Fluctuation and dissipation in Brownian motion

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surable properties; to "be responsive under one's finger," "have a beautiful sound which carries," "speak clearly and easily," and other such expressions which are no doubt meaningful but have not been translated into physical properties, cannot be considered to answer the question I am posing.

VI. CONCLUDING MEDITATION

Many years ago, just after I finished presenting a colloquium on piano physics at Michigan State University, an anonymous student handed me a sketch he had made while listening to me. It was entitled "Great Moments in Physics #42: Galileo Begins the Study of Musical Instruments," and showed the great man himself dropping a piano and a saxophone side by side from the top of the Leaning Tower of Pisa.

At moments of discouragement, I have been known to look at that picture and wonder just how far we have come from such an apocryphal beginning. Yet the truth is that we have come an enormous distance. The trouble is that the nature of research is forever to be doing something that we do not know how to do and, as soon as we have learned how to do it, to stop doing it and look for a new problem; this means that a researcher's mind is forever fixed on what has *not* been achieved—which, by the standards of the world, means being condemned to a life of perpetual discouragement. That this is not the way that we researchers perceive it is one of the great miracles of human creativity, and the primary reason that we love our work as much as we do.

ACKNOWLEDGMENTS

That this paper is completely devoid of references is entirely a matter of the style in which it was presented, and is by no means meant to imply that all the results quoted are the author's own. On the contrary, I am, like every scientist, infinitely indebted both to the myriad of others who have contributed to our knowledge and understanding, and to the generally open and sharing atmosphere that exists in our profession, making good research possible. By contrast, the many occasions on which I have, in the course of this presentation, gotten out on a limb are occasions for which I myself accept complete responsibility. I would like also to acknowledge the support of the National Science Foundation, which over the years has furthered my research immensely. Readers who would like to pursue the subject to a deeper level will do well to begin with the book *The Physics of Musical Instruments* by N. H. Fletcher and T. D. Rossing, which contains not only excellent presentations but extensive bibliographies as well.

Fluctuation and dissipation in Brownian motion

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An analysis of Brownian motion based upon a "Langevin equation" form of Newton's second law provides a physically motivated introduction to the theory of continuous Markov processes, which in turn illuminates the subtle mathematical underpinnings of the Langevin equation. But the Langevin approach to Brownian motion requires one to *assume* that the collisional forces of the bath molecules on the Brownian particle artfully resolve themselves into a "dissipative drag" component and a "zero-mean fluctuating" component. A physically more plausible approach is provided by a simple discrete-state jump Markov process that models in a highly idealized way the immediate effects of individual molecular collisions on the velocity of the Brownian particle. The predictions of this jump Markov process model in the continuum limit are found to precisely duplicate the predictions of the Langevin equation, thereby validating the critical two-force assumption of the Langevin approach.

I. INTRODUCTION

Brownian motion is the motion of a macroscopically small but microscopically large particle that is subject only to the collisional forces exerted by the molecules of a surrounding fluid. If M denotes the particle's mass and $V(t)$ its instantaneous velocity, then the traditional way of analyzing Brownian motion[1-4] is to begin with the Newton's second law equation

$$M\frac{dV(t)}{dt}=-\gamma V(t)+f\Gamma(t). \quad (1)$$

Here γ is a positive constant called the drag coefficient, $\Gamma(t)$ is an entity called the Gaussian white noise process (which will be discussed more fully later), and f is a constant whose value remains to be specified. The physical interpretation of Eq. (1) is that the particle is subject to two kinds of forces: a steady dissipative drag force

$-\gamma V(t)$, and a zero-mean temporally uncorrelated fluctuating force $f\Gamma(t)$. A detailed analysis of Eq. (1) ultimately reveals that, in order to satisfy the thermodynamic equipartition of energy condition

$$\tfrac{1}{2}M\langle V^2(\infty)\rangle=\tfrac{1}{2}k_BT, \tag{2}$$

where k_B is Boltzmann's constant, T is the absolute temperature of the bath, and the angular brackets denote an averaging operation, the constant f must be assigned the value

$$f=(2\gamma k_BT)^{1/2}. \tag{3}$$

This monotonic relation between the fluctuating force coefficient f and the dissipative drag coefficient γ is an expression of the fluctuation-dissipation theorem. It implies that fluctuation and dissipation are intimately related, and that one cannot be present without the other.

The fact that the two force terms on the right side of Eq. (1) are not independent of each other, as one might have at first supposed, points up a disconcerting feature of the foregoing approach to the Brownian motion problem: Because Eq. (1) *assumes* that the forces of the fluid molecules on the particle neatly resolve themselves into a dissipative drag component and a zero-mean fluctuating component, it can offer no insight on how that resolution comes about. To gain such insight, one should instead begin with a plausible model of thermally moving molecules impinging on the particle and then infer the existence of the alleged forces. The main purpose of this paper is to demonstrate a fairly simple way of accomplishing this task.

In Sec. II we shall place Eq. (1) in the more general context of continuous Markov process theory, which will allow a much deeper understanding of its mathematical structure, and we shall then describe a novel way of deducing its solution $V(t)$. For the sake of completeness, we shall go on to show how this solution $V(t)$ leads to the fluctuation-dissipation relation (3), as well as the experimentally important result of Einstein for the Brownian particle's asymptotic mean-square displacement $\langle X^2(t\to\infty)\rangle$. In Sec. III we shall present an altogether different approach to the Brownian motion problem. There we shall cast $V(t)$ as a *jump* Markov process which models, in a highly simplified way, the expected effects of individual molecular impingements on the particle. We shall show that the solution $V(t)$ to our jump Markov model in the continuum limit exactly reproduces the solution $V(t)$ obtained in Sec. II, yielding specific formulas for the Eq. (1) parameters γ and f in terms of the parameters of our jump model. This agreement can then be taken as a vindication of the critical two-force assumption of Eq. (1). As this program of analysis suggests, the model presented in Sec. III is not intended to replace Eq. (1), but rather to justify it.

II. THE LANGEVIN EQUATION AND CONTINUOUS MARKOV PROCESSES

Equation (1) is an example of a "Langevin equation." In order to properly interpret and solve such an equation, it is necessary to know a few basic facts about the Gaussian or *normal* random variable.

We say that Y is a "normal random variable with mean m and variance σ^2," and write $Y=\mathbf{N}(m,\sigma^2)$, if and only if the probability that a sampling of Y will yield a value in the infinitesimal interval $[y,y,+dy)$ is given by

$$\text{Prob}\{Y\in[y,y+dy)\}=\frac{1}{(2\pi\sigma^2)^{1/2}}\exp\left(-\frac{(y-m)^2}{2\sigma^2}\right)dy. \tag{4}$$

Two important properties of the normal random variable are the following: First, if β is any constant, then

$$\beta\mathbf{N}(m,\sigma^2)=\mathbf{N}(\beta m,\beta^2\sigma^2). \tag{5}$$

And second, if the two random variables $\mathbf{N}(m_1,\sigma_1^2)$ and $\mathbf{N}(m_2,\sigma_2^2)$ are statistically independent of each other, meaning that knowing the value of one does not help us to predict the value of the other, then

$$\mathbf{N}(m_1,\sigma_1^2)+\mathbf{N}(m_2,\sigma_2^2)=\mathbf{N}(m_1+m_2,\sigma_1^2+\sigma_2^2). \tag{6}$$

Proofs of these two properties may be found in most any textbook on statistics.[5]

The Gaussian white noise process $\Gamma(t)$ appearing in Eq. (1) is formally defined by

$$\Gamma(t)\equiv\lim_{dt\to0^+}\mathbf{N}(0,1/dt). \tag{7}$$

This definition, which we shall see later is not as whimsically arbitrary as it might at first appear, implies that the mean of $\Gamma(t)$ is zero, that the variance of $\Gamma(t)$ diverges like $1/dt$ as $dt\to0^+$, and that $\Gamma(t_1)$ and $\Gamma(t_2)$ are statistically independent if $t_1\neq t_2$. All of these properties are implicit in the often quoted pair of formulas

$$\langle\Gamma(t)\rangle=0 \quad\text{and}\quad \langle\Gamma(t_1)\Gamma(t_2)\rangle=\delta(t_1-t_2),$$

but it is the more complete definition (7) that will be required for our analysis here.

The ill-behaved nature of $\Gamma(t)$ implicit in the definition (7) means that $dV(t)/dt$ in Eq. (1) will likewise be ill behaved. The way to get around this difficulty is to back off from the limit $dt=0$ and write Eq. (1) in the differential form

$$V(t+dt)-V(t)=-\frac{\gamma}{M}V(t)dt+\frac{f}{M}\Gamma(t)dt. \tag{8}$$

Here, dt is to be regarded as a real variable on the interval $[0,\epsilon]$, where the positive number ϵ is arbitrarily close to zero but otherwise unimportant. Using the definition (7) of $\Gamma(t)$ and the normal property (5), we can write the factor $\Gamma(t)dt$ in the last term of the above equation as

$$\Gamma(t)dt=\mathbf{N}(0,1/dt)dt=(1/dt)^{1/2}\mathbf{N}(0,1)dt,$$

$$\Gamma(t)dt=\mathbf{N}(0,1)(dt)^{1/2}. \tag{9}$$

Substituting this into Eq. (8) and transposing the $V(t)$ term, we obtain

$$V(t+dt)=V(t)-\frac{\gamma}{M}V(t)dt+\frac{f}{M}\mathbf{N}(0,1)(dt)^{1/2}. \tag{10}$$

With the understanding that dt is a "positive infinitesimal," Eq. (10) is fully equivalent to Eq. (1). In a moment we shall show how to solve Eq. (10) for $V(t)$. But first, let us place our considerations here in the more general context of continuous Markov process theory.

A *Markov process* is basically any function of time whose value at time $t+dt$ can be *probabilistically* predicted from its value at time t, but in a way that cannot be sharpened by taking cognizance of its values at times earlier than t. A Markov process $V(t)$ is said to be of the *continuous* type if the difference $V(t+dt)-V(t)$ behaves as much like a "smooth infinitesimal" as is possible. It turns out[4] that this seemingly vague condition implies a surprisingly specific formula relating $V(t+dt)$ to $V(t)$, namely

$$V(t+dt)=V(t)+A(V(t),t)dt + D^{1/2}(V(t),t)\mathbf{N}(0,1)(dt)^{1/2}. \tag{11}$$

In this equation, which is called the *Langevin equation* of the continuous Markov process $V(t)$, $A(v,t)$ and $D(v,t)$ are any two differentiable functions of their arguments, with D being non-negative. One thus has considerable latitude in choosing forms for the two functions A and D. But any attempt to alter the structure of Eq. (11)—for instance, by replacing the normal random variable $\mathbf{N}(0,1)$ with a zero-mean *uniform* random variable or by changing the respective exponents of dt in either of the last two terms on the right—will invariably result in a formula that cannot consistently be applied for *all* sufficiently small $dt > 0$.

The Langevin equation (11) is essentially a recipe for computing the value of $V(t+dt)$ from the value of $V(t)$: one merely substitutes for $\mathbf{N}(0,1)$ a sample value of that random variable and then carries out the indicated arithmetic operations. The increment $V(t+dt)-V(t)$ is seen to be composed of two terms, one being proportional to dt and the other being proportional to $(dt)^{1/2}$. It is not legitimate to discard the dt term relative to the $(dt)^{1/2}$ term simply because the former is of higher order in dt than the latter. The reason is that the factor $\mathbf{N}(0,1)$ that multiplies $(dt)^{1/2}$ is about as often negative as positive, with the result that the cumulative effect of the third term over a *succession* of dt increments is diminished to a level comparable to that of the second term. In the imagery of a well known fable, the dt term in Eq. (11) might be called the *tortoise* term, because it's slow but steady, while the $\mathbf{N}(0,1)(dt)^{1/2}$ term might be called the *hare* term, because it's fast but erratic. More conventionally, the second term on the right side of Eq. (11) is called the drift term, and $A(v,t)$ the *drift function*, while the third term on the right side of Eq. (11) is called the diffusion term, and $D(v,t)$ the *diffusion function*. Continuous Markov processes differ from one another only to the extent that their drift and diffusion functions differ. For the Brownian motion Langevin Eq. (10), we evidently have $A(v,t)=-(\gamma/M)v$ and $D(v,t)=(f/M)^2$.

That the process defined by the general Langevin Eq. (11) satisfies the fundamental *continuity* condition, $V(t+dt)\to V(t)$ as $dt\to 0^+$, is quite clear from that equation. But *differentiability* is another matter, as can be seen by algebraically rearranging Eq. (11) to read

$$\frac{V(t+dt)-V(t)}{dt}=A(V(t),t)+\frac{D^{1/2}(V(t),t)\mathbf{N}(0,1)}{(dt)^{1/2}}. \tag{12}$$

Clearly, the limit of this equation as $dt\to 0^+$ does not exist in any conventional mathematical sense if the diffusion function D is not identically zero. Of course, if $D\equiv 0$ then the limit gives the ordinary differential equation $dV(t)/dt = A(V(t),t)$; in that case the "randomness," which enters exclusively through the random variable $\mathbf{N}(0,1)$, never appears, and $V(t)$ will be a purely *deterministic* process, the stuff of ordinary differential calculus. But a truly *stochastic* continuous Markov process, namely one for which the diffusion function D is positive, is an example of a function that is everywhere continuous but nowhere differentiable.

Most physicists, however, are loath to abjure the familiar notation of differential calculus. So they, in effect, invoke theorem (5) to write $(dt)^{-1/2}\mathbf{N}(0,1)=\mathbf{N}(0,1/dt)$, and then take the limit $dt\to 0^+$ in Eq. (12) with the help of the definition (7) to obtain

$$\frac{dV(t)}{dt}=A(V(t),t)+D^{1/2}(V(t),t)\Gamma(t). \tag{13}$$

This equation too is called the Langevin equation, and is actually more commonly encountered than the form (11). But, being a relation between two mathematically ill-defined entities, namely $dV(t)/dt$ and $\Gamma(t)$, Eq. (13) is probably best regarded as a *mnemonic* for the form (11)—in the same way that Eq. (1) is a mnemonic for Eq. (10). In any case, it should now be clear that the curious definition (7) of the white noise process $\Gamma(t)$ is a consequence of ensuring that Eq. (13) be compatible with Eq. (11), which in turn is the *only* consistent update formula for a process $V(t)$ that is both past-forgetting (Markovian) and continuous.[4]

The Brownian motion Langevin equation (10), because of its relatively simple form, can actually be solved for $V(t)$ by merely invoking the two normal theorems (5) and (6). Thus, if $V(t)$ has the sure value V_0 at time $t=0$, then since $V_0=\mathbf{N}(V_0,0)$ Eq. (10) gives for $t=0$

$$\begin{aligned}V(dt)&=\left(1-\frac{\gamma dt}{M}\right)\mathbf{N}(V_0,0)+\left(\frac{f}{M}\right)\mathbf{N}(0,1)(dt)^{1/2}\\&=\mathbf{N}\left(V_0\left(1-\frac{\gamma dt}{M}\right),0\right)+\mathbf{N}\left(0,\left(\frac{f}{M}\right)^2 dt\right)\\&=\mathbf{N}\left(V_0\left(1-\frac{\gamma dt}{M}\right),\left(\frac{f}{M}\right)^2 dt\right),\end{aligned}$$

where the second step has invoked rule (5) and the last step has invoked rule (6). Noting that

$$\left(\frac{f}{M}\right)^2 dt\equiv\frac{f^2}{2\gamma M}\left[1-\left(1-\frac{2\gamma dt}{M}\right)\right], \tag{14}$$

we thus have

$$V(dt)=\mathbf{N}\left(V_0\left(1-\frac{\gamma dt}{M}\right),\frac{f^2}{2\gamma M}\left[1-\left(1-\frac{2\gamma dt}{M}\right)\right]\right). \tag{15}$$

A straightforward induction argument, which is detailed in the Appendix, expands this result to

$$V(Kdt)=\mathbf{N}\left(V_0\left(1-\frac{\gamma dt}{M}\right)^K,\frac{f^2}{2\gamma M}\left[1-\left(1-\frac{2\gamma dt}{M}\right)^K\right]\right) \tag{16}$$

for K any positive integer. Now, for any finite $t>0$, let us choose $K=t/dt$, so that $Kdt=t$ and $dt=t/K$. Since dt is infinitesimally small, then K will be infinitely large, so

$$\left(1-\frac{\gamma dt}{M}\right)^K=\left(1-\frac{(\gamma/M)t}{K}\right)^K=e^{-(\gamma/M)t}$$

and

$$\left(1-\frac{2\gamma dt}{M}\right)^K=\left(1-\frac{2(\gamma/M)t}{K}\right)^K=e^{-2(\gamma/M)t}.$$

Equation (16) therefore becomes, for $K=t/dt$,

$$V(t)=\mathbf{N}\left(V_0e^{-(\gamma/M)t},\frac{f^2}{2\gamma M}(1-e^{-2(\gamma/M)t})\right). \tag{17}$$

This is the solution of Eq. (10), and hence also of Eq. (1).

In Sec. III we shall show how a formula for $V(t)$ of the form (17) can be derived *without* making the overt assumption that the forces of the bath molecules on the Brownian particle conveniently resolve themselves into the two components appearing on the right side of Eq. (1). But before doing that, let us quickly review the major physical implications of Eq. (17). First of all, the long-time limit of Eq. (17) evidently gives

$$V(\infty)=\mathbf{N}\left(0,\frac{f^2}{2\gamma M}\right). \tag{18}$$

This implies that

$$\langle V^2(\infty)\rangle=\frac{f^2}{2\gamma M}.$$

So the thermodynamic requirement (2) will be satisfied if and only if

$$\frac{f^2}{2\gamma M}=\frac{k_BT}{M}. \tag{19}$$

Solving Eq. (19) for f yields the anticipated fluctuation-dissipation relation (3), which connects the two Langevin parameters γ and f. The implication of Eqs. (18) and (19), that the equilibrium velocity $V(\infty)$ of the Brownian particle is a normal random variable with mean zero and variance k_BT/M, can be seen from the normal density formula (4) to be the same as saying that the asymptotic velocity of the Brownian particle is distributed in a Maxwell–Boltzmann fashion.

Although a continuous Markov process will not generally have a proper derivative, it will have a proper *integral*.[6] In the case of our process $V(t)$, that integral is of course the physically important *position* $X(t)$ of the Brownian particle. With $V(t)$ defined by the Langevin Eq. (10), we may conveniently define the integral process $X(t)$ by the companion equation

$$X(t+dt)=X(t)+V(t)dt. \tag{20}$$

By using Eqs. (10) and (20) in tandem, it is possible to analytically calculate all the moments of both $V(t)$ and $X(t)$. Since experimental investigations of Brownian motion traditionally focus on the long-time behavior of $\langle X^2(t)\rangle$, let us see how $\langle X^2(t\to\infty)\rangle$ can be most expeditiously deduced.

Squaring Eq. (20) and dropping terms of order >1 in dt, we get

$$X^2(t+dt)=X^2(t)+2X(t)V(t)dt.$$

Averaging, and then transposing the first term on the right, dividing through by dt, and letting $dt\to0^+$, we get

$$\frac{d}{dt}\langle X^2(t)\rangle=2\langle X(t)V(t)\rangle. \tag{21}$$

To calculate the right side of Eq. (21), we first multiply Eqs. (10) and (20) together, dropping terms of order >1 in dt. This gives

$$X(t+dt)V(t+dt)=X(t)V(t)-\frac{\gamma}{M}X(t)V(t)dt$$
$$+\frac{f}{M}X(t)\mathbf{N}(0,1)(dt)^{1/2}+V^2(t)dt.$$

Averaging this equation, noting as we do that $\mathbf{N}(0,1)$ is statistically independent of $X(t)$ and satisfies $\langle\mathbf{N}(0,1)\rangle=0$, we obtain after the usual algebraic rearrangement and $dt\to0^+$ limit

$$\frac{d}{dt}\langle X(t)V(t)\rangle=-\frac{\gamma}{M}\langle X(t)V(t)\rangle+\langle V^2(t)\rangle. \tag{22}$$

Since we know $\langle V^2(t)\rangle$ [because Eq. (17) tells us the mean and variance of $V(t)$], we could now proceed to first solve Eq. (22) for $\langle X(t)V(t)\rangle$, and then use that result to solve Eq. (21) for $\langle X^2(t)\rangle$. But as we are presently interested only in the $t\to\infty$ limit, we may proceed more directly by simply observing from condition (2) that

$$\langle V^2(t\to\infty)\rangle=\frac{k_BT}{M}.$$

Using this result, we may immediately deduce from the differential Eq. (22) that its solution $\langle X(t)V(t)\rangle$ must approach a constant value $\langle X(\infty)V(\infty)\rangle$ which satisfies

$$0=-\frac{\gamma}{M}\langle X(\infty)V(\infty)\rangle+\frac{k_BT}{M};$$

thus,

$$\langle X(\infty)V(\infty)\rangle=\frac{k_BT}{\gamma}.$$

It therefore follows from Eq. (21) that $\langle X^2(t)\rangle$ ultimately increases *linearly* with t according to

$$\langle X^2(t\to\infty)\rangle=2\left(\frac{k_BT}{\gamma}\right)t. \tag{23}$$

Experimental studies of Brownian motion find that $\langle X^2(t\to\infty)\rangle$ is indeed directly proportional to t, and the constant of proportionality is operationally defined to be twice the diffusion coefficient of the Brownian particle. The result in Eq. (23), which predicts the diffusion coefficient to be k_BT/γ, is the famous result obtained by Einstein in 1905, and also by Smoluchowski in 1906 and Langevin in 1908.[7] Our present derivation of this result is closest in spirit to the derivation of Langevin.

III. A MOLECULAR IMPINGEMENT MODEL OF BROWNIAN MOTION

In Sec. II we began by assuming the validity of the Langevin Eq. (1), with its explicit dissipative drag force $-\gamma V(t)$ and zero-mean fluctuating force $f\Gamma(t)$, and we then derived the result (17) for $V(t)$. Now we want to present a way of calculating $V(t)$ that proceeds, not from Eq. (1), but rather from an idealized model of the direct effects of molecular impingements on the particle's velocity. An *exact* way of doing this when the surrounding fluid is an ideal "one-dimensional gas" has been described else-

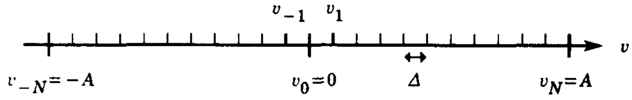

Fig. 1. Illustrating the permissible states (24) of our proposed continuous-time random walk model for the velocity $V(t)$ of a particle undergoing Brownian motion.

where.[8] Here, however, we shall focus on a mathematically simpler *approximate* model that manages to capture the essential features of an exact analysis.

We propose to model $V(t)$ as a *jump* Markov process (sometimes also called a "continuous-time random walk") over the $2N+1$ discrete states

$$v_n=(A/N)n \quad (n=0,\pm 1,\ldots,\pm N), \tag{24}$$

where A is some positive number. As illustrated in Fig. 1, these states $\{v_n\}$ cover the velocity range from $-A$ to A in discrete steps of size

$$\Delta=A/N. \tag{25}$$

We shall eventually let $N\to\infty$ in such a way that $A\to\infty$ and $\Delta\to 0$, so that the allowable values of $V(t)$ will become virtually unrestricted.

The motion of $V(t)$ over its allowed states $\{v_n\}$ consists of random steps of size $\pm\Delta$, these steps being taken at random times and in a past-forgetting (Markovian) manner. Such behavior can be characterized by two "stepping functions" $W_+(v)$ and $W_-(v)$, which are defined so that

$$W_\pm(v_n)dt \equiv \text{the probability, given } V(t)=v_n, \text{ that } V(t+dt) \text{ will equal } v_{n\pm 1}. \tag{26}$$

Our first task will be to find forms for these two functions that characterize in a plausible way the effect on the particle's velocity of the naturally occurring molecular impingements. Since symmetry considerations dictate that

$$W_-(-v)=W_+(v), \tag{27}$$

then we may focus our efforts on finding a form for the function $W_+(v)$.

Suppose first that the particle is at *rest*. Then in the next infinitesimal time interval dt, there will be a certain probability that some molecule will strike the particle's backside sufficiently hard to increase the particle's velocity from zero to $v_1=\Delta$. Let us assume that this probability can be written in the form Bdt, where B is some positive constant:

$$Bdt \equiv \text{the probability that the particle, at rest at time } t, \text{ will acquire velocity } v_1=\Delta \text{ in the next infinitesimal time interval } [t,t+dt). \tag{28}$$

We may reasonably expect B to be an *increasing* function of the average kinetic energy of the bath molecules, and a *decreasing* function of the particle's mass M and the velocity step size Δ; however, we shall be content here to let B be phenomenologically defined by the statement (28). Comparison with the definition (26) shows that

$$W_+(0)=B. \tag{29}$$

Since our model assumes that the velocity of the particle can never exceed the value A, then we must have

$$W_+(A)=0. \tag{30}$$

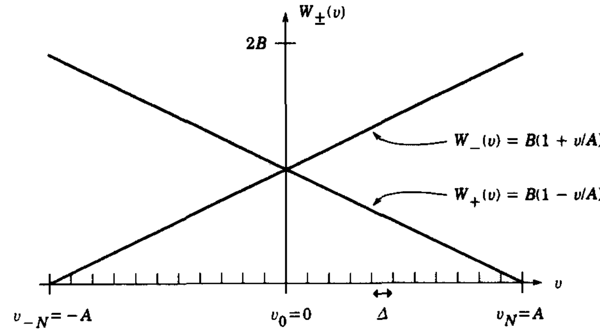

Fig. 2. Graphs of the stepping functions $W_\pm(v)$ in Eqs. (31).

And it is clear on physical grounds that, for any $v<A$, $W_+(v)$ must be a *steadily* decreasing function of v; because, if the particle's forward speed is increased, then the likelihood that the particle will be struck from behind by a gas molecule hard enough to further augment its forward speed by Δ should surely decrease. To keep our model simple, let us assume that $W_+(v)$ is a *linearly* decreasing function of v. This linearity assumption and the conditions (29) and (30) suffice to determine $W_+(v)$ completely

$$W_+(v)=B(1-v/A) \quad (-A\leqslant v\leqslant A). \tag{31a}$$

The symmetry relation (27) then gives for $W_-(v)$

$$W_-(v)=B(1+v/A) \quad (-A\leqslant v\leqslant A). \tag{31b}$$

Plots of these two functions are shown in Fig. 2.

We now have a fully defined jump Markov process model for the particle's velocity $V(t)$. Our model contains two parameters A and B, and we shall later have to decide how these two parameters should depend upon the parameter N that controls the total number of velocity states. For now, though, let us deduce the consequences of this model.

Our analysis will focus on the function

$$P(v_n,t) \equiv \text{the probability that } V(t)=v_n, \text{ given that } V(0)=V_0. \tag{32}$$

To derive a time-evolution equation for this function, we begin by using the definitions (32) and (26), along with the multiplication and addition laws of probability, to infer the following expression for the probability that $V(t+dt)$ will equal v_n:

$$\begin{aligned} P(v_n,t+dt) &= P(v_{n-1},t)\times W_+(v_{n-1})dt \\ &\quad + P(v_{n+1},t)\,W_-(v_{n+1})dt \\ &\quad + P(v_n,t)\{1-[W_+(v_n)dt+W_-(v_n)dt]\}. \end{aligned} \tag{33}$$

The first term on the right is the probability that $V(t)=v_{n-1}$ and then an up-going step occurs in the next dt; the second term is the probability that $V(t)=v_{n+1}$ and then a down-going step occurs in the next dt; and the third term is the probability that $V(t)=v_n$ and then *no* step occurs in the next dt. All other routes to $V(t+dt)=v_n$ from time t involve more than one velocity jump in time $[t,t+dt)$, and consequently will be of order >1 in dt. Upon transposing the term $P(v_n,t)$, dividing through by dt and then passing to the limit $dt\to 0^+$, we obtain

$$\frac{\partial}{\partial t}P(v_n,t)=P(v_{n+1},t)W_-(v_{n+1})-P(v_n,t)W_+(v_n)$$
$$+P(v_{n-1},t)W_+(v_{n-1})-P(v_n,t)W_-(v_n). \tag{34}$$

Substitution of the formulas (31) for the functions $W_\pm(v)$, followed by some simple algebraic rearrangement, then gives

$$\frac{\partial}{\partial t}P(v_n,t)=(B/A)[v_{n+1}P(v_{n+1},t)-v_{n-1}P(v_{n-1},t)]$$
$$+B[P(v_{n-1},t)-2P(v_n,t)+P(v_{n+1},t)]$$
$$(-N\leqslant n\leqslant N). \tag{35}$$

In preparation for taking the limit $N\to\infty$, we use the fact that $\Delta=A/N$ to write Eq. (35) as

$$\frac{\partial}{\partial t}P(v_n,t)=\frac{2B}{N}\left(\frac{v_{n+1}P(v_{n+1},t)-v_{n-1}P(v_{n-1},t)}{2\Delta}\right)$$
$$+\frac{BA^2}{N^2}\left(\frac{P(v_{n-1},t)-2P(v_n,t)+P(v_{n+1},t)}{\Delta^2}\right)$$
$$(-N\leqslant n\leqslant N). \tag{36}$$

Now, as mentioned earlier, we intend to arrange things so that $\Delta\to 0$ and $A\to\infty$ when $N\to\infty$. Assuming that those conditions are fulfilled, then the limit $N\to\infty$ brings Eq. (36) into the form of the partial differential equation

$$\frac{\partial}{\partial t}P(v,t)=C_1\frac{\partial}{\partial v}[vP(v,t)]+C_2\frac{\partial^2}{\partial v^2}P(v,t)$$
$$(-\infty\leqslant v\leqslant\infty), \tag{37}$$

where we have put

$$C_1\equiv\lim_{N\to\infty}\frac{2B}{N}, \tag{38a}$$

$$C_2\equiv\lim_{N\to\infty}\frac{BA^2}{N^2}. \tag{38b}$$

As may be verified by direct differentiation, the solution to Eq. (37) that satisfies the required initial condition $P(v,0)=\delta(v-V_0)$ is

$$P(v,t)=[2\pi(C_2/C_1)(1-e^{-2C_1t})]^{-1/2}$$
$$\times\exp\left(-\frac{(v-V_0e^{-C_1t})^2}{2(C_2/C_1)(1-e^{-2C_1t})}\right). \tag{39}$$

Comparing the form of this solution to Eq. (4), we immediately deduce that

$$V(t)=\mathbf{N}(V_0e^{-C_1t},(C_2/C_1)(1-e^{-2C_1t})). \tag{40}$$

We now observe that this solution will be *physically* sensible only if C_1 and C_2 are both finite, positive numbers. But according to Eq. (38a), C_1 can be finite and positive only if $B\propto N$ as $N\to\infty$. And given that, Eq. (38b) tells us that C_2 can be finite and positive only if $A^2\propto N$ as $N\to\infty$. Thus we conclude that our two model parameters A and B must scale with N according to

$$N\to\infty:\begin{cases}A=aN^{1/2} & \text{(41a)}\\ B=bN & \text{(41b)}\end{cases},$$

where the positive constants a and b are now our *new* model parameters.

Before proceeding, let us verify that these scaling formulas are satisfactory. First, Eq. (41a) implies that A does indeed satisfy the required condition $A\to\infty$ as $N\to\infty$. Second, Eqs. (41a) and (25) together give, for $N\to\infty$,

$$\Delta=aN^{-1/2}, \tag{42}$$

which in turn implies that Δ satisfies the required condition $\Delta\to 0$ as $N\to\infty$. And finally, the implication of Eq. (41b) that B increases with N is entirely plausible; because, increasing N decreases the step size Δ, and that in turn should increase the probability (28).

Substituting Eqs. (41) into Eqs. (38), we find that

$$C_1=2b;\quad C_2=ba^2. \tag{43}$$

Therefore, our formula (40) for $V(t)$ becomes

$$V(t)=\mathbf{N}(V_0e^{-2bt},(a^2/2)(1-e^{-4bt})). \tag{44}$$

This is the solution of our jump Markov process model of Brownian motion in the continuum limit of $\Delta\to 0$ and $A\to\infty$.

When we compare our model solution (44) with the solution (17) of the Langevin Eq. (1), we observe that the two solutions will be identical provided that

$$2b=\frac{\gamma}{M};\quad \frac{a^2}{2}=\frac{f^2}{2\gamma M}.$$

Solving these two relations simultaneously for γ and f, we conclude that our jump Markov process model, in the continuum limit, *predicts the existence* of a dissipative drag force $-\gamma V(t)$ and a zero-mean fluctuating force $f\Gamma(t)$, where γ and f are given in terms of our model parameters a and b by

$$\gamma=2Mb, \tag{45a}$$

$$f=Ma(2b)^{1/2}. \tag{45b}$$

To this point, we have not invoked the thermodynamic requirement (2). To do so, we first note that the $t\to\infty$ limit of Eq. (44) gives

$$V(\infty)=\mathbf{N}(0,a^2/2).$$

So satisfaction of requirement (2) demands that $a^2/2=k_BT/M$, or

$$a=(2k_BT/M)^{1/2}. \tag{46}$$

With this result, our formulas (45) for γ and f become

$$\gamma=2Mb, \tag{47a}$$

$$f=(4Mbk_BT)^{1/2}. \tag{47b}$$

Now only the single model parameter b ($=B/N$) remains. The fact that γ and f both increase with b, and vanish only when $b=0$, is an expression of the fluctuation-dissipation theorem: the dissipative drag force and the zero-mean fluctuating force are concomitants.

IV. CONCLUDING REMARKS

The jump Markov process model of Brownian motion presented in Sec. III is based on the premise that the particle's velocity evolves with time according to the probabilistic rules set forth in Eqs. (26) and (31). Although those rules are obviously somewhat contrived, they nevertheless describe reasonably well the sort of behavior that we should expect from the hypothesis that a fluid consists of "molecules in random motion." By solving this discrete-state random walk model for $V(t)$ in the continuum-state limit, and then comparing the solution with the solution to the traditional Langevin Eq. (1), we were able to demonstrate the mathematical equivalence of these two approaches to Brownian motion. This equivalence can be regarded as providing a microphysical rationale for the key assumption of the Langevin Eq. (1), namely, that the net effect of the surrounding fluid on the particle can be described as a simple superposition of a steady dissipative drag force and a zero-mean temporally uncorrelated fluctuating force. Once this assumption has been justified, the Langevin Eq. (1), viewed in its natural setting of continuous Markov process theory, provides a truly exquisite mathematical description of the Brownian motion phenomenon.

APPENDIX: INDUCTION PROOF OF EQ. (16)

Equation (15) shows that Eq. (16) is valid for $K=1$, so it remains only to show that Eq. (16) implies validity when K is replaced by $K+1$. We have

$$V((K+1)dt)=V(Kdt+dt)$$

$$=\left(1-\frac{\gamma\,dt}{M}\right)V(Kdt)+\left(\frac{f}{M}\right)\mathbf{N}(0,1)(dt)^{1/2}\quad\text{[by Eq. (10)]}$$

$$=\left(1-\frac{\gamma\,dt}{M}\right)V(Kdt)+\mathbf{N}\left(0,\left(\frac{f}{M}\right)^2dt\right)\quad\text{[by Eq. (5)]}$$

$$=\left(1-\frac{\gamma\,dt}{M}\right)\mathbf{N}\left(V_0\left(1-\frac{\gamma dt}{M}\right)^K,\frac{f^2}{2\gamma M}\left[1-\left(1-\frac{2\gamma dt}{M}\right)^K\right]\right)+\mathbf{N}\left(0,\frac{f^2}{2\gamma M}\left[1-\left(1-\frac{2\gamma dt}{M}\right)\right]\right)$$

[by Eqs. (16) and (14)]

$$V[(K+1)dt]=\mathbf{N}\left(V_0\left(1-\frac{\gamma dt}{M}\right)^{K+1},\frac{f^2}{2\gamma M}\left\{\left(1-\frac{\gamma dt}{M}\right)^2\left[1-\left(1-\frac{2\gamma dt}{M}\right)^K\right]+\left[1-\left(1-\frac{2\gamma dt}{M}\right)\right]\right\}\right)$$

[by Eqs. (5) and (6)].

Now since dt is an infinitesimal, then

$$\left(1-\frac{\gamma dt}{M}\right)^2=1-\frac{2\gamma dt}{M}\equiv x.$$

So the variance argument in last expression is equal to

$$\frac{f^2}{2\gamma M}[x(1-x^K)+(1-x)]=\frac{f^2}{2\gamma M}(1-x^{K+1}).$$

Therefore,

$$V((K+1)dt)=\mathbf{N}\left(V_0\left(1-\frac{\gamma dt}{M}\right)^{K+1},\ \frac{f^2}{2\gamma M}\left[1-\left(1-\frac{2\gamma dt}{M}\right)^{K+1}\right]\right).$$

Since this expression is precisely Eq. (16) with K replaced by $K+1$, then our induction proof of that equation is completed.

[1]F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw–Hill, New York, 1965), Chap. 15.
[2]N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981), Chap. VIII.
[3]C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences* (Springer-Verlag, Berlin, 1983), Chap. 1.
[4]D. T. Gillespie, *Markov Processes: An Introduction for Physical Scientists* (Academic, San Diego, 1991), Chap. 3.
[5]Or, see Ref. 4, Chap. 1.
[6]But notice that the integral of a continuous Markov process cannot itself be a continuous Markov process, because the integral by definition *does* have a proper derivative.
[7]See Ref. 3, Sec. 1.2 for discussions of, and quotations from, the original Brownian motion papers of Einstein and Langevin.
[8]Reference 4, Sec. 4.5.