(t+\tfrac{1}{2}dt),t+\tfrac{1}{2}dt)(\tfrac{1}{2}dt)^{1/2}. \qquad (2.3c)$$

By simply substituting Eq. (2.3b) into the right side of Eq. (2.3c) and then invoking the hypothesized smoothness of the two functions A and X, we obtain, to lowest order in dt,

$$X(t+dt)=X(t)+2^{1/2}A(X(t),t)(dt)^{1/2}. \qquad (2.3d)$$

But this is inconsistent with Eq. (2.3a).

An *acceptable* process update formula, when applied first from t to $t+\alpha dt$ $(0<\alpha<1)$ and then from $t+\alpha dt$ to $t+dt$, must always give the same result, at least to lowest order in dt, as when it is applied directly from t to $t+dt$. In Sec. II C we shall see that this *self-consistency condition* constrains *all* continuous memoryless deterministic processes to evolve according to equations of the form (2.1) and (2.2), and constrains all continuous memoryless *stochastic* processes (i.e., all continuous Markov processes) to evolve according to *generalizations* of Eqs. (2.1) and (2.2) known as "Langevin equations."

We remark that our restriction here to processes that are "memoryless" is not quite so limiting as it might appear. Many nonmemoryless processes are *components*, or functions of one or more components, of *multivariate memoryless* processes. In the deterministic case this is a direct consequence of the fact that any nth order ordinary differential equation for a variable $u(t)$ can always be written as a *first-order* ordinary differential equation for the n-dimensional vector variable $v(t)\equiv[u(t),u'(t),...,u^{(n-1)}(t)]$. An example is the position x and velocity $v\equiv dx/dt$ of a classical particle in a force field $F(x,v,t)$: neither x nor v is in general memoryless by itself, but the *pair* $[v(t),x(t)]$ satisfies time-evolution equations of the forms (2.1) and (2.2), and hence constitutes a *bivariate* continuous memoryless deterministic process. Our focus in what follows will be exclusively on

univariate memoryless processes, but we should keep in mind that many nonmemoryless processes in nature should be treatable by extending our univariate results to the multivariate case.

B. Random variables

As will become clear in the sequel, a continuous Markov process $X(t)$ is what is known as a "random variable." Although most students of physics have some knowledge of random variables, that knowledge tends to be spotty and confused, and occasionally even erroneous. We shall briefly summarize here the *minimum* that one needs to know about random variables in order to understand continuous Markov process theory. All of this material is widely known and treated in many introductory textbooks. The particular didactic approach taken here is one that has been presented in detail by the author elsewhere,[12] but probably most students will be content to accept these random variable results without proofs.

We say that Y is a *random variable* with *density function* P if and only if $P(y)dy$ equals the probability that a "sampling" of Y will yield a value between y and $y+dy$; symbolically, $P(y)dy=\mathrm{Prob}\{Y\in[y,y+dy)\}$. Notice that we distinguish between the *random variable* Y and the possible *values* y which that random variable may exhibit when it is sampled.

The density function P specifies the shape of the normalized frequency histogram that one would get by plotting a very large number of samples of Y. A random variable is completely specified by its density function, and there are as many different random variables Y as there are different histogram shape functions P. If P depends parametrically on one or more variables a, so that $P=P(y;a)$, then we may write $Y=Y(a)$.

The *average* of any function h with respect to the random variable Y is denoted by $\langle h(Y)\rangle$, and is defined (when it exists) by

$$\langle h(Y)\rangle\equiv\lim_{M\to\infty}\frac{1}{M}\sum_{i=1}^{M}h(y^{(i)}), \qquad (2.4a)$$

where $y^{(i)}$ is the value obtained in the ith sampling of Y. This average can also be computed as

$$\langle h(Y)\rangle=\int_{-\infty}^{\infty}h(y)P(y)dy. \qquad (2.4b)$$

The average $\langle Y^k\rangle$ is called the kth *moment* of Y. The first moment $\langle Y\rangle$ is also called the *mean* of Y, and is sometimes written mean$\{Y\}$. The *variance* of Y is defined by

$$\mathrm{var}\{Y\}\equiv\langle(Y-\langle Y\rangle)^2\rangle\equiv\langle Y^2\rangle-\langle Y\rangle^2, \qquad (2.5)$$

and, when it exists, is always nonnegative. The square root of var$\{Y\}$ is called the *standard deviation* of Y, and is written sdev$\{Y\}$; it provides a usually reasonable measure of the spread or dispersion of the sample values of Y about its mean. If var$\{Y\}$ happens to be zero, then every sampling of Y will yield the same value, and we say that Y is a *sure* variable; the density function of a sure variable is a Dirac delta function centered at the (unique) sample value.

We say that n random variables $Y_1,...,Y_n$ have *joint density function* P if and only if $P(y_1,...,y_n)dy_1...dy_n$ $=\mathrm{Prob}\{Y_i\in[y_i,y_i+dy_i)$ for all $i=1$ to $n\}$. Averages with

respect to $Y_1,...,Y_n$ are defined analogously to the univariate case; in particular, the *covariance* of two random variables Y_1 and Y_2 is by definition the average

$$\text{cov}\{Y_1,Y_2\} \equiv \langle (Y_1 - \langle Y_1 \rangle)(Y_2 - \langle Y_2 \rangle) \rangle$$

$$\equiv \langle Y_1 Y_2 \rangle - \langle Y_1 \rangle \langle Y_2 \rangle. \qquad (2.6)$$

We say that Y_1 and Y_2 are *statistically independent* if and only if a knowledge of the result of sampling one will not help us to predict the result of sampling the other. In that case their joint density function $P(y_1,y_2)$ factors into two single-variate functions, and $\text{cov}\{Y_1,Y_2\} = 0$.

Random variables can be functionally manipulated just like ordinary sure variables. If $Y_1,...,Y_n$ are n random variables with joint density function P, and if f is some ordinary function, then $Z \equiv f(Y_1,...,Y_n)$ is *defined* to be the random variable whose sample values $z^{(i)}$ can be computed from simultaneous sample values of $Y_1,...,Y_n$ according to $z^{(i)} = f(y_1^{(i)},...,y_n^{(i)})$. The trick lies in computing the density function Q of this random variable Z. It turns out that[12] Q is given in terms of the two functions P and f by the formula

$$Q(z) = \int_{-\infty}^{\infty} dy_1 ... \int_{-\infty}^{\infty} dy_n P(y_1,...,y_n)$$

$$\times \delta(z - f(y_1,...,y_n)), \qquad (2.7)$$

where δ is the Dirac delta function. This formula describes the functional transformation of an n-variate random variable **Y** to a univariate random variable **Z**, and it has a straightforward generalization to the case where the dimension of **Z** is any positive integer m: One merely replaces the single delta function with a product of m delta functions, one for each component of **Z**. But the integrations in these formulas are often quite difficult to carry out. Much of practical random variable theory consists of obtaining and cataloging the many specific consequences of such integrations. One easy (and instructive) consequence of the one-to-one transformation formula is that the mean of the random variable $Z = h(Y)$ is identically equal to the average of the function h with respect to the random variable Y as given by Eq. (2.4b). Happily, we shall not have to compute any integrals like Eq. (2.7) here, but we shall quote below three widely known results of such computations that we shall require later.

Of all possible random variables Y, the one that will be most important for our work here is the *normal* (or Gaussian) random variable $Y = N(m,\sigma^2)$. Its density function is given by

$$P(y) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left(- \frac{(y-m)^2}{2\sigma^2} \right), \qquad (2.8)$$

and one can show by using the foregoing formulas that this random variable has mean m and variance σ^2. The normal random variable $N(0,1) \equiv N$ will be referred to as the *unit normal random variable*. The normal random variable $N(m,0)$ is just the "sure" variable m.

Three well-known functional transformation results involving normal random variables that we shall need later are the following: First, for any two numbers α and β, we have

$$\alpha + \beta N(m,\sigma^2) = N(\alpha + \beta m, \beta^2 \sigma^2). \qquad (2.9)$$

This is a straightforward consequence of Eq. (2.7) for the case $n=1$. In particular, recalling our definition of the unit normal random variable $N \equiv N(0,1)$, we have

$$\alpha + \beta N = N(\alpha, \beta^2). \qquad (2.10)$$

Second, if $N(m_1,\sigma_1^2)$ and $N(m_2,\sigma_2^2)$ are *statistically independent*, then

$$N(m_1,\sigma_1^2) + N(m_2,\sigma_2^2) = N(m_1 + m_2, \sigma_1^2 + \sigma_2^2). \qquad (2.11)$$

This is a more complicated consequence of Eq. (2.7) for the case $n=2$. Actually, the additivity of the means and variances expressed in Eq. (2.11) holds for statistically independent random variables of *any* kind; however, among all classes of random variables whose means and variances exist, only *normal* random variables preserve their class under addition.

It follows from theorems (2.9) and (2.11) that *any linear combination of statistically independent normal random variables is itself normal.* (This statement is also true even if the normal random variables being summed are *not* statistically independent, but the variances then do not combine so simply.)

Finally we have the celebrated *central limit theorem*. This theorem asserts that the sum of any K statistically independent (but not necessarily normal) random variables with finite means and variances will *become* a normal random variable in the limit $K \to \infty$. This theorem can be proved using Eq. (2.7), but the proof is rather involved.[12]

C. Continuous Markov processes and their Langevin equations

A *stochastic process X* is a random variable whose density function depends parametrically on time t; so if t_1 and t_2 are two different instants of time, then $X(t_1)$ and $X(t_2)$ are in general two different random variables. We say that X is a *continuous memoryless* stochastic process, or a *continuous Markov* process, if and only if the following three conditions obtain:

(i) The increment in X from any time t to any infinitesimally later time $t + dt$ is "memoryless," in the sense that it depends *only* on t, dt and the value of X at t; thus, we can meaningfully define the *conditional increment in X* by

$$\Xi(dt;x,t) \equiv X(t+dt) - X(t), \quad given \text{ that } X(t) = x. \qquad (2.12)$$

(ii) The random variable $\Xi(dt;x,t)$ depends "smoothly" on the three variables dt, x, and t.

(iii) X is "continuous" in the sense that $\Xi(dt;x,t) \to 0$ as $dt \to 0$ for all x and t.

The following theorem shows that the above three conditions conspire with the earlier mentioned requirement of *self-consistency* to impose a surprisingly rigid mathematical form on $\Xi(dt;x,t)$. That form will give us the stochastic generalizations of Eqs. (2.1) and (2.2).

Theorem. The defining conditions (i), (ii), and (iii) of a continuous Markov process X imply that the conditional increment (2.12) must have the analytical form

$$\Xi(dt;x,t) = A(x,t)dt + D^{1/2}(x,t)N(t)(dt)^{1/2}. \qquad (2.13)$$

Here $A(x,t)$ and $D(x,t)$ can be any two smooth functions of their arguments, with $D(x,t)$ non-negative. And $N(t)$ is a temporally uncorrelated unit normal random variable; i.e., $N(t) = N(0,1)$, and $N(t)$ is statistically independent of $N(t')$ if $t \neq t'$.

The proof of this theorem is given in the Appendix. In brief, the proof divides the time interval $[t, t+dt)$ into $n \to \infty$ equal subintervals, and then demands that the increments in those subintervals sum to give the increment over the full

interval. The normality of random variable $N(t)$ then arises from the central limit theorem. The peculiar square root of dt, which we earlier found to be untenable in the context of Eqs. (2.3), arises here because in adding statistically independent random variables one must add variances rather than standard deviations.

If we insert into the left side of Eq. (2.13) the definition of Ξ in Eq. (2.12), which definition also allows us to replace x with $X(t)$, we immediately obtain

$$X(t+dt)=X(t)+A(X(t),t)dt$$
$$+D^{1/2}(X(t),t)N(t)(dt)^{1/2}. \qquad (2.14)$$

This equation is called the *(standard form) Langevin equation* for the process X. It is evidently an "update" formula for X, and so is the stochastic generalization of Eq. (2.2).[13] To actually use the Langevin equation as an update formula, one would simply substitute for $N(t)$ a sample value of the random variable $N(0,1)$ and then carry out the indicated arithmetic operations. But, as we shall see later, Eq. (2.14) also has other uses. The function $A(x,t)$ in Eq. (2.14) is called the *drift function* of the process, and the function $D(x,t)$ is called the *diffusion function*.

That the process X defined by the update formula (2.14) is continuous, memoryless, and stochastic is apparent from the form of that equation. It follows from the constructive nature of the proof given in the Appendix that the update formula (2.14) satisfies the self-consistency condition mentioned just after Eq. (2.3d), and moreover that it is not possible to alter the form of Eq. (2.14) without violating that self-consistency condition.

It might seem that we ought to *drop* the dt term from the right side of Eq. (2.14) on the grounds that it will be vanishingly small compared to the $(dt)^{1/2}$ term. But that would be wrong; because, the $(dt)^{1/2}$ term is multiplied by the random variable $N(t)$, which, being about as often negative as positive, greatly diminishes the effect of the $(dt)^{1/2}$ term over a *succession* of many dt increments. The *long-range* effects of the weak-but-steady dt term and the strong-but-erratic $(dt)^{1/2}$ term are comparable if the functions A and D are of comparable magnitudes.

Clearly, the functional forms of A and D completely define the process X. A comparison of the two update formulas (2.14) and (2.2) shows that the processes of ordinary differential calculus comprise the $D\equiv 0$ subclass of continuous Markov processes; thus, all continuous memoryless deterministic processes have update formulas of the form (2.2).

Although the process X defined by the Langevin equation (2.14) is continuous, it is *not* in general differentiable. This can be seen by rearranging Eq. (2.14) to read

$$\frac{X(t+dt)-X(t)}{dt}=A(X(t),t)+\frac{D^{1/2}(X(t),t)N(t)}{(dt)^{1/2}}. \qquad (2.15)$$

Obviously, the $dt\to 0$ limit of this equation will not exist in any conventional sense unless the diffusion function D vanishes identically, in which case X would be deterministic. A truly stochastic continuous Markov process is an example of a function that is everywhere continuous but nowhere differentiable. In spite of this fact, the heuristic appeal of the notion of the derivative is so great that it has become customary to *pretend* that dX/dt exists even when $D(x,t)>0$. This is accomplished through the following ruse: Since theorem (2.9) implies that

$$\frac{N(t)}{(dt)^{1/2}}=(dt)^{-1/2}\mathbf{N}(0,1)=\mathbf{N}(0,(dt)^{-1}),$$

then if we define the *Gaussian white noise* process $\Gamma(t)$ by

$$\Gamma(t)\equiv \lim_{dt\to 0} \mathbf{N}(0,1/dt), \qquad (2.16)$$

we may formally take the $dt\to 0$ limit in Eq. (2.15) and obtain

$$\frac{dX(t)}{dt}=A(X(t),t)+D^{1/2}(X(t),t)\Gamma(t). \qquad (2.17)$$

We call Eq. (2.17) the *(white noise form) Langevin equation*. It is obviously the stochastic generalization of the deterministic differential equation (2.1).

Two "averaged"[14] properties of Gaussian white noise, as defined in Eq. (2.16), are

$$\langle \Gamma(t) \rangle = 0 \qquad (2.18a)$$

and

$$\langle \Gamma(t)\Gamma(t+t') \rangle = \delta(t'). \qquad (2.18b)$$

The validity of Eq. (2.18b) for $t'\neq 0$ follows from the temporally uncorrelated nature of the zero-mean process $\Gamma(t)$. The validity of Eq. (2.18b) for $t'=0$ follows from the variance of $\Gamma(t)$ being $1/dt$ in the limit $dt\to 0$, in which limit we can write heuristically $\delta(0)dt=1$, or $1/dt=\delta(0)$.

We shall more frequently use the following "averaged"[14] properties of $N(t)$, which hold because $N(t)$ is a zero-mean, unit-variance random variable that is statistically independent of $N(t')$ for all $t'\neq t$, and thus also statistically independent of $X(t')$ for all $t'\leq t$:

$$\langle N(t) \rangle = 0, \qquad (2.19a)$$

$$\langle N^2(t) \rangle = 1, \qquad (2.19b)$$

$$\langle X(t')N(t) \rangle = 0 \quad \text{for all } t'\leq t. \qquad (2.19c)$$

It should be obvious from Eq. (2.16) that the white-noise form Langevin equation (2.17) is on much shakier mathematical ground than the standard-form Langevin equation (2.14), which is an ironic reversal of roles with respect to the deterministic formulas (2.1) and (2.2). In fact, there is virtually nothing that can be done with the white-noise Langevin equation that cannot be done with more mathematical rigor using the standard-form Langevin equation; so, it might well be argued that Eq. (2.17) is nothing more than a mnemonic for Eq. (2.14). But old habits die hard, and for many physical applications we find this mnemonic to be useful.

By means of a rather lengthy argument,[15] it can be shown that the density function $P(x;t)$ of the random variable $X(t)$ defined by Eqs. (2.14) and (2.17) satisfies the partial differential equation

$$\frac{\partial}{\partial t}P(x;t)=-\frac{\partial}{\partial x}[A(x,t)P(x;t)]$$
$$+\frac{1}{2}\frac{\partial^2}{\partial x^2}[D(x,t)P(x;t)]. \qquad (2.20)$$

This is called the *(forward) Fokker–Planck equation*. Like the Langevin equation, it serves as a time-evolution equation for the continuous Markov process X with drift function A and diffusion function D. We shall not discuss this equation, since we do not require it for our work here.

Finally we note that, although a continuous Markov process X does not have a mathematically proper derivative, it does have a mathematically proper integral. The time integral Y of X may be defined formally through the differential equation

$$\frac{dY(t)}{dt} = X(t), \qquad (2.21a)$$

or more practically through the equivalent update formula

$$Y(t+dt) = Y(t) + X(t)dt. \qquad (2.21b)$$

Since Y by definition has a mathematically proper derivative, then Y is not itself a continuous Markov process; this can also be seen by noting that the update formula (2.21b) for Y does not have the Langevin form (2.14). (But it turns out that a continuous Markov process X and its integral Y *together* form a *bivariate* continuous Markov process.)

D. The Ornstein–Uhlenbeck process

A continuous Markov process whose drift and diffusion functions have the forms

$$A(x,t) = -\frac{1}{\tau}x \quad \text{and} \quad D(x,t) = c, \qquad (2.22)$$

where τ and c are two positive constants, is called an *Ornstein–Uhlenbeck* (O–U) process with *relaxation time* τ and *diffusion constant* c. The O–U process turns out to be central to the mathematical descriptions of both Brownian motion and Johnson noise. And conveniently, the O–U process happens to be one of the relatively few continuous Markov processes that admits a complete, closed-form analysis. The following detailed review of that analysis may seem at times to be a bit tedious, but the effort spent in assimilating it will be repaid later.

Equations (2.14) and (2.17) evidently give

$$X(t+dt) = X(t) - \frac{1}{\tau}X(t)dt + c^{1/2}N(t)(dt)^{1/2} \qquad (2.23)$$

and

$$\frac{dX(t)}{dt} = -\frac{1}{\tau}X(t) + c^{1/2}\Gamma(t) \qquad (2.24)$$

as the equivalent forms of the Langevin equation for the O–U process. To solve these equations subject to the initial condition $X(t_0) = x_0$, we begin by observing that Eq. (2.23) expresses $X(t+dt)$ as a linear combination of $X(t)$ and $N(t)$. For $t = t_0$ those two random variables are statistically independent and normal [since $X(t_0) = x_0 = N(x_0, 0)$], so by Eqs. (2.9) and (2.11), $X(t_0 + dt)$ must be normal. Then $X(t_0 + 2dt)$, being a linear combination of the two statistically independent normal random variables $X(t_0 + dt)$ and $N(t_0 + dt)$, must also be normal. By induction, we infer that *the O–U process $X(t)$ is normal for all $t > t_0$.*

To find the mean and variance of the normal random variable $X(t)$, we begin by taking an average[14] over Eq. (2.23). This gives, because of Eq. (2.19a),

$$\langle X(t+dt) \rangle = \langle X(t) \rangle - \frac{1}{\tau}\langle X(t) \rangle dt.$$

Transposing the first term on the right, dividing through by dt and then letting $dt \to 0$, we obtain an ordinary differential equation for $\langle X(t) \rangle$ whose solution for the initial condition $\langle X(t_0) \rangle = x_0$ is

$$\langle X(t) \rangle = x_0 e^{-(t-t_0)/\tau} \qquad (t \geq t_0). \qquad (2.25)$$

Next we square Eq. (2.23). This gives, to first order in dt,

$$X^2(t+dt) = X^2(t) - \frac{2}{\tau}X^2(t)dt + 2c^{1/2}X(t)N(t)(dt)^{1/2}$$

$$+ cN^2(t)dt.$$

Averaging this equation using Eqs. (2.19b) and (2.19c) gives

$$\langle X^2(t+dt) \rangle = \langle X^2(t) \rangle - \frac{2}{\tau}\langle X^2(t) \rangle dt + cdt.$$

This equation can be converted into a differential equation for $\langle X^2(t) \rangle$ that can easily be solved subject to the initial condition $\langle X^2(t_0) \rangle = x_0^2$. From that solution and the formula for $\langle X(t) \rangle$ in Eq. (2.25), we calculate $\text{var}\{X(t)\} = \langle X^2(t) \rangle - \langle X(t) \rangle^2$ to be

$$\text{var}\{X(t)\} = \frac{c\tau}{2}(1 - e^{-2(t-t_0)/\tau}) \qquad (t \geq t_0). \qquad (2.26)$$

So, since $X(t)$ is normal, we conclude that the O–U process with relaxation time τ, diffusion constant c and initial condition $X(t_0) = x_0$ is

$$X(t) = N\left(x_0 e^{-(t-t_0)/\tau}, \frac{c\tau}{2}(1 - e^{-2(t-t_0)/\tau}) \right) \qquad (t \geq t_0). \qquad (2.27)$$

Notice that the relaxation time τ characterizes the time scale over which the mean and variance of $X(t)$ "relax" to their respective asymptotic $(t \to \infty)$ values of 0 and $c\tau/2$.

The *auto-covariance* of $X(t)$ is defined to be

$$\text{cov}\{X(t_1), X(t_2)\} \equiv \langle X(t_1)X(t_2) \rangle - \langle X(t_1) \rangle \langle X(t_2) \rangle$$

$$(t_0 \leq t_1 \leq t_2). \qquad (2.28)$$

For the O–U process, the second term on the right can be computed from Eq. (2.25). To compute the first term, we first put $t = t_2$ and $dt = dt_2$ in Eq. (2.23), and then multiply through by $X(t_1)$:

$$X(t_1)X(t_2 + dt_2) = X(t_1)X(t_2) - \frac{1}{\tau}X(t_1)X(t_2)dt_2$$

$$+ c^{1/2}X(t_1)N(t_2)(dt_2)^{1/2}.$$

Averaging this equation using Eq. (2.19c) leads to the differential equation

$$\frac{d}{dt_2}\langle X(t_1)X(t_2) \rangle = -\frac{1}{\tau}\langle X(t_1)X(t_2) \rangle.$$

We can now compute $\langle X(t_1)X(t_2) \rangle$ as the solution to this differential equation subject to the initial condition $\langle X(t_1)X(t_2 = t_1) \rangle = \langle X^2(t_1) \rangle$, which can be computed from Eqs. (2.25) and (2.26). Upon substituting that result into Eq. (2.28), we find after a bit of algebra that

$$\text{cov}\{X(t_1), X(t_2)\} = \frac{c\tau}{2}e^{-(t_2-t_1)/\tau}(1 - e^{-2(t_1-t_0)/\tau})$$

$$(t_0 \leq t_1 \leq t_2). \qquad (2.29)$$

This result shows that if $t_1 - t_0 \gg \tau$, then $X(t_1)$ and $X(t_2)$ will be highly correlated or effectively uncorrelated according to

whether t_2-t_1 is much less than or much greater than τ. So the relaxation time τ may also be regarded as a "decorrelation time" for the O–U process X.

By proceeding similarly with the *pair* of update formulas (2.23) and (2.21b), we can derive explicit formulas for the mean and variance of the *integral* Y of the O–U process X. It is easy to deduce from either of Eqs. (2.21) that

$$d\langle Y(t)\rangle/dt=\langle X(t)\rangle,$$

whence, taking $Y(t_0)=0$,

$$\langle Y(t)\rangle=\int_{t_0}^{t}\langle X(t')\rangle dt'.$$

But computing var$\{Y(t)\}$ is rather more involved: By squaring Eq. (2.21b), retaining only terms up to first order in dt, and then averaging, we find that

$$\langle Y^2(t)\rangle=2\int_{t_0}^{t}\langle Y(t')X(t')\rangle dt'.$$

To compute the integrand here, we first multiply Eqs. (2.21b) and (2.23) together and then average the result. Again retaining only terms up to first order in dt, and using the fact that $N(t)$ is statistically independent of $Y(t)$, we find that

$$\frac{d}{dt}\langle Y(t)X(t)\rangle=-\frac{1}{\tau}\langle Y(t)X(t)\rangle+\langle X^2(t)\rangle.$$

Since $\langle X^2(t)\rangle$ is known from Eqs. (2.25) and (2.26), we can solve this differential equation explicitly for the initial condition $\langle Y(t_0)X(t_0)\rangle=0$. Substituting that solution into the preceding equation and then integrating will finally give an explicit formula for $\langle Y^2(t)\rangle$. The result of all these algebraically straightforward but somewhat tedious calculations is

$$\langle Y(t)\rangle=x_0\tau(1-e^{-(t-t_0)/\tau}) \quad (t\geqslant t_0), \qquad (2.30)$$

$$\text{var}\{Y(t)\}=c\tau^2\left[(t-t_0)-2\tau(1-e^{-(t-t_0)/\tau})\right.$$
$$\left.+\frac{\tau}{2}(1-e^{-2(t-t_0)/\tau})\right] \quad (t\geqslant t_0). \qquad (2.31)$$

The O–U process $X(t)$ is said to be a *stable* process because, according to Eq. (2.27),

$$X^*(t)\equiv\lim_{t_0\to-\infty}X(t)=N(0,c\tau/2) \qquad (2.32)$$

exists and is independent of both t and x_0. But notice that what ultimately stops changing with time here is *not* the *value* of the process, but only our *statistical predictions* about that value. Taking the $t_0\to-\infty$ limit of Eqs. (2.28) and (2.29), we get for the auto-covariance of $X^*(t)$,

$$\text{cov}\{X^*(t_1),X^*(t_2)\}=\langle X^*(t_1)X^*(t_2)\rangle$$
$$=\frac{c\tau}{2}e^{-(t_2-t_1)/\tau} \quad (t_1\leqslant t_2). \qquad (2.33)$$

Equations (2.30) and (2.31) show that the integral $Y(t)$ of an O–U process satisfies

$$\lim_{t-t_0\to\infty}\langle Y(t)\rangle=x_0\tau, \qquad \lim_{t-t_0\to\infty}\text{var}\{Y(t)\}=c\tau^2(t-t_0).$$
$$(2.34)$$

The asymptotic t dependency of var$\{Y(t)\}$ shows that $Y(t)$ here is *not* a stable process.

Finally, we want to establish two important theorems for O–U processes. Both of these theorems involve also *another* species of continuous Markov process, namely the one for which

$$A(x,t)=0 \quad \text{and} \quad D(x,t)=c, \qquad (2.35)$$

where c is a positive constant. Such a process is called a *driftless Wiener process* with diffusion constant c. For this process, the Langevin equations (2.14) and (2.17) evidently read

$$X(t+dt)=X(t)+c^{1/2}N(t)(dt)^{1/2}, \qquad (2.36a)$$

and

$$\frac{dX(t)}{dt}=c^{1/2}\Gamma(t). \qquad (2.36b)$$

By using the same kind of reasoning that led from the O–U Langevin equation (2.23) to the result (2.27), one can deduce from Eq. (2.36a) that the driftless Wiener process with diffusion constant c and initial condition x_0 is the normal random variable[16]

$$X(t)=N(x_0,c(t-t_0)) \quad (t_0\leqslant t). \qquad (2.37)$$

Now for the two promised theorems about O–U processes. First we have the rather obvious

Infinite-tau Limit Theorem. The O–U process $X(t)$ with relaxation time τ and diffusion constant c becomes, in the limit $\tau\to\infty$, the driftless Wiener process with diffusion constant c.

Proof: Observe that, in the limit $\tau\to\infty$, the O–U process characterizing functions (2.22) become identical to the driftless Wiener process characterizing functions (2.35). Or, observe that the $\tau\to\infty$ limit of the O–U formula (2.27) gives the driftless Wiener formula (2.37).

Somewhat more intriguing is the

Zero-tau Limit Theorem. The O–U process $X(t)$ with relaxation time τ and diffusion constant c becomes, in the limit

$$\tau\to0 \quad \text{with} \quad \tau c^{1/2}\equiv\epsilon(\text{constant}), \qquad (2.38)$$

the process $\epsilon\Gamma(t)$, where $\Gamma(t)$ is the Gaussian white noise process. And the time integral $Y(t)$ of $X(t)$ becomes in this limit the driftless Wiener process with diffusion constant ϵ^2.

Proof: A simple algebraic rearrangement of the O–U Langevin equation (2.24) gives

$$\tau\frac{dX(t)}{dt}+X(t)=\tau c^{1/2}\Gamma(t).$$

The limit (2.38) of this equation evidently gives $X(t)=\epsilon\Gamma(t)$. The definition (2.21a) of $Y(t)$ then reads $dY(t)/dt=\epsilon\Gamma(t)$, and a comparison of this with Eq. (2.36b) shows $Y(t)$ to be the driftless Wiener process with diffusion constant ϵ^2. (Section II E contains another proof of this theorem.)

E. Spectral density functions

One advantage that a Markovian description of Brownian motion and Johnson noise offers over a standard statistical thermodynamics description is some insight into the frequency spectrums of the fluctuations. We now review the mathematical basis of spectral analysis.

Let $X_s(t)$ be a zero-mean stochastic process, *not necessarily Markovian*, that is *stationary* in the sense that all of its

moments are independent of t. We call the constant second moment $\langle X_s^2(t) \rangle \equiv \sigma^2$ the *intensity* of $X_s(t)$. As is discussed in many textbooks,[7] the auto-covariance

$$\langle X_s(t)X_s(t+t') \rangle \equiv C_X(t') \quad (t' \geqslant 0), \qquad (2.39)$$

will, as a consequence of the stationarity of $X_s(t)$, be independent of t, and will have a positive-frequency Fourier amplitude $S_X(\nu)$ which is such that[17]

$$C_X(t') = \int_0^\infty S_X(\nu) \cos(2\pi\nu t')d\nu \quad (t' \geqslant 0), \qquad (2.40)$$

and

$$S_X(\nu) = 4 \int_0^\infty C_X(t') \cos(2\pi\nu t')dt' \quad (\nu \geqslant 0). \qquad (2.41)$$

Now, by setting $t'=0$ in both of Eqs. (2.39) and (2.40), we deduce that

$$\langle X_s^2(t) \rangle = \int_0^\infty S_X(\nu)d\nu. \qquad (2.42)$$

This is the *Wiener–Khintchine theorem*: $S_X(\nu)d\nu$ gives the portion of the intensity $\langle X_s^2(t) \rangle = \sigma^2$ of $X_s(t)$ that is due to (positive) frequencies between ν and $\nu + d\nu$. For that reason, the function $S_X(\nu)$ in Eq. (2.41) is called the *spectral density function* of the stationary process $X_s(t)$. But it is important to note that $S_X(\nu)$ gives the frequency spectrum of $\langle X_s^2(t) \rangle$, not of $X_s(t)$ itself.

Two examples will be relevant to our work here.

(a) The process $\alpha\Gamma(t)$, where α is any constant and $\Gamma(t)$ is the Gaussian white noise process, is a zero-mean, infinite-intensity, stationary process, by virtue of definition (2.16) and theorem (2.9). From Eqs. (2.39) and (2.18b), we compute the auto-covariance of $\alpha\Gamma(t)$ as

$$C_{\alpha\Gamma}(t') = \langle \alpha\Gamma(t)\alpha\Gamma(t+t') \rangle$$

$$= \alpha^2 \langle \Gamma(t)\Gamma(t+t') \rangle = \alpha^2 \delta(t').$$

Substituting this into Eq. (2.41) and then integrating over t' from 0 (not $-\infty$) to ∞, we get

$$S_{\alpha\Gamma}(\nu) = 2\alpha^2 \quad (\nu \geqslant 0). \qquad (2.43)$$

The fact that the spectral density function of $\alpha\Gamma(t)$ is a constant for *all* frequencies $\nu \geqslant 0$ is consistent with the fact that $\alpha\Gamma(t)$ is an infinite-intensity process. This frequency-independent intensity spectrum is of course the reason why the process $\Gamma(t)$ is referred to as "white" noise.

(b) The fully relaxed O–U process $X^*(t)$ in Eq. (2.32) is evidently a zero-mean stationary stochastic process with intensity $c\tau/2$. Its auto-covariance is, according to Eqs. (2.39) and (2.33),

$$C_X(t') = (c\tau/2)e^{-t'/\tau}.$$

Substituting this into Eq. (2.41) and then integrating over t', we obtain for the spectral density function of the fully relaxed O–U process with relaxation time τ and diffusion constant c,

$$S_X(\nu) = \frac{2c\tau^2}{1+(2\pi\tau\nu)^2} \quad (\nu \geqslant 0). \qquad (2.44)$$

It is easy to verify that the integral of this function over all $\nu > 0$ indeed gives the intensity $c\tau/2$. This function is graphed for $\tau = c = 1$ in Fig. 1. The linear graph in Fig. 1(a) displays the "density" property described by Eq. (2.42), but the log–
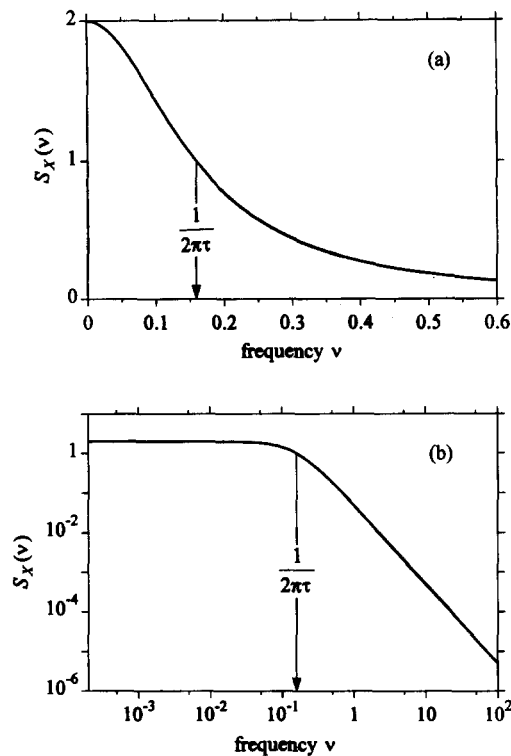

Fig. 1. Plots of the spectral density function (2.44) of the fully relaxed O–U process $X^*(t)$ with relaxation time $\tau=1$ and diffusion constant $c=1$. In the *linear* plot (a), the area under the curve inside any frequency band $[\nu_1,\nu_2]$ is numerically equal to the portion of the intensity $\langle X^{*2}(t) \rangle = c\tau/2$ due to frequencies within that band. The *log–log* plot (b) shows a "knee" at frequency $1/2\pi\tau$, below which the curve tends to slope 0 and above which it tends to slope -2.

log graph in Fig. 1(b) is more often referenced. It shows a leg-like curve with a "knee" at frequency $\nu = (2\pi\tau)^{-1}$, below which the curve has slope 0 and above which it has slope -2. The frequency region below the knee is called the "white noise" region, and the frequency region above the knee is called the "$1/f^2$ noise" region.

In the limit (2.38), where $\tau \to 0$ with $\tau c^{1/2} = \epsilon$, in which limit the "fully relaxed" condition $(t-t_0) \gg \tau$ will be satisfied for *any* $t > t_0$, Eq. (2.44) evidently becomes $S_X(\nu) = 2\epsilon^2$. This, by Eq. (2.43), implies that in this limit the O–U process $X(t)$ becomes $\epsilon\Gamma(t)$; thus we have another proof of the zero-tau limit theorem of Sec. II D.

Since the intensity of the *driftless Wiener* process (2.37) is proportional to $(t-t_0)$, then that process is *not* asymptotically stationary; so we should *not* expect the driftless Wiener process to have a spectral density function. But Eq. (2.44) seems to belie this, implying as it does that the spectral density function of a fully relaxed O–U process approaches in the limit $\tau \to \infty$ $(c/2\pi^2)\nu^{-2}$. That would seem to imply, by the infinite-tau limit theorem in Sec. II D, that a driftless Wiener process should have spectral density function $(c/2\pi^2)\nu^{-2}$, i.e., that it should be a pure $1/f^2$ noise process. But this assertion is really tenable only as a high-frequency approximation to Eq. (2.44); because, an O–U process cannot be considered to be "fully relaxed" unless $(t-t_0) \gg \tau$, and that condition obviously becomes problematic in the required limit $\tau \to \infty$.

Finally, we note that the definition (2.39) implies that $C_{\alpha X}(t') = \alpha^2 C_X(t')$. This allows us to deduce from Eq. (2.41) the often used result,

$$S_{\alpha X}(\nu) = \alpha^2 S_X(\nu). \qquad (2.45)$$

F. Numerically simulating an Ornstein–Uhlenbeck process

The key to making a numerical simulation of an O–U process X is to find a way of calculating, from the value of X at any time t, its value at a slightly later time $t + \Delta t$. An obvious but *approximate* way of doing this is to simply "finitize" dt in the standard-form O–U Langevin equation (2.23). That gives

$$X(t + \Delta t) \doteq X(t) - \tau^{-1} X(t)\Delta t + c^{1/2}n(\Delta t)^{1/2}. \qquad (2.46)$$

Here, n denotes a *sample value* of the random variable $N = N(0,1)$, and the dot over the equal sign denotes an *approximate* equality that becomes exact in the limit $\Delta t \to 0$.

For many continuous Markov processes, a finitized Langevin update formula such as Eq. (2.46) is about the best that one can do, and one must consequently worry about just how small Δt needs to be in order to get results that are sufficiently accurate. But for O–U processes there is an update formula that is *exact* for *any* positive value of Δt. To derive that formula, we begin by replacing, in Eq. (2.27), t by $t + \Delta t$, t_0 by t, and thus also x_0 by $X(t)$; that gives

$$X(t + \Delta t) = N\left(X(t)e^{-(1/\tau)\Delta t}, \frac{c\tau}{2}(1 - e^{-(2/\tau)\Delta t}) \right).$$

Now invoking theorem (2.10), and letting n denote, as in Eq. (2.46), a sample value of $N \equiv N(0,1)$, we infer from this last equation that

$$X(t + \Delta t) = X(t)e^{-(1/\tau)\Delta t} + \left[\frac{c\tau}{2}(1 - e^{-(2/\tau)\Delta t}) \right]^{1/2} n. \qquad (2.47)$$

This is the *exact O–U update formula*. It is not difficult to show that if $\Delta t \ll \tau$, then Eq. (2.47) reduces, to first order in Δt, to the approximate update formula (2.46).

Both update formulas (2.46) and (2.47) evidently require the ability to generate sample values n of the unit normal random variable $N(0,1)$. A simple exact method of doing this, albeit in a pairwise fashion, is as follows:[18] First generate two statistically independent sample values r_1 and r_2 of the unit-interval *uniform* random variable; then calculate the two values

$$s = [2\ln(1/r_1)]^{1/2} \quad \text{and} \quad \theta = 2\pi r_2; \qquad (2.48a)$$

and finally, take as two *statistically independent* samples of $N(0,1)$ the values

$$n_1 = s\cos\theta \quad \text{and} \quad n_2 = s\sin\theta. \qquad (2.48b)$$

Given some value x_0 of X at some initial time t_0, a simulation of the process is realized by iteratively applying either the approximate updating formula (2.46) or the exact updating formula (2.47) to compute values of X at times $t_0 + \Delta t$, $t_0 + 2\Delta t$, $t_0 + 3\Delta t$, etc. If a fixed value for Δt is used throughout the simulation run, then the coefficients of $X(t)$ and n in both updating formulas will not change with time, so there will be no advantage in using the approximate updating formula instead of the exact updating formula. The exact updating formula (2.47) and the exact procedure (2.48)

for generating values of n give us the following *exact simulation algorithm* for an O–U process with relaxation time τ and diffusion constant c:

Step 1. Specify values for τ, c, x_0, Δt, and some stopping time t_{stop}. Also specify a "starting seed" for the unit-interval uniform random number generator used to compute values for r_1 and r_2 in formulas (2.48a).

Step 2. Compute the constant coefficients of $X(t)$ and n in the updating formula (2.47):

$$\mu \equiv e^{-(1/\tau)\Delta t}, \qquad (2.49a)$$

$$\sigma \equiv \left[\frac{c\tau}{2}(1 - e^{-(2/\tau)\Delta t}) \right]^{1/2}. \qquad (2.49b)$$

Step 3. Set $X = x_0$ and $t = 0$.

Step 4. Write out for plotting the point (t, X).

Step 5. Replace t by $t + \Delta t$. Terminate the simulation if the new t exceeds t_{stop}.

Step 6. Using the algorithm of Eqs. (2.48), generate a random value for n.

Step 7. In accordance with Eqs. (2.47) and (2.49), replace X by $\mu X + \sigma n$.

Step 8. Return to Step 4.

In Fig. 2 we show the results of three numerical simulations of the O–U process, all computed using the foregoing exact algorithm. For the simulations of Fig. 2(a), we took $\tau = c = 1$, $x_0 = 0$ and $\Delta t = 10^{-3}$. The jagged curve is the simulated $X(t)$ trajectory—its theoretically predicted spectral density function is plotted in Fig. 1—and the dotted curves show the one-standard deviation envelope $\langle X(t)\rangle \pm \text{sdev}\{X(t)\}$ predicted by Eqs. (2.25) and (2.26). Notice that we have *not* connected successive trajectory points with any kind of smooth curve, because that would erroneously suggest that the trajectory is differentiable. That no smoothing of the trajectory occurs on a finer time scale is illustrated by the simulation of Fig. 2(b), for which the time step Δt has been reduced to 10^{-4}. In Fig. 2(c) we have reduced τ to 10^{-3} and increased c to 10^6, thereby leaving the combination $\epsilon = \tau c^{1/2}$ with the same value 1 as it had in Fig. 2(a), and we have used the same time step size $\Delta t = 10^{-3}$ as in Fig. 2(a). As predicted by the zero-tau limit theorem (2.38), the process $X(t)$ has begun to take on the appearance of the Gaussian white noise process $\Gamma(t)$, at least when viewed on the ordinate and abscissa scales of Fig. 2(a).

Finally, we note that it is easy to incorporate into the above algorithm an *approximate* simulation of the *integral Y* of X. This is done simply by finitizing the dt update formula (2.21b). We would of course have to initialize $Y = 0$ in step 3. And in the main simulation loop we would want to plot out the point (t, Y) in step 4, and replace Y by $Y + X\Delta t$ (*before* updating X) in step 7. The price to be paid for using this simple updating formula for Y is that Δt now *must* be taken "small," i.e., much less than τ.[19] Figure 3 shows such a simulation of Y generated in conjunction with the simulation of X in Fig. 2(a), for which $\Delta t = 0.001 \ll \tau$. Notice that, unlike the X trajectory in Fig. 2(a), this Y trajectory *is* smooth (i.e., differentiable). The dotted curves in Fig. 3 show the one-standard deviation envelope $\langle Y(t)\rangle \pm \text{sdev}\{Y(t)\}$ as predicted by Eqs. (2.30) and (2.31).

III. BROWNIAN MOTION

In this section we shall see how continuous Markov process theory, as set forth in Sec. II, can be coupled with two
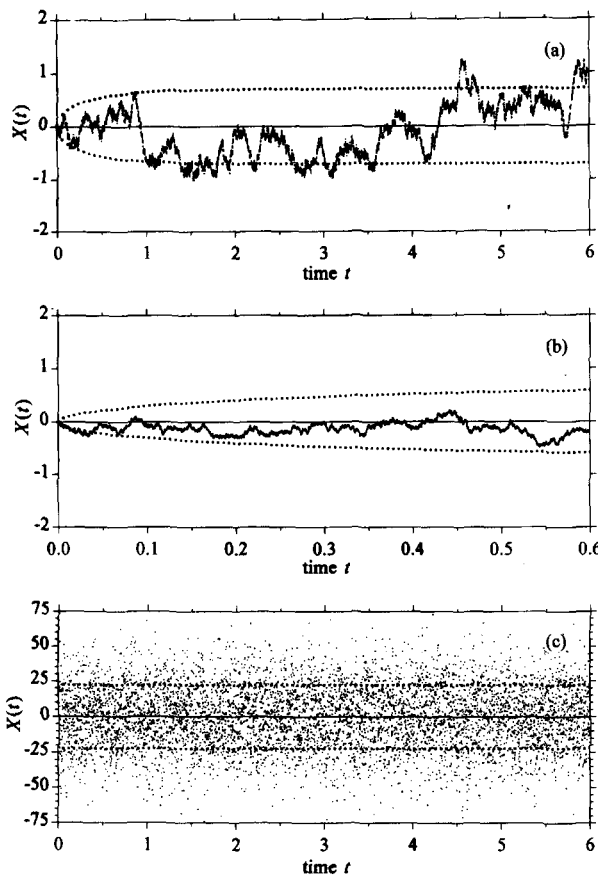
Fig. 2. Results of three numerical simulations of the O–U process $X(t)$, all made using the *exact* simulation procedure described in Sec. II F. The dotted curves show the theoretically predicted one-standard deviation envelope $\langle X(t) \rangle \pm \text{sdev}\{X(t)\}$ as calculated from Eqs. (2.25) and (2.26). For (a) we have taken $\tau = c = 1$, $x_0 = 0$ and $\Delta t = 10^{-3}$. For (b) we have reduced Δt to 10^{-4} in order to demonstrate the absence of any smoothing of the $X(t)$ trajectory on a finer time scale. For (c) we have taken $\tau = 10^{-3}$ and $c = 10^6$, so that $\tau c^{1/2}$ has the same value as in (a), in order to illustrate the approach of $X(t)$ to Gaussian white noise in the limit (2.38).

simple physical assumptions to give a very concise yet remarkably complete description of the phenomenon of Brownian motion. But first let us recall how this phenomenon was originally analyzed by Albert Einstein in his landmark paper of 1905.[1,20]

A. Traditional approaches

Einstein began by supposing time to increase in discrete steps of size δt, which, though infinitesimally small from a macroscopic point of view, was microscopically large enough that many fluid molecules will typically collide with the Brownian particle in a time δt. The net changes in the Brownian particle's *position* during successive time steps δt should therefore be practically statistically independent. Einstein adroitly argued that, under these conditions, the function $n(x,t)$, defined so that $n(x,t)dx$ gives the average number of Brownian particles with x coordinates between x and $x + dx$ at time t, should satisfy the equation

$$\frac{\partial n(x,t)}{\partial t} = D \frac{\partial^2 n(x,t)}{\partial x^2}, \tag{3.1}$$

wherein the constant D was defined by Einstein to be the mean-square displacement of the particle in time δt divided by $2\,\delta t$.

Equation (3.1) is called the *diffusion equation*. It is usually deduced in elementary kinetic theory courses by first defining D through the purely phenomenological formula

$$j_x(x,t) = -D \frac{\partial n(x,t)}{\partial x}$$

for the "average flux" of Brownian particles in the x direction, and then simply substituting this relation into the particle conservation equation,

$$\frac{\partial n(x,t)}{\partial t} = -\frac{\partial j_x(x,t)}{\partial x}.$$

Einstein solved his Eq. (3.1) for the initial condition $n(x,t_0) \propto \delta(x)$, and thereby deduced that the mean-square displacement of the Brownian particle evolves with time according to

$$\langle X^2(t) \rangle = 2D(t - t_0) \quad (t - t_0 \text{ ``large''}). \tag{3.2}$$

The "large-t" restriction on this formula, although often ignored, is important; it is a consequence not of Eq. (3.1), but rather of the "coarse-grained time" assumption Einstein used to derive that equation. Historically, the experimental confirmation of formula (3.2) provided a clinching piece of evidence for the kinetic molecular hypothesis. But *granting* that hypothesis, we now take formula (3.2) to be the *phenomenological definition* of the *diffusion coefficient* D.

Three years after Einstein's work, Paul Langevin presented a quite different analysis of Brownian motion,[3] which however led to the same key result (3.2). Whereas Einstein's analysis had focused directly on the *position* of the Brownian particle, Langevin's analysis began by considering the particle's *velocity*, a variable that Einstein had scrupulously avoided. The exposition of Brownian motion given below is essentially a modernized version of Langevin's analysis, and it will allow us to see, among other things, how the two very different approaches of Einstein and Langevin are logically connected.

B. Langevin's hypothesis

Our analysis proceeds from the *hypothesis* that the net force exerted on the Brownian particle in the x direction at

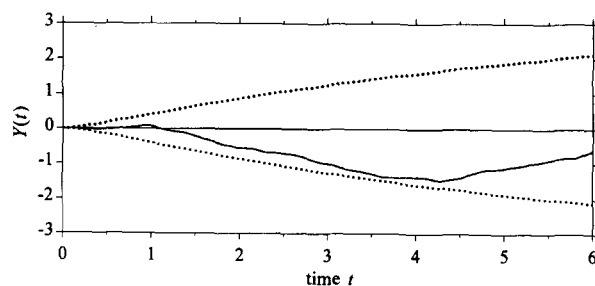Fig. 3. Results of an *approximate* simulation of the integral $Y(t)$ of the O–U process $X(t)$, done in conjunction with the simulation in Fig. 2(a). Note that this trajectory, unlike its companion $X(t)$ trajectory in Fig. 2(a), is "smooth." The dotted curves show the theoretically predicted one-standard deviation envelope $\langle Y(t) \rangle \pm \text{sdev}\{Y(t)\}$ as calculated from Eqs. (2.30) and (2.31).

time t by the surrounding fluid molecules can be effectively resolved into two components: (i) a *dissipative drag* force $-\gamma V(t)$, where $V(t)$ is the x component of the velocity of the particle at time t and $\gamma > 0$ is some unspecified drag coefficient; and (ii) a zero-mean, temporally uncorrelated *randomly fluctuating* force $F(t)$, which is assumed to be statistically independent of $V(t')$ for all $t' \leq t$. Letting m denote the Brownian particle's mass, we therefore have by Newton's second law,

$$m\,\frac{dV(t)}{dt} = -\gamma V(t) + F(t). \tag{3.3}$$

As we shall see shortly, the drag force $-\gamma V(t)$ and the fluctuating force $F(t)$ are *not* independent of each other. This is a consequence of the fact that the molecular collisions that give rise to those two forces simply cannot be separated into one kind of collision that gives rise *only* to a drag effect and another kind that gives rise *only* to a fluctuating effect. The hypothesis that the molecular forces on the Brownian particle can nevertheless be effectively resolved into two such components is essentially due to Langevin. An elementary argument in support of Langevin's hypothesis has recently been given in this journal,[21] but of course the ultimate justification for this hypothesis will have to be found in the accuracy of its predictions.

Writing Eq. (3.3) as

$$\frac{dV(t)}{dt} = -\frac{1}{m/\gamma}\,V(t) + m^{-1}F(t), \tag{3.4}$$

we immediately perceive a close resemblance to the white-noise form Langevin equation (2.24) of an Ornstein–Uhlenbeck (O–U) process; indeed, if we define the positive constant τ by

$$m/\gamma \equiv \tau, \tag{3.5a}$$

and then require the fluctuating force component $F(t)$ to be such that

$$F(t) = mc^{1/2}\Gamma(t), \tag{3.5b}$$

where c is some positive constant and $\Gamma(t)$ is the Gaussian white noise process (2.16), then Eq. (3.4) would read

$$\frac{dV(t)}{dt} = -\frac{1}{\tau}\,V(t) + c^{1/2}\Gamma(t). \tag{3.6}$$

It would then follow that $V(t)$ is an O–U process with relaxation time τ and diffusion constant c. [Note that the diffusion *constant c* of the O–U process $V(t)$ is *not* the same as the diffusion *coefficient* D of the Brownian particle; this will become obvious in Eq. (3.13b).]

We now observe that requirement (3.5b) is *not* an additional assumption! Langevin's hypothesis that $F(t)$ is a zero-mean random function that is independent of $V(t')$ for all $t' \leq t$ renders Eq. (3.4) a *manifestly memoryless* time-evolution equation for $V(t)$. Our theorem at Eq. (2.13) then implies that the only way for such an equation to be *mathematically self-consistent* is for $F(t)$ to be proportional to an independent Gaussian white noise process. If $F(t)$ in Eq. (3.4) were *not* proportional to Gaussian white noise, then the right side of that equation should be exhibiting some overt dependence on $V(t')$ for $t' < t$.

C. Fixing τ and c

To express the two constants τ and c in Eqs. (3.6) wholly in terms of physically measurable parameters, we need only note two facts: First, since the Brownian particle is surrounded by a fluid that is assumed to be in thermal equilibrium at absolute temperature T, then according to classical statistical thermodynamics the velocity of the particle must eventually be distributed in a Maxwell–Boltzmann fashion; thus, the particle's x velocity component $V(t)$ must asymptotically approach a *normal* random variable with mean zero and variance kT/m, where k is Boltzmann's constant [see Eq. (2.8) with $y = v$, $m = 0$ and $\sigma^2 = kT/m$]:

$$V(t \to \infty) = \mathbf{N}(0, kT/m). \tag{3.7}$$

Second, the x displacement of the particle in time $(t - t_0)$, namely

$$X(t) \equiv \int_{t_0}^{t} V(t')\,dt', \tag{3.8}$$

must exhibit the experimentally observed diffusive behavior described by Eq. (3.2); i.e.,

$$\langle X^2(t \to \infty)\rangle = 2D(t - t_0). \tag{3.9}$$

But since $V(t)$ is an O–U process with relaxation time τ and diffusion constant c, then it follows from the O–U property (2.32) that the asymptotic form of $V(t)$ will be

$$V(t \to \infty) = \mathbf{N}(0, c\tau/2). \tag{3.10}$$

Furthermore, it follows from the asymptotic formulas (2.34) for the integral of an O–U process that the mean of the square of the time-integral $X(t)$ of $V(t)$ will satisfy

$$\langle X^2(t \to \infty)\rangle = c\tau^2(t - t_0). \tag{3.11}$$

Clearly, the two O–U properties (3.10) and (3.11) will agree with the respective physical requirements (3.7) and (3.9) if and only if

$$c\tau/2 = kT/m \tag{3.12a}$$

and

$$c\tau^2 = 2D. \tag{3.12b}$$

Upon solving Eqs. (3.12) simultaneously for τ and c, we obtain

$$\tau = \frac{Dm}{kT} \tag{3.13a}$$

and

$$c = \frac{2}{D}\left(\frac{kT}{m}\right)^2 \tag{3.13b}$$

for the relaxation time and diffusion constant of the O–U process $V(t)$.

We may now infer from the solution (2.27) of the O–U Langevin equation, after taking note of Eq. (3.12a), that, given some initial condition $V(t_0)=v_0$, $V(t)$ for any $t>t_0$ will be the normal random variable

$$V(t)=\mathbf{N}\left(v_0 e^{-(t-t_0)/\tau},\ \frac{kT}{m}(1-e^{-2(t-t_0)/\tau})\right) \quad (t\geq t_0).$$
(3.14)

Exact expressions for the mean and variance of the integral $X(t)$ of $V(t)$ follow from the integral O–U formulas (2.30) and (2.31), after taking note of Eq. (3.12b):

$$\langle X(t)\rangle=v_0\tau(1-e^{-(t-t_0)/\tau}),$$
(3.15a)

$$\mathrm{var}\{X(t)\}=2D\left[(t-t_0)-2\tau(1-e^{-(t-t_0)/\tau})\right.$$
$$\left.+\frac{\tau}{2}(1-e^{-2(t-t_0)/\tau})\right].$$
(3.15b)

Numerical simulations of $V(t)$ and $X(t)$, leading to graphs such as those in Fig. 2 for $V(t)$, and in Fig. 3 for $X(t)$, can be constructed by employing the algorithm described in Sec. II F.

D. Implications of the theory

Now let us examine the implications of the foregoing Markovian picture of Brownian motion. First, by substituting Eq. (3.13a) into Eq. (3.5a) and then solving for γ, we obtain

$$\gamma=kT/D.$$
(3.16)

This experimentally verified formula, originally discovered by Einstein, relates the drag coefficient γ to the diffusion coefficient D.

A second implication emerges upon substituting Eq. (3.13b) into Eq. (3.5b), and then eliminating D in favor of γ using Eq. (3.16):

$$F(t)=(2kT\gamma)^{1/2}\Gamma(t).$$
(3.17)

This is called the *fluctuation–dissipation formula*. It quantifies the earlier mentioned intimate connection between the drag coefficient γ and the fluctuating force $F(t)$: The fluctuating force is an increasing function of γ, and vanishes identically if and only if γ vanishes. An alternative expression of this relationship can be obtained by noting that Eq. (3.17) implies that

$$\langle F(t)F(t+t')\rangle=(2kT\gamma)\langle\Gamma(t)\Gamma(t+t')\rangle=2kT\gamma\delta(t'),$$

where the last step invokes Eq. (2.18b)]. Integrating over all t' and then solving for γ, we get

$$\gamma=\frac{1}{2kT}\int_{-\infty}^{\infty}\langle F(t)F(t+t')\rangle dt',$$
(3.18)

a formula that is often encountered in the literature.[7] Equation (3.18) in turn leads to Eq. (3.17) under our present hypothesis that $F(t)$ is some constant times Gaussian white noise.

Another important integral relation, which sometimes gets confused with Eq. (3.18), is provided by formula (2.33) for the asymptotic auto-covariance of an O–U process. That formula implies for our O–U process $V(t)$ that

$$\lim_{t\to\infty}\langle V(t)V(t+t')\rangle=\frac{c\tau}{2}e^{-t'/\tau}=\frac{D}{\tau}e^{-t'/\tau} \quad (t'\geq 0),$$

where the last step follows from Eq. (3.12b). Integrating this over all $t'>0$ evidently gives

$$D=\int_0^\infty\left[\lim_{t\to\infty}\langle V(t)V(t+t')\rangle\right]dt'.$$
(3.19)

This is called the *velocity auto-covariance formula* for the diffusion coefficient. The quantity in square brackets is usually referred to as the "equilibrium auto-covariance" of $V(t)$. Some physicists might prefer to take Eq. (3.19), instead of Eq. (3.9), as the definition of D.

Finally, the asymptotic or equilibrium mean power dissipated by the Brownian particle via the three orthogonal components of the drag force is

$$P_{\mathrm{eq}}=3\langle[\gamma V(\infty)]V(\infty)\rangle=3\gamma\langle V^2(\infty)\rangle=\frac{3\gamma kT}{m},$$
(3.20)

where the last step follows from the equipartition theorem of classical statistical thermodynamics, or equivalently from relation (3.7). Statistical thermodynamics tells us nothing about the *frequency spectrum* of this dissipated power, but our Markovian formulation does: Recalling the formula (2.44) for the spectral density function of an asymptotic O–U process, and remembering that the spectral density function of any stationary process describes the spectrum of the *mean of the square* of that process, we conclude from Eq. (3.20) that the frequency spectrum of the equilibrium power dissipated by the Brownian particle is

$$P_{\mathrm{eq}}(\nu)=3\gamma S_V(\nu)=3\gamma\left(\frac{2c\tau^2}{1+(2\pi\tau\nu)^2}\right),$$

$$P_{\mathrm{eq}}(\nu)=\frac{(12)kT}{1+(2\pi m\nu/\gamma)^2} \quad (\nu\geq 0),$$
(3.21)

where the last step makes use of Eqs. (3.12b), (3.16), and (3.5a). It is easy to verify that the integral of Eq. (3.21) over all $\nu>0$ gives $3\gamma kT/m$, in agreement with Eq. (3.20). As we saw in Fig. 1(b), Eq. (3.21) implies that the average equilibrium dissipated power is "white" for frequencies well below $\gamma/2\pi m$, and "$1/f^2$" for frequencies well above $\gamma/2\pi m$.

E. Connection with Einstein's analysis

To understand the mathematical connection between the foregoing Langevin treatment of Brownian motion and Einstein's pioneering treatment of 1905, we begin by observing from Eq. (3.14) that τ characterizes the time scale of the motion of the O–U process $V(t)$. But the numerical value of τ, as computed from the formula (3.13a), will usually be very small from a macroscopic viewpoint. In that case, we can invoke the zero-tau limit theorem (2.38) for O–U processes to the following effect: Since $\tau\approx 0$, while the product

$$\tau c^{1/2}=\frac{Dm}{kT}\left[\frac{2}{D}\left(\frac{kT}{m}\right)^2\right]^{1/2}=(2D)^{1/2}$$
(3.22)

is nonzero, we can *approximate* the O–U process $V(t)$ as $\tau c^{1/2}$ times Gaussian white noise:

$$V(t)\approx(2D)^{1/2}\Gamma(t).$$
(3.23)

Furthermore, we can approximate the corresponding time integral of $V(t)$ by a driftless Wiener process [see Eqs. (2.35)–

(2.37)] whose diffusion constant is the square of $\tau c^{1/2}$:

$$X(t) \approx N(0, 2D(t - t_0)).\qquad(3.24)$$

Equation (3.24) is precisely Einstein's 1905 result: It implies that the mean-square position of the Brownian particle obeys Einstein's formula (3.2); moreover, the probability density function for the normal random variable (3.24) exactly satisfies the diffusion equation (3.1).

So we see that Einstein's classic analysis of Brownian motion, with its "coarse-grained time" assumption, is equivalent to a $\tau \approx 0$ *approximation* of our Langevin analysis. The precise connection between the two analyses is provided by the zero-tau limit theorem (2.38). Einstein's approach is usually quite adequate for the purpose of describing the *position* of a Brownian particle: His key result (3.2) coincides with the $(t - t_0) \gg \tau$ approximation of the exact formula (3.15b), and there is usually no need to take note of the concomitant $(t - t_0) \ll \tau$ approximation of that formula, which approximation can easily be shown to be

$$\langle X^2(t) \rangle \approx \frac{2D}{3\tau^2}(t - t_0)^3 \quad (t - t_0 \ll \tau).\qquad(3.25)$$

But, as regards the *velocity* of a Brownian particle, the approximate formula (3.23) implied by the Einstein approach is rather crude. And in cases where the temporal characteristics of the velocity happen to be important (as they might be for instance in estimating the magnetic field produced by a hydrated ion in an aqueous solution), it would be advisable to use the exact formula (3.14) for $V(t)$ instead of the approximate $\tau \approx 0$ formula (3.23).

IV. JOHNSON NOISE

The first experimental study of thermally generated electrical noise in a conductor was reported by J. B. Johnson[4] in a 1928 issue of the *Physical Review*. In the very next article of that issue, H. Nyquist[5] gave a theoretical explanation of Johnson's results. One might be tempted to infer from this timing of events that the theory underlying Johnson noise is "immediately obvious," but that is not so. We shall consider here the problem of describing Johnson noise in a rigid wire loop of self-inductance L and resistance R at absolute temperature T. Although this problem can be *formally* disposed of simply by replacing, in the Brownian motion results of Sec. III, the Brownian particle's velocity and mass with the loop's current and self-inductance, that approach tends to obscure the unique physical characteristics of the electrical problem. For example, whereas in the Brownian motion problem the key phenomenological parameter is the particle diffusion coefficient D, the natural phenomenological parameter in the electrical problem is the loop resistance R, which is not the electrical analog of D. We shall see below that the mathematical formalism of Sec. II makes it just as easy, and considerably more revealing, to start afresh.

A. Langevin's hypothesis for an electrical circuit

Interactions between the conducting electrons and the thermally vibrating atomic lattice of the wire give rise to a temporally varying electromotive force (emf) in the loop, called the *thermal emf*. Following Langevin (but *not* Nyquist),[22] we hypothesize that this thermal emf can be regarded as the sum of two separate emfs, namely a "retarding emf" and a "random emf": The retarding emf always opposes the instantaneous electrical current $I(t)$ in the loop,

and is assumed to have the Ohm's law form, $-RI(t)$. The random emf, denoted by $V(t)$, is assumed to be statistically independent of the current $I(t')$ for all $t' \leq t$, and to fluctuate randomly about a mean of zero. But as we shall see shortly, the two components $-RI(t)$ and $V(t)$ of the thermal emf are not entirely independent of each other. This is a consequence of the fact that the microphysical processes that are ultimately responsible for the thermal emf, and which we do *not* undertake to examine here, cannot be separated into one class that is solely responsible for $-RI(t)$ and another class that is solely responsible for $V(t)$. We shall refer to the negative of the retarding emf, namely $RI(t)$, as the *dissipative voltage*, and to the random emf $V(t)$ as the *Johnson emf*.

According to Faraday's law, any temporal variation in the current $I(t)$ will give rise to an induced emf, $-d[LI(t)]/dt = -L \, dI(t)/dt$, in the loop. The requirement that the integral of the electric potential around the loop must vanish therefore gives us the *circuit equation*:

$$-RI(t) + V(t) - L\frac{dI(t)}{dt} = 0.\qquad(4.1)$$

Taking the *average* of Eq. (4.1) gives, on account of the assumed property $\langle V(t) \rangle = 0$,

$$-L\frac{d\langle I(t) \rangle}{dt} = R\langle I(t) \rangle.\qquad(4.2)$$

This equation describes what we would observe experimentally in those common situations where the fluctuations in the current caused by $V(t)$ are so small that measured values of $I(t)$ are indistinguishable from its mean $\langle I(t) \rangle$. Equation (4.2) thus affords a way, at least in principle, of experimentally determining the phenomenological parameter R once L has been calculated, say, from Neumann's formula.[23]

Defining the positive constant τ by

$$L/R \equiv \tau,\qquad(4.3a)$$

and then assuming (for reasons to be explained below) that the Johnson emf $V(t)$ can be written

$$V(t) = Lc^{1/2}\Gamma(t),\qquad(4.3b)$$

where $\Gamma(t)$ is Gaussian white noise and c is some positive constant, we can evidently bring Eq. (4.1) into the form

$$\frac{dI(t)}{dt} = -\frac{1}{\tau}I(t) + c^{1/2}\Gamma(t).\qquad(4.4)$$

Comparing this with the O–U Langevin equation (2.24), we may immediately conclude that $I(t)$ is an O–U process with relaxation time τ and diffusion constant c. We can also see that the seemingly arbitrary assumption in Eq. (4.3b) is in fact *already implicit* in our initial hypotheses. For, according to our theorem at Eq. (2.13), the circuit equation (4.1), which is memoryless by the Langevin hypothesis, will be mathematically self-consistent *only* if $V(t)$ is proportional to independent Gaussian white noise.

B. Fixing c

With the relaxation time τ given in terms of the known parameters L and R, it remains only to determine the value of the diffusion constant c. Defining the fully relaxed or "equilibrium" current $I^*(t)$ by

$$I^*(t) \equiv \lim_{t_0 \to -\infty} I(t),\qquad(4.5)$$

where $I(t_0)$ is some sure initial value i_0, then the stipulation that the system be in thermal equilibrium at absolute temperature T requires, by the equipartition theorem of classical statistical thermodynamics,[24] that the equilibrium energy of the loop current be

$$\langle \tfrac{1}{2} L I^{*2}(t) \rangle = \tfrac{1}{2} kT, \tag{4.6a}$$

where k is Boltzmann's constant. Thus, we must have

$$\langle I^{*2}(t) \rangle = kT/L. \tag{4.6b}$$

But from Eq. (2.32), we know that a fully relaxed O–U process $I^*(t)$ with relaxation time τ and diffusion constant c satisfies

$$\langle I^{*2}(t) \rangle = c\tau/2. \tag{4.7}$$

The last two equations show that c must be equal to $2kT/L\tau$. This, together with Eq. (4.3a), gives

$$\tau = L/R \tag{4.8a}$$

and

$$c = 2kTR/L^2 \tag{4.8b}$$

as the relaxation time and diffusion constant of the O–U process $I(t)$.

We can now infer from the basic O–U result (2.27), after noting from Eqs. (4.8) that $c\tau/2 = kT/L$, that given the initial condition $I(t_0) = i_0$, the current at any time $t > t_0$ will be

$$I(t) = \mathbf{N}\left(i_0 e^{-(R/L)(t-t_0)}, \frac{kT}{L}(1 - e^{-2(R/L)(t-t_0)}) \right)$$

$$(t \geq t_0). \tag{4.9}$$

Exact numerical simulations of $I(t)$, leading to graphs such as those shown in Fig. 2, can be constructed by employing the algorithm described in Sec. II F.

C. Implications of the theory

Substituting Eq. (4.8b) into Eq. (4.3b) evidently gives

$$V(t) = (2kTR)^{1/2} \Gamma(t). \tag{4.10}$$

This is the *fluctuation–dissipation formula*. It reveals explicitly the earlier mentioned intimate connection that exists between the fluctuating Johnson emf $V(t)$ and the dissipative resistance R: they are concomitant, each being an increasing function of the other. Another way of expressing this interdependence follows by first noting that Eq. (4.10) implies

$$\langle V(t)V(t+t') \rangle = (2kTR)\langle \Gamma(t)\Gamma(t+t') \rangle = 2kTR\,\delta(t'),$$

where the second equality follows from the Gaussian white noise property (2.18b). Integrating over all t' and then solving for R yields

$$R = \frac{1}{2kT} \int_{-\infty}^{\infty} \langle V(t)V(t+t') \rangle dt', \tag{4.11}$$

a formula that is often encountered in the literature.[7] Equation (4.11) in turn leads to Eq. (4.10) under our present hypothesis that $V(t)$ is some constant times Gaussian white noise.

Another integral formula involving R, which superficially resembles Eq. (4.11), can be deduced as follows: The fully relaxed O–U formula (2.33) implies that for any $t' > 0$,

$$\langle I^*(t)I^*(t+t') \rangle = \frac{c\tau}{2} e^{-t'/\tau} = \frac{c\tau^2}{2}\frac{1}{\tau} e^{-t'/\tau}$$

$$= \frac{kT}{R}\frac{1}{\tau} e^{-t'/\tau},$$

where the last step has invoked Eqs. (4.8). Integrating over $t' > 0$ and then solving for R^{-1} gives

$$R^{-1} = \frac{1}{kT} \int_0^{\infty} \langle I^*(t)I^*(t+t') \rangle dt'. \tag{4.12}$$

This, like Eq. (3.19), is an "auto-covariance transport formula." It expresses a transport coefficient, in this case the electrical conductance R^{-1} of the circuit, as a time integral over the auto-covariance of some equilibrium microscopic current, in this case the equilibrium electrical current in the circuit. We shall refer to Eq. (4.12) as the *conductance formula*.

Because the Johnson emf is proportional to Gaussian white noise, its intensity is *infinite*:

$$\langle V^2(t) \rangle = \infty. \tag{4.13}$$

By contrast, the intensity of the fully relaxed dissipative voltage $RI^*(t)$ is *finite*; because, $\langle [RI^*(t)]^2 \rangle = R^2 \langle I^{*2}(t) \rangle$, and this, by Eq. (4.6b), is the finite quantity

$$\langle [RI^*(t)]^2 \rangle = kTR^2/L. \tag{4.14}$$

We can gain more insight into the relationship between the Johnson emf $V(t)$ and the fully relaxed dissipative voltage $RI^*(t)$ by examining their spectral density functions.

Equation (4.10) tells us that the Johnson emf $V(t)$ is equal to the Gaussian white noise process multiplied by the constant $a = (2kTR)^{1/2}$. It therefore follows from Eq. (2.43) that the spectral density function of $V(t)$ is given by $2a^2$; hence, the famous *Nyquist formula*,[5]

$$S_V(\nu) = 4kTR \quad (\nu \geq 0). \tag{4.15}$$

$S_V(\nu)d\nu$ gives, by definition, the portion of $\langle V^2(t) \rangle$ due to frequencies in the frequency band $[\nu, \nu + d\nu)$. The constancy of $S_V(\nu)$ is of course consistent with the infinite result (4.13).

The spectral density function of the fully relaxed dissipative voltage $RI^*(t)$ can, by Eq. (2.45), be calculated as the product of R^2 times the spectral density function of $I^*(t)$:

$$S_{RI}(\nu) = R^2 S_I(\nu). \tag{4.16}$$

Since $I^*(t)$ is a fully relaxed O–U process, its spectral density function is given by Eq. (2.44). Substituting therein for τ and c from Eqs. (4.8), we get

$$S_I(\nu) = \frac{4kT}{R}\left(\frac{1}{1 + (2\pi L\nu/R)^2} \right) \quad (\nu \geq 0). \tag{4.17}$$

Combining Eqs. (4.16) and (4.17), we conclude that the spectral density function of the fully relaxed dissipative voltage $RI^*(t)$ is

$$S_{RI}(\nu) = \frac{4kTR}{1 + (2\pi L\nu/R)^2} \quad (\nu \geq 0). \tag{4.18}$$

$S_{RI}(\nu)d\nu$ gives, by definition, the portion of $\langle [RI^*(t)]^2 \rangle$ that is contributed by frequencies in the band $[\nu, \nu + d\nu)$. It is straightforward to show that the integral of Eq. (4.18) over all $\nu > 0$ equals kTR^2/L, in agreement with Eq. (4.14). We note that $S_{RI}(\nu)$ is equal to $S_V(\nu)$ at $\nu = 0$, but falls off from

that "Nyquist value" in the manner of Fig. 1(a) as ν increases from zero. Apparently, the self-inductance of the loop "filters out" the higher frequencies. A log–log plot of $S_{RI}(\nu)$ gives a leglike curve, as in Fig. 1(b), with a "knee" at frequency $R/2\pi L$. Below that knee the curve has slope 0, and above it has slope -2; so the frequency region below $R/2\pi L$ is called the white noise region, and the frequency region above $R/2\pi L$ is called the $1/f^2$ noise region.

The spectral density function $S_I(\nu)$ in Eq. (4.17) is by definition such that

$$\langle I^{*2}(t) \rangle = \int_0^\infty S_I(\nu) d\nu. \tag{4.19}$$

Since $(L/2)\langle I^{*2}(t) \rangle$ is the *average energy* of the equilibrium current in the loop, then by multiplying Eq. (4.19) through by $L/2$ we may conclude from Eq. (4.17) that the frequency spectrum of the average energy of the equilibrium current is described by the function

$$E(\nu) = \frac{L}{2} S_I(\nu) = \frac{2kTL}{R}\left(\frac{1}{1+(2\pi L\nu/R)^2}\right) \quad (\nu \geq 0). \tag{4.20}$$

It is easy to verify that the integral of this function over all $\nu > 0$ gives $\frac{1}{2}kT$, in agreement with Eq. (4.6a). And since $\langle RI^{*2}(t) \rangle$ is the *average power* that is dissipated by the equilibrium current in the loop, then by multiplying Eq. (4.19) through by R we may deduce that the frequency spectrum of the average dissipated power of the equilibrium current is described by the function

$$D(\nu) \equiv R S_I(\nu) = \frac{4kT}{1+(2\pi L\nu/R)^2} \quad (\nu \geq 0). \tag{4.21}$$

As expected, the integral of this function over all $\nu > 0$ gives the total dissipated power, kTR/L.

Finally, in the limit $L \to 0$ Eqs. (4.8) show that τ will approach 0 and c will approach ∞, but

$$\epsilon \equiv \tau c^{1/2} = \frac{L}{R} \frac{(2kTR)^{1/2}}{L} = \left(\frac{2kT}{R}\right)^{1/2}$$

will remain constant. We can therefore invoke the zero-tau limit theorem (2.38) to obtain

$$\lim_{L \to 0} I(t) = \left(\frac{2kT}{R}\right)^{1/2} \Gamma(t), \tag{4.22}$$

where $\Gamma(t)$ is Gaussian white noise. Note that if we multiply Eq. (4.22) through by R and then invoke the fluctuation–dissipation formula (4.10), we get

$$\lim_{L \to 0} [RI(t)] = V(t), \tag{4.23}$$

which is just what we should expect from the original circuit equation (4.1). So, in the limit $L \to 0$, the practical distinction between the dissipative voltage $RI(t)$ and the Johnson emf $V(t)$ disappears. Another consequence of the zero-tau limit theorem here is that the *integral* of $I(t)$, namely the net charge $Q(t)$ conveyed in some sense around the loop between times t_0 and t, becomes a driftless Wiener process with diffusion constant $2kT/R$; so as $L \to 0$, $Q(t)$ becomes, by Eq. (2.37), a normal random variable with mean 0 and standard deviation $[2kT(t-t_0)/R]^{1/2}$.

V. CONCLUSIONS

The analyses of Brownian motion and Johnson noise presented here are predicated on two assumptions: First, the driving force in each case can be regarded as the sum of a "retarding force" that is linear in the process and a "random force" that is independent of the process; this is Langevin's hypothesis. Second, each process asymptotically satisfies classical statistical thermodynamics. We showed, in Secs. III and IV, how all the principal results of classical Brownian motion and Johnson noise can be rigorously and efficiently derived from these two assumptions by using the mathematical machinery of continuous Markov process theory.

A focused tutorial on continuous Markov process theory was presented in Sec. II. Our exposition there utilized a streamlined approach to random variables, and it emphasized, through the theorem at Eq. (2.13), that continuous Markov process theory is a natural extension of ordinary differential calculus which has a remarkable *logical rigidity*. In particular, in the Langevin equation (2.14), we cannot summarily replace $N(t)$ with some nonnormal random variable, or $(dt)^{1/2}$ with some other dt dependence, and still have an update formula for a continuous memoryless stochastic process that is *self-consistent*.

Also implied here is a cautionary note for any proposed "non-Markovian generalization" of the Langevin equation: Even if a process update rule incorporates some memory effect, the update rule *still* must satisfy the self-consistency condition, that its application to the two successive intervals $[t, t+\alpha dt]$ and $[t+\alpha dt, t+dt]$ gives, for any $\alpha \in (0,1)$, the same result as its application to the interval $[t, t+dt]$, at least to lowest order in dt. Devising a nonmemoryless (non-Markovian) update formula that satisfies this self-consistency condition is not an easy task; indeed, as we saw in Eqs. (2.3), it is not hard to write down a process update formula that *looks* legitimate but is actually *self-contradicting*. Many non-Markovian processes are *components*, or functions of one or more components, of *multivariate Markov* processes, and it may be that such non-Markovian processes are best analyzed from that point of view. Whether *all* non-Markovian processes of physical interest can be so analyzed appears to be a question that has not yet been seriously addressed.

APPENDIX: DERIVATION OF EQ. (2.13)

To prove the theorem at Eq. (2.13), we begin by dividing the interval $[t, t+dt]$ into $n > 1$ subintervals of equal length dt/n by means of the points $t_i = t + i(dt/n)$, $(i=0$ to $n)$. We then have

$$X(t+dt) - X(t) = X(t_n) - X(t_0) = \sum_{i=1}^{n} [X(t_i) - X(t_{i-1})]$$

$$X(t+dt) - X(t) = \sum_{i=1}^{n} [X(t_{i-1} + dt/n) - X(t_{i-1})]. \tag{A1}$$

Equation (A1) shows that the conditional increment Ξ defined in Eq. (2.12) must satisfy, for any $n > 1$, the *self-consistency* relation

$$\Xi(dt;X(t),t) = \sum_{i=1}^{n} \Xi(dt/n;X(t_{i-1}),t_{i-1}).\qquad \text{(A2)}$$

Conditions (ii) and (iii), stated just below Eq. (2.12), tell us that Ξ is a smooth function of its last two arguments while X is continuous in t. So, since dt can be made so small that all the t_i's are arbitrarily close to t, then, at least to lowest order in dt, we can *replace* in Eq. (A2)

$$t_{i-1} \to t \quad \text{and} \quad X(t_{i-1}) \to X(t) \equiv x.\qquad \text{(A3)}$$

We therefore conclude that, at least to lowest order in dt and for all $n > 1$,

$$\Xi(dt;x,t) = \sum_{i=1}^{n} \Xi_i(dt/n;x,t).\qquad \text{(A4)}$$

The terms $\Xi_1, \ldots, \Xi_n$ in Eq. (A4) are n *copies* of the random variable $\Xi(dt/n;x,t)$. They are *statistically independent* copies, since condition (i) stipulates that each increment is independent of what happened in the past. The (identical) first and second moments of these n random variables will be assumed to be *finite* on account of condition (iii), which implies that sample values of $\Xi(dt;x,t)$ will tend to be close to zero whenever dt is close to zero. So, by taking n to be *arbitrarily large*, we may conclude by the central limit theorem that the sum $\Xi(dt;x,t)$ of these n statistically independent random variables must be *normal*.

Since $\Xi(dt;x,t)$ is normal, then so are the n statistically independent random variables $\Xi_i(dt/n;x,t)$. It therefore follows from Eq. (A4) and the normal sum theorem (2.11) that

$$\text{mean}\{\Xi(dt;x,t)\} = \sum_{i=1}^{n} \text{mean}\{\Xi_i(dt/n;x,t)\}$$

$$= n \times \text{mean}\{\Xi(dt/n;x,t)\},\qquad \text{(A5a)}$$

$$\text{var}\{\Xi(dt;x,t)\} = \sum_{i=1}^{n} \text{var}\{\Xi_i(dt/n;x,t)\}$$

$$= n \times \text{var}\{\Xi(dt/n;x,t)\}.\qquad \text{(A5b)}$$

Now, an easily proved theorem of elementary calculus says that if $h(z)$ is any smooth function of z that satisfies $h(z) = nh(z/n)$ for *all* positive integers n, then it must be true that $h(z) = Cz$, where C is independent of z.[25] Applying this theorem to Eqs. (A5) with dt replacing z, which is justified because condition (ii) requires $\Xi(dt;x,t)$ to be smooth in dt, we conclude that both the mean and the variance of the conditional increment $\Xi(dt;x,t)$ must be *directly proportional to dt*:

$$\text{mean}\{\Xi(dt;x,t)\} = A(x,t)dt,\qquad \text{(A6a)}$$

$$\text{var}\{\Xi(dt;x,t)\} = D(x,t)dt.\qquad \text{(A6b)}$$

Here, A and D may depend (*smoothly* by hypothesis) on x and t, but *not* on dt. Also, since both $\text{var}\{\Xi(dt;x,t)\}$ and dt are nonnegative, then D must be everywhere nonnegative.

Recalling now the previously established normality of $\Xi(dt;x,t)$, we have

$$\Xi(dt;x,t) = \mathbf{N}(A(x,t)dt, D(x,t)dt).\qquad \text{(A7)}$$

Equation (2.13) now follows simply by invoking Eq. (2.10), and recognizing that the past-forgetting property described in condition (i) demands that the unit normal $N(t)$ used for each increment must be statistically independent of the unit normals used for all earlier increments.

[1] A. Einstein, "Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen," Ann. Phys. **17**, 549–560 (1905); "Zur Theorie der Brownschen Bewegung," Ann. Phys. **19**, 371–381 (1906).

[2] M. von Smoluchowski, "Zarys teoryi kinetycznej ruchu Browna i roztworów metnych," Ann. Phys. **21**, 756–780 (1906).

[3] P. Langevin, "Sur la théorie du mouvement brownien," Comptes Rendues **146**, 530–533 (1908).

[4] J. B. Johnson, "Thermal agitation of electricity in conductors," Phys. Rev. **32**, 97–109 (1928).

[5] H. Nyquist, "Thermal agitation of electric charge in conductors," Phys. Rev. **32**, 110–113 (1928).

[6] Six especially important physics-oriented articles are available in *Selected Papers on Noise and Stochastic Processes*, edited by N. Wax (Dover, New York, 1954). Three of those papers deserve particular mention in the present context: G. E. Uhlenbeck and L. S. Ornstein, "On the theory of Brownian motion," Phys. Rev. **36**, 823–841 (1930); S. Chandrasekhar, "Stochastic problems in physics and astronomy," Rev. Mod. Phys. **15**, 1–89 (1943); and Ming Chen Wang and G. E. Uhlenbeck, "On the theory of Brownian motion II," Rev. Mod. Phys. **17**, 323–342 (1945).

[7] F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw-Hill, New York, 1965), Chap. 15.

[8] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981).

[9] C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences* (Springer-Verlag, New York, 1985).

[10] D. T. Gillespie, *Markov Processes: An Introduction for Physical Scientists* (Academic, San Diego, 1992).

[11] R. L. Stratonovich, *Nonlinear Nonequilibrium Thermodynamics I: Linear and Nonlinear Fluctuation–Dissipation Theorems* (Springer-Verlag, New York, 1992).

[12] D. T. Gillespie, "A theorem for physicists in the theory of random variables," Am. J. Phys. **51**, 520–533 (1983). See also Ref. 10, Chap. 1.

[13] An entirely equivalent version of the Langevin equation (2.14) can be obtained by replacing $N(t)(dt)^{1/2}$ on the right side with $dW(t)$, which is defined to be the temporally uncorrelated normal random variable with mean 0 and variance dt. This replacement is justified simply by noting from theorem (2.9) that $(dt)^{1/2}N(0,1) = N(0,dt)$. The $dW(t)$ version of the Langevin equation is very common in the literature; however, once one gets over the false notion that it is against the law to take the square root of an infinitesimal variable, the $N(t)(dt)^{1/2}$ version will be seen to be more transparent and easier to work with.

[14] For any time-dependent random variable $Y(t)$, the "average" $\langle Y^n(t) \rangle$ can, according to Eq. (2.4b), be computed as $\int y^n P(y;t)dy$, where $P(y;t)$ is the time-dependent density function of $Y(t)$. To compute $\langle Y^n(t) \rangle$ using the equivalent sampling formula (2.4a), we evidently must obtain a large number M of sample values of $Y(t)$. We can do that *either* by running the process M times up to the instant t, *or* by running M identical copies of the process once up to the instant t. The latter procedure shows that $\langle Y^n(t) \rangle$ is what is commonly referred to as an "ensemble average."

[15] See, e.g., Ref. 10, Chaps. 2–3.

[16] The driftless Wiener process with $c = 1$—i.e., the continuous Markov process with $A = 0$ and $D = 1$—is called the *special Wiener process*, and is denoted by $W(t)$. Equation (2.36a) implies that $dW(t) = N(t)(dt)^{1/2} = N(0,dt)$, a relation already mentioned in Ref. 13. Equation (2.36b) gives $dW(t)/dt = \Gamma(t)$, and that prompts some people to call $W(t)$ "the integral of Gaussian white noise." But since the mathematical credentials of $W(t) \equiv N(x_0, t-t_0)$ are more solid than those of $\Gamma(t)$, we have chosen to proceed the other way around and define $\Gamma(t)$ to be the derivative of $W(t)$ [see Eq. (2.16)]. Electrical engineers tend to call the special Wiener process "Brownian motion," and denote it by $B(t)$; however, we shall see in Sec. III that that appellation would be more appropriately applied to the *integral* of the Ornstein–Uhlenbeck process. (The Ornstein–Uhlenbeck process in turn is often called by electrical engineers "the Markov process," giving yet another example of the confusing intradisciplinary mismatch in terminology.)

[17] The consistency of Eqs. (2.40) and (2.41) can be formally verified by

substituting Eq. (2.41) into Eq. (2.40), switching the order of integration, and then invoking the delta function formula

$$\delta(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i\omega x} \, d\omega = 2\int_{0}^{\infty} \cos(2\pi\nu x) d\nu.$$

[18]A rigorous derivation of the widely known formulas (2.48), which is sometimes called the Box–Muller algorithm, may be found in Ref. 10, Sec. 1.8, which also describes a simple procedure for generating values for r_1 and r_2 in Eqs. (2.48a).

[19]It is possible to construct a simulation algorithm for O–U processes that is exact for *both* Y and X, but to do that we need a little more random variable theory than is given in Sec. II B.

[20]A nice account of Einstein's work in Ref. 1 is given in Ref. 9, pp. 2–6.

[21]D. T. Gillespie, "Fluctuation and dissipation in Brownian motion," Am. J. Phys. **61**, 1077–1083 (1993). A more sophisticated argument is given in Ref. 10, Secs. 4.5 and 4.6.

[22]Nyquist's original analysis in Ref. 5 proceeds quite differently from the analysis that we give here; see Ref. 7, p. 592.

[23]J. R. Reitz and F. J. Milford, *Foundations of Electromagnetic Theory* (Addison-Wesley, Reading, MA, 1960), p. 176.

[24]See, e.g., Ref. 7, Sec. 7.5.

[25]A short proof of this "lemma" may be found in Ref. 10, p. 114.

Photon counting statistics—Undergraduate experiment

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A photon counting experiment for student physics laboratory is described. It is designed to illustrate the probabilistic nature of the photodetection process itself as well as statistical fluctuations of light. The setup enables the student to measure photon count distributions for both coherent and pseudothermal light sources yielding Poisson and Bose–Einstein distributions, respectively. © *1996 American Association of Physics Teachers.*

I. INTRODUCTION

Photon counting is a technique commonly used to measure extremely low light fluxes. A photomultiplier tube (PMT) of proper design is used to convert light into an electrical signal. (The most important property of photon counting PMT is very high gain at the first dynode. This allows one to distinguish between the pulse[1] resulting from electrons ejected from the photocathode and those coming from the dynodes.) Light impinging on the photocathode of the PMT ejects electrons from it. Assuming that the gain of the PMT itself and that of the following electronics is high enough, one can distinguish individual electrical pulses, each of them corresponding to a single photoelectron. The electrical pulses from the PMT are fed into a discriminator and pulses with amplitudes higher than a given threshold value are counted. Those are usually referred to as photon counts. This way one can count the number of electrical pulses, ideally each of them corresponding to a single photoelectron. Since for each photoelectron created one photon of the light field has to be destroyed, the method is commonly called photon counting. Thus, in this oversimplified picture, one can think of the method as a way to prove the existence of photons. This is not true. Actually there is no need to quantize the electromagnetic field in order to explain all the features of the photoelectric process. All that is necessary is an assumption that the light interacts with matter which is described quantum mechanically. This leads to what is commonly referred to as a semiclassical description,[2] a model in which light is described as a classical electromagnetic wave and the atomic system, the photocathode in our case, quantum mechanically. The question of whether one has to invoke the quantum nature of electromagnetic field at all has been disputed ever

since Einstein introduced the notion of the "light quantum" in 1905 in his paper describing the external photoelectric process.[3] Several different models for the photon have been proposed starting with a simple particle model and ending with what is known as Dirac's model,[4] each of them presenting its own difficulties in interpretation. Amazingly enough, the answer to this question has been settled quite recently when experiments on photon antibunching[5] and squeezed states (for a review on the topic see, for example, the paper by Walls[1]) proved that at least in some cases a quantum mechanical description of the light is necessary. Since in the experiments described in this paper the light can be perfectly described in a classical way, the semiclassical picture will be used henceforth.

The noise present in photon counting can be separated into two terms. The first is of a technical nature and is caused predominantly by electrical pulses created by amplification of electrons thermally released from the photocathode or the first dynode, which cannot be distinguished from the pulses corresponding to photoelectrons. Those pulses are present even if there is no light falling on the photocathode and for this reason are called dark counts. The dark count rate can be minimized by proper design and cooling of the PMT. Currently, even modestly priced systems have a dark count rate as small as a few counts per second. We will assume throughout this paper that the dark counts can be neglected altogether. The second contribution to the noise in the photon counting experiment and the only one considered in this paper is of a fundamental nature and cannot be eliminated. Again, it can be divided into two parts caused by the stochastic nature of photoelectric process and light intensity fluctuations, respectively. Using the semiclassical model mentioned above one finds that for a constant intensity of light reaching