Physical time and physical space in general relativity

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This paper comments on the physical meaning of the line element in general relativity. We emphasize that, generally speaking, physical spatial and temporal coordinates (those with direct metrical significance) exist only in the immediate neighborhood of a given observer, and that the physical coordinates in different reference frames are related by Lorentz transformations (as in special relativity) even though those frames are accelerating or exist in strong gravitational fields.
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"Now it came to me:...the independence of the gravitational acceleration from the nature of the falling substance, may be expressed as follows: In a gravitational field (of small spatial extension) things behave as they do in space free of gravitation,... This happened in 1908. Why were another seven years required for the construction of the general theory of relativity? The main reason lies in the fact that it is not so easy to free oneself from the idea that coordinates must have an immediate metrical meaning." [1] *Albert Einstein*

I. INTRODUCTION

I often start my lecture on the meaning of the line element in general relativity (GR) with the provocative statement, "In general relativity the Galilean transformation of classical mechanics is just as valid as the Lorentz transformation of special relativity, because *all* space–time coordinate transformations are equally valid in general relativity" (the democracy of coordinate systems). This statement invariably sparks a heated discussion about what is physically significant in GR and serves to introduce the following comments on the meaning of the spacetime metric. I claim no new results in the following—only that this approach to understanding the metric has been useful for the author and may be of some help to others who attempt the daunting task of teaching GR to undergraduates.

II. REFERENCE FRAMES

For our purpose, a reference frame will be defined as a collection of fiducial observers distributed over space and moving in some prescribed manner. Each fiducial observer (FO) is assigned space coordinates x^i $(i=1,2,3)$ that do not change. [The FOs are "at rest" ($x^i=$constant) in these coordinates.] Each FO carries a standard measuring rod and a standard clock that measure proper length and proper time at his/her location. The basic data of GR are the results of *local measurements* made by the FOs. (These are the "10,000 local witnesses" in the words of Taylor and Wheeler.[2])

In special relativity, the FOs usually sit on a rigid lattice of Cartesian coordinates in inertial space; there is a different set of FOs in a different inertial reference frame. In Schwarzschild space, the FOs reside at constant values of the Schwarzschild space coordinates (r,θ,ϕ), and, in an expanding universe, the FOs sit at constant values of the comoving Robertson–Walker space coordinates (r,θ,ϕ). In this case, the distance between FOs changes with time even though each FO is "at rest" ($r,\theta,\phi=$constant) in these coordinates.

III. PHYSICAL SPACE

What is the distance $d\ell$ between neighboring FOs separated by coordinate displacement dx^i when the line element has the general form

$$ds^2=g_{\alpha\beta}\,dx^\alpha\,dx^\beta? \tag{1}$$

Einstein's answer is simple.[3] Let a light (or radar) pulse be transmitted from the first FO (observer A) to the second FO (observer B) where it is reflected and returns to A. If the time between transmission and reception of the reflected pulse (as measured by A's standard clock) is $d\tau_A$, then the Einstein distance between A and B is defined by the radar formula $d\ell=c\,d\tau_A/2$. Using the general metric (1) for light $(ds^2=0)$, we find in the Appendix that this distance $d\ell$ is given by the spatial metric

$$d\ell^2=\gamma_{ij}\,dx^i\,dx^j, \tag{2}$$

where

$$\gamma_{ij}=g_{ij}-\frac{g_{0i}g_{0j}}{g_{00}}. \tag{3}$$

We call Eq. (2) the metric of *physical space* for the given reference frame. It measures proper distance at the point of interest, that is, local radar distance is the same as proper distance (the distance measured with a standard ruler).

Notice that, when $g_{0i}\neq 0$, the spatial metric components γ_{ij} are not simply the spatial components g_{ij} of the full metric $g_{\alpha\beta}$. The example of Einstein's rotating disk with measuring rods along the diameter and circumference, as depicted in Fig. 1, nicely illustrates the use of Eqs. (2) and (3).

In inertial space with polar coordinates (r_0,θ_0) and origin at the center of the disk, the space–time metric reads $ds^2=-c^2\,dt^2+dr_0^2+r_0^2\,d\theta_0^2$. A transformation to the frame rotating with the disk at angular velocity Ω ($r=r_0$, $\theta=\theta_0+\Omega t$) puts the metric into the form

$$ds^2=-\left[1-\left(\frac{\Omega r}{c}\right)^2\right]c^2\,dt^2+2\left(\frac{\Omega}{c}\right)r^2(c\,dt)d\theta+dr^2+r^2\,d\theta^2, \tag{4}$$

with nonzero metric components $g_{00}=-[1-(\Omega r/c)^2]$, $g_{0\theta}=g_{\theta 0}=-\Omega r^2/c$, $g_{rr}=1$, and $g_{\theta\theta}=r^2$ on coordinates

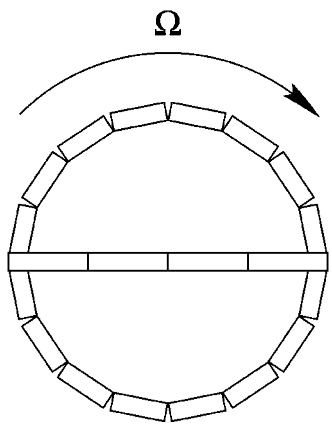

Fig. 1. Einstein disk rotating about its center O with angular velocity Ω (viewed from inertial space). Standard measuring rods are laid end-to-end along the diameter and circumference of the disk and move with the disk.

$x^\mu=(ct,r,\theta)$. A glance at the metric (4) might miss that the spatial geometry in this reference frame is non-Euclidean. But application of Eqs. (2) and (3) gives the spatial metric

$$d\ell^2=dr^2+\frac{r^2\,d\theta^2}{1-(\Omega r/c)^2}, \tag{5}$$

which clearly shows no change of radial distance ($dr=dr_0$) and an increase in measured circumferential distance [from $r\,d\theta$ to $r\,d\theta/(1-v^2/c^2)^{1/2}$] due to the Lorentz contraction of measuring rods placed end-to-end on the circumference and hence moving in the direction of their lengths with speed $v=\Omega r$.

When three-space is flat, we can transform to Cartesian space coordinates for which the spatial metric takes the Euclidean form $d\ell^2=\delta_{ij}\,dx^i\,dx^j$, and the coordinates x^i measure distance directly. We shall refer to such coordinates as "physical" space coordinates. But, when three-space is curved, global physical coordinates do not exist. We can, however, always transform to local Cartesian coordinates $d\tilde{x}^1$, $d\tilde{x}^2$, $d\tilde{x}^3$ in the immediate neighborhood of any chosen FO, in which case the local space metric takes the form

$$d\ell^2=(d\tilde{x}^1)^2+(d\tilde{x}^2)^2+(d\tilde{x}^3)^2. \tag{6}$$

(Here and in the following we use a tilde to denote differentials that may not be exact and, therefore, may not be integrable to global functions.) The proper distances $d\tilde{x}^1$, $d\tilde{x}^2$, $d\tilde{x}^3$ are local physical coordinates measured from the given FO, and are written as differentials to suggest that they cannot be large. They are finite but small enough that curvature effects are negligible over the region they span. When the spatial metric is diagonal to begin with [$d\ell^2=\gamma_{11}(dx^1)^2+\gamma_{22}(dx^2)^2+\gamma_{33}(dx^3)^2$], the local physical spatial coordinates are

$$d\tilde{x}^1=\sqrt{\gamma_{11}}dx^1, \tag{7a}$$

$$d\tilde{x}^2=\sqrt{\gamma_{22}}dx^2, \tag{7b}$$

$$d\tilde{x}^3=\sqrt{\gamma_{33}}dx^3. \tag{7c}$$

But such differentials are not exact and cannot be integrated to give global Cartesian coordinates (if they could, the space would not be curved).

IV. PHYSICAL TIME

Physical time $\tilde{t}$ in the immediate neighborhood of a particular fiducial observer O is that function of position coordinates x^i and coordinate time $x^0=ct$ that, when used to measure speed $d\ell/d\tilde{t}$, places the speed of light (the one-way speed) at the invariant value c in all directions.

Now the general line element (1) can be written in terms of the physical space metric (2) as

$$ds^2=-c^2\,d\tilde{t}^2+d\ell^2, \tag{8}$$

where $d\tilde{t}$ is defined by

$$d\tilde{t}\equiv\sqrt{-g_{00}}dt-\frac{g_{0i}\,dx^i}{c\sqrt{-g_{00}}}. \tag{9}$$

The proof of Eq. (8) is by direct expansion of this equation using Eqs. (2), (3), and the definition (9) of $d\tilde{t}$. We have used the notation $d\tilde{t}$ because for light ($ds^2=0$), Eq. (8) shows that $d\tilde{t}$ is the time differential that gives the speed of light $d\ell/d\tilde{t}$ the value c in all directions, that is, $d\tilde{t}$ is indeed the differential of physical time.

At observer O [at $dx^i=0$ in Eq. (9)], the physical time increases at the same rate as the proper time τ_0 at that location ($d\tilde{t}=\sqrt{-g_{00}}dt=d\tau_0$), and, at the coordinate displacement dx^i from O, the physical time is synchronized with the clock at O by the Einstein synchronization procedure.[3] By the Einstein synchronization procedure, we mean the process in which, when the clock at O reads time $\tilde{t}_0$ and this time is transmitted over the coordinate displacement dx^i of length $d\ell$ at the speed of light, the physical time at the end of this journey has the value $\tilde{t}=\tilde{t}_0+dl/c$, that is, it contains the retardation correction $d\ell/c$ required to account for the finite and invariant propagation speed of the time signal. [That the definition (9) of the physical time differential $d\tilde{t}$ is consistent with the Einstein synchronization procedure follows from the fact that $d\tilde{t}$ is the time differential that gives the speed of light the value c in all directions.] Another way of saying this is that, in the immediate neighborhood of the given fiducial observer O, the condition $d\tilde{t}=0$ (or $\tilde{t}=$constant) defines a hypersurface of simultaneity.

The all important point of this discussion is that, as a temporal coordinate, the physical time $\tilde{t}$ is a very special one. It is the time actually used by the fiducial observer for measurements in his local reference frame. If it were not so, this observer would not measure the local light speed c. Thus the notion that all fiducial observers measure the same speed c for light (Einstein's postulate) is equivalent to the notion that all fiducial observers use the physical time $\tilde{t}$ in their local reference frames for the measurement of that speed. Also notice that, when we use local Cartesian coordinates $d\tilde{x}^i$ at a particular FO, the local line element (8) takes the Minkowski form

$$ds^2=-c^2\,d\tilde{t}^2+(d\tilde{x}^1)^2+(d\tilde{x}^2)^2+(d\tilde{x}^3)^2. \tag{10}$$

Equation (10) does not imply that the local frame $d\tilde{x}^\mu$ is inertial. The observer O can be moving arbitrarily, and there can be a gravitational acceleration in this frame. [The local metric $g_{\alpha\beta}(O)=\eta_{\alpha\beta}$ is Minkowskian, but the derivatives of the metric $\partial g_{\alpha\beta}/\partial \tilde{x}^{\mu}$ are not necessarily zero at this point.]

We use phrases such as "the local reference frame" or "immediate neighborhood," because the differential of physical time in Eq. (9) may not be an exact differential. If $d\tilde{t}$ is not exact, Eq. (9) defines the physical time only in a small neighborhood about O. That is to say, unless they are very close to one another, different FOs use different physical times for their interpretation of nature.

Only when the differential of physical time is exact does there exist a single global physical temporal coordinate $\tilde{t}(x^0,x^i)$ for all FOs. In this case, specific values of the physical time $[\tilde{t}(x^0,x^i)=\text{constant}]$ define global hypersurfaces of simultaneity in the reference frame under consideration. Notice that, when $d\tilde{t}$ is exact, its integral $\tilde{t}(x^0,x^i)$ is a global temporal coordinate for which the line element (8), namely

$$ds^2=-c^2\,d\tilde{t}^2+\gamma_{ij}\,dx^i\,dx^j, \tag{11}$$

is in Gaussian normal form, that is, $g_{00}=-1$ and $g_{0i}=g_{i0}=0$. This form of the metric implies that the FOs of such a reference frame are freely falling [$x^i=\text{constant}$ are geodesics of the metric (11)], and the clocks of all the FOs run at the same rate and remain synchronized (no gravitational time dilation in this reference frame). These are the "comoving" (or "synchronous") reference frames.[4] Only in such frames does a global physical time exist. Examples include the Robertson–Walker metrics of cosmology, the comoving coordinates used by Oppenheimer and Snyder in their early studies of gravitational collapse,[5] and the inertial frames of special relativity. In all other reference frames (other than comoving ones) the differential of physical time is not exact, and there does not exist a global physical time coordinate. Gravitational time dilation is a symptom of the lack of a global physical time $\tilde{t}(x^0,x^i)$.

When the physical time differential is not exact, it often is possible to make it exact by means of an integrating factor [$d\overline{t}=d\tilde{t}/\mathcal{R}$ is exact for some function $\mathcal{R}(x^0,x^i)$]. In this case, integration gives what we call a "synchronous temporal coordinate" $\overline{t}(x^0,x^i)$, and the metric (8) is written in terms of this time coordinate as

$$ds^2=-\mathcal{R}^2c^2\,d\overline{t^2}+\gamma_{ij}dx^i\,dx^j, \tag{12}$$

where $\mathcal{R}=d\tau_0/d\overline{t}$ is the rate of a fiducial clock on the time scale $\overline{t}$. Clearly $\mathcal{R}$ describes gravitational time dilation. A most important feature of the metric (12) is that equal values of the temporal coordinate $\overline{t}$, say $\overline{t}_A=\overline{t}_B$ for widely separated events A and B, implies that these events occur simultaneously (according to the Einstein definition) in the given reference frame. This result follows because the condition for neighboring events to be simultaneous ($d\tilde{t}=0$) can be written as $d\overline{t}=0$ and $d\overline{t}$ is integrable. Hence, the locus $\overline{t}=\text{constant}$ is a global hypersurface of simultaneity. Examples of this case include the Schwarzschild metric in Schwarzschild coordinates and any other time-orthogonal metric with metric coefficients $g_{0i}=0$.

The final and probably most interesting case occurs when the physical time differential (9) is not exact and cannot be made exact by applying an integrating factor. In such reference frames, the question of whether widely separated events are simultaneous has no unambiguous answer. In this case, there does not exist *any* temporal coordinate $t(x^0,x^i)$ with the property that equal values of this coordinate for separated events implies simultaneity of those events. We call such frames "asynchronous" frames. It is bad enough (or "good enough" because it is true) that judgments of simultaneity are different in different reference frames (as in special relativity), but for the concept of simultaneity of separated events to become meaningless in a single reference frame (even a rigid one) is even more difficult to swallow (but equally true). That our notion of simultaneity at a distance loses meaning in an asynchronous reference frame is, in the author's view, one of the most counterintuitive ideas in all of physics, and surprisingly little emphasis is given to it in many textbooks on general relativity.

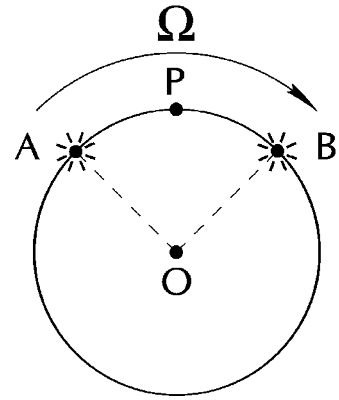

Fig. 2. On Einstein's rotating disk, two firecrackers explode at points A and B on the disk's edge equidistant from observers O and P. Observer O sees the two flashes at the same instant and concludes the explosions occurred simultaneously. Observer P, moving with the disk, sees the flash from B before that from A and concludes the explosions are not simultaneous. Thus, the question of simultaneity is ambiguous in this rigid rotating reference frame.

We can begin to understand how the concept of simultaneity becomes ambiguous by returning to Einstein's rotating disk. Consider an observer at the center O of the disk and another at P on the edge of the rotating disk. Let two firecrackers explode at points A and B on the disk's edge equidistant from observers O and P as in Fig. 2, and let the explosions be timed so that observer O sees the flashes at the same instant and concludes, therefore, that the flashes occur simultaneously. Observer P, who is also "at rest" in the rotating frame, sees the light from B before that from A because (from the vantage point of inertial space) he is moving toward the light coming from B and away from the light coming from A. Observer P concludes that the firecracker at B exploded before the one at A because he observed it first and the points A and B (where burn marks are left on the disk) are at equal distance from him. In this way we see how two observers, both at rest in a rigid reference frame, can disagree about the simultaneity of two events, and thus render the concept of simultaneity ambiguous.

The hallmark of an asynchronous reference frame is that the metric components g_{0i} not be zero, that is, that the metric tensor not be time-orthogonal. Examples of such metrics include rotating reference frames, the Kerr metric[6] in Boyer–Lindquist coordinates[7] representing a rotating black hole, and the Gödel metric[8] representing a model rotating uni-

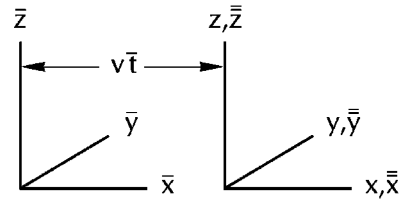

Fig. 3. Inertial reference frame labeled K (and $\bar{\bar{K}}$) with coordinates x, y, z (or $\bar{\bar{x}}$, $\bar{\bar{y}}$, $\bar{\bar{z}}$) moves out along the $\bar{x}$-axis of inertial frame $\bar{K}$ with velocity $v=d\bar{x}/d\bar{t}$.

verse. There is, of course, a time coordinate t in these metrics, but we must understand that the equality of this time, say $t_A=t_B$ for separated events A and B, does not mean that these events are simultaneous. (One wonders why such a thing is called a "time coordinate" at all, because it does not have the most fundamental features one associates with the word "time.") No doubt it is the conceptual problems surrounding the failure of simultaneity in such frames that motivates many authors to eliminate the metric terms containing g_{oi} by transforming to a different reference frame in which these terms are zero. But, if one wishes to work in a rotating frame (such as the frame rigidly attached to earth), or if one wishes to study certain "frame dragging" effects, these terms are necessarily present and give rise to such interesting effects as Coriolis forces, the gravitomagnetic field, and the Sagnac effect.[9,10] In fact, the experimental demonstration of the Sagnac effect (the different light travel times for propagation in opposite directions around a closed path in a rotating frame) using a ring-laser gyro[11] or the global positioning system[12] may be interpreted as a verification of the failure of simultaneity in rotating reference frames, because such an effect would be inconsistent with the invariant light speed c if a global physical time (or even a global synchronous temporal coordinate) existed.

V. THE LORENTZ TRANSFORMATION

Let us return now to the statement made in Sec. I that "In general relativity the Galilean transformation is just as valid as the Lorentz transformation." Specifically, consider the Galilean transformation

$$x=\bar{x}-v\bar{t}, \tag{13a}$$

$$y=\bar{y}, \tag{13b}$$

$$z=\bar{z}, \tag{13c}$$

$$t=\bar{t}, \tag{13d}$$

from an inertial frame $\bar{K}$ with metric,

$$ds^2=-c^2d\bar{t}^2+d\bar{x}^2+d\bar{y}^2+d\bar{z}^2, \tag{14}$$

to another inertial frame K as depicted in Fig. 3. The time $\bar{t}$ is the time on synchronized clocks at rest in frame $\bar{K}$, and frame K moves in the positive direction along the $\bar{x}$ axis at speed $v=d\bar{x}/d\bar{t}$ as measure by the FOs in $\bar{K}$.

We sometimes hear that Galileo's transformation (13) is "wrong" and Einstein's transformation (the Lorentz transformation) is "right." But surely this statement cannot be correct when general relativity allows arbitrary space–time coordinate transformations, and Eq. (13) is a perfectly valid description of the motion of frame K from the point of view of the fiducial observers in $\bar{K}$. What then is it that makes the Lorentz transformation preferable to the Galilean transformation? The answer is that the Lorentz transformation is expressed in terms of the physical time and physical space coordinates in frame K instead of the coordinates (ct,x,y,z) used in the Galilean transformation, which are actually physical times and physical distances in frame $\bar{K}$. This property of the Lorentz transformation is convenient (but not necessary) for the calculation of physical quantities in the new frame.

Let us show this explicitly. The metric (14), written in terms of the Galilean coordinates $x^\mu=(ct,x,y,z)$ reads

$$ds^2=-(1-v^2/c^2)c^2\,dt^2+2\left(\frac{v}{c}\right)dx(c\,dt)+dx^2+dy^2+dz^2, \tag{15}$$

that is, the nonzero metric components are $g_{00}=-(1-v^2/c^2)$, $g_{0x}=g_{x0}=v/c$, and $g_{xx}=g_{yy}=g_{zz}=1$. Therefore the spatial metric, Eqs. (2) and (3), is

$$d\ell^2=\frac{dx^2}{1-v^2/c^2}+dy^2+dz^2. \tag{16}$$

Equation (16) can be expressed in terms of the physical space differentials $d\bar{\bar{x}}=d\tilde{x}=dx/\sqrt{1-v^2/c^2}$, $d\bar{\bar{y}}=d\tilde{y}=dy$, and $d\bar{\bar{z}}=d\tilde{z}=dz$ which, in this case, are exact differentials that integrate to global physical space coordinates

$$\bar{\bar{x}}=\frac{x}{\sqrt{1-v^2/c^2}}, \tag{17a}$$

$$\bar{\bar{y}}=y, \tag{17b}$$

$$\bar{\bar{z}}=z \tag{17c}$$

for frame $\bar{\bar{K}}(=K)$. We can also express the metric (15) in frame $\bar{\bar{K}}$ in terms of the physical time differential in this frame, Eq. (9), which also is an exact differential that integrates to the global physical time

$$\bar{\bar{t}}=\tilde{t}=\sqrt{1-v^2/c^2}\,t-\frac{vx}{c^2\sqrt{1-v^2/c^2}}. \tag{18}$$

If we substitute the transformations to physical variables (17) and (18) into the Galilean transformation (13), we obtain the Lorentz transformation

$$\bar{\bar{x}}=\frac{\bar{x}-v\bar{t}}{\sqrt{1-v^2/c^2}}, \tag{19a}$$

$$\bar{\bar{y}}=\bar{y}, \tag{19b}$$

$$\bar{\bar{z}}=\bar{z}, \tag{19c}$$

$$\bar{\bar{t}}=\frac{\bar{t}-v\bar{x}/c^2}{\sqrt{1-v^2/c^2}}. \tag{19d}$$

Observe that the Galilean transformation, far from being wrong, is a fully correct kinematic description of the motion of frame $\bar{\bar{K}}(=K)$ in terms of the space and time variables of frame $\bar{K}$, and as soon as we express the Galilean transformation in terms of the physical time $\bar{\bar{t}}$ and physical space coordinates $\bar{\bar{x}}$, $\bar{\bar{y}}$, $\bar{\bar{z}}$ of frame $\bar{\bar{K}}$, it becomes the Lorentz transformation.

VI. THE UNIVERSAL LORENTZ TRANSFORMATION

The local physical coordinates in different reference frames are related by a Lorentz transformation regardless of the motion of those frames. To see that this is so, consider two arbitrary reference frames (two sets of fiducial observers in relative motion) with space–time coordinates x^μ and y^μ connected by the coordinate transformation $y^\mu = y^\mu(x^\nu)$ or its inverse $x^\mu = x^\mu(y^\nu)$. At a particular fiducial observer X in the x-frame, we construct physical coordinates $d\tilde{x}^\mu = (c\,d\tilde{t}, d\tilde{x}^1, d\tilde{x}^2, d\tilde{x}^3)$ for which the metric (8) at this point takes the Minkowski form $ds^2 = \eta_{\alpha\beta}\, d\tilde{x}^\alpha\, d\tilde{x}^\beta$. We think of $d\tilde{x}^\mu$ as finite but small coordinate values covering a limited neighborhood about observer X. At the same event in the y-frame we similarly construct local physical coordinates $d\tilde{y}^\mu$ measured from the fiducial observer Y of that frame who, at the instant under consideration, is at the same place as observer X. The metric at observer Y is also Minkowskian in the local physical coordinates of this observer, $ds^2 = \eta_{\alpha\beta}\, d\tilde{y}^\alpha\, d\tilde{y}^\beta$, and there may be a gravitational field in either or both of these frames.

The overall transformation (from local physical coordinates $d\tilde{x}^\mu$ to coordinates x^α, then to coordinates y^ν, and finally to local coordinates $d\tilde{y}^\beta$) takes the Minkowski tensor $\eta_{\mu\nu}$ into the same Minkowski tensor and, therefore, can only be a Lorentz transformation Λ^α_β:

$$d\tilde{y}^\alpha = \Lambda^\alpha_\beta\, d\tilde{x}^\beta, \tag{20}$$

with

$$\eta_{\alpha\beta} = \eta_{\mu\nu}\Lambda^\mu_\alpha \Lambda^\nu_\beta. \tag{21}$$

Hence, the *differential* Lorentz transformation (20) is not limited to inertial frames, but applies to arbitrary spacetime coordinate transformations. (It applies to accelerating frames and frames in arbitrarily strong gravitational fields.) It follows that all of the tensors of special relativity in Minkowski coordinates, such as the electromagnetic field tensor $F^{\mu\nu}$ ($F^{01}=E^1$, $F^{02}=E^2$, $F^{03}=E^3$, $F^{12}=B^3$, $F^{23}=B^1$, $F^{31}=B^2$, $F^{\alpha\beta} = -F^{\beta\alpha}$), the four-momentum $P^\mu = m\, d\tilde{x}^\mu / d\tau$, the stress-energy tensor $T^{\mu\nu}$, transform as in special relativity between local physical reference frames.

A. Example: Transformation to a falling reference frame in Schwarzschild space

Start with the static Schwarzschild metric,

$$ds^2 = -(1 - r_s/r)c^2\, dt^2 + \frac{dr^2}{1 - r_s/r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{22}$$

describing a nonrotating black hole of Schwarzschild radius $r_s = 2GM/c^2$ in the Schwarzschild reference frame with coordinates (ct, r, θ, ϕ), and transform to a frame that falls radially inward from rest at infinity. For a fiducial observer at rest in Schwarzschild coordinates ($r, \theta, \phi =$ constants), the local physical coordinate differentials are

$$d\tilde{t} = \sqrt{1 - r_s/r}\, dt, \tag{23a}$$

$$d\tilde{x} = r\, d\theta, \tag{23b}$$

$$d\tilde{y} = r \sin\theta\, d\phi, \tag{23c}$$

$$d\tilde{z} = dr / \sqrt{1 - r_s/r}, \tag{23d}$$

where we have taken the $\tilde{z}$-axis in the outward radial direction, the $\tilde{x}$-axis in the $d\theta$-direction, and the $\tilde{y}$-axis in the $d\phi$-direction. We think of $c\,d\tilde{t}$, $d\tilde{x}$, $d\tilde{y}$, $d\tilde{z}$ as finite but small Minkowski coordinates measured from the given fiducial observer. Clearly the Schwarzschild metric (22) takes the form $ds^2 = -c^2\, d\tilde{t}^2 + d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2$ in these local coordinates.

Now consider freely falling fiducial observers (a different reference frame) who fall radially inward from rest at $r = \infty$. For this initial condition, the time equation of motion derived from the metric (22) reads

$$\left(1 - \frac{r_s}{r}\right)\frac{dt}{d\bar{t}} = 1, \tag{24}$$

where $d\bar{t}$ is the differential of proper time at the falling observer. The radial equation of motion (once integrated) is

$$\frac{dr}{d\bar{t}} = -\sqrt{\frac{r_s c^2}{r}} = -\sqrt{\frac{2GM}{r}}. \tag{25}$$

Fortuitously, this relativistic equation is the same as the corresponding Newtonian equation. [If readers are unfamiliar with the derivation of Eqs. (24) and (25), they may consult Ref. 2 where these results are simply derived as Eqs. (19) and (32) on pp. 3–22.] If we use Eqs. (23a), (23d), and (24) in Eq. (25), we find that the physical velocity of the fall is

$$v = \frac{d\tilde{z}}{d\tilde{t}} = -\sqrt{\frac{r_s c^2}{r}}. \tag{26}$$

Equation (26) is the velocity that enters the differential Lorentz transformation

$$d\tilde{\tilde{t}} = \frac{d\tilde{t} - v\, d\tilde{z}/c^2}{\sqrt{1 - v^2/c^2}}, \tag{27a}$$

$$d\tilde{\tilde{x}} = d\tilde{x}, \tag{27b}$$

$$d\tilde{\tilde{y}} = d\tilde{y}, \tag{27c}$$

$$d\tilde{\tilde{z}} = \frac{d\tilde{z} - v\, d\tilde{t}}{\sqrt{1 - v^2/c^2}}, \tag{27d}$$

to the local physical coordinates ($c\tilde{\tilde{t}}$, $d\tilde{\tilde{x}}$, $d\tilde{\tilde{y}}$, $d\tilde{\tilde{z}}$) of the falling fiducial observer. Notice that the physical fall velocity (26) approaches c as r approaches the Schwarzschild radius r_s, and, for $r < r_s$, the Lorentz transformation (27) fails because there can be no fiducial observers at rest in Schwarzschild coordinates ($r =$ constant) at these values of r.

As an example we note that, exactly as in special relativity, the differential Lorentz transformation leads to the velocity transformation

$$\tilde{\tilde{u}}^x = \frac{\tilde{u}^x \sqrt{1 - v^2/c^2}}{1 - v\tilde{u}^z/c^2}, \tag{28a}$$

$$\tilde{\tilde{u}}^y = \frac{\tilde{u}^y \sqrt{1-v^2/c^2}}{1-v\tilde{u}^z/c^2}, \tag{28b}$$

$$\tilde{\tilde{u}}^z = \frac{\tilde{u}^z - v}{1-v\tilde{u}^z/c^2}, \tag{28c}$$

from the Schwarzschild physical components $\tilde{u}^i \equiv d\tilde{x}^i/d\tilde{t}$ to the physical velocity components $\tilde{\tilde{u}}^i \equiv d\tilde{\tilde{x}}^i/d\tilde{\tilde{t}}$ in the freely falling frame.

Similarly, the physical components of the electric and magnetic fields, charge, and current densities, and the components of the stress-energy tensor all transform under the Lorentz transformation (27) as they do in special relativity, so long as we use local physical coordinates in the two frames of interest.

VII. CONCLUSION

The above arguments attempt to make it clear that an essential difference between the special and general theories of relativity is that, in the former, there exist *global* physical coordinates (the Minkowski coordinates) but, in the latter physical coordinates (coordinates with direct metrical significance and a Minkowski metric) exist only in the immediate neighborhood of each fiducial observer. But aside from this essential difference, the Lorentz transformation still applies in general relativity for transformations between local physical coordinate frames in arbitrary relative motion and in arbitrary gravitational fields. The differential Lorentz transformation is not limited to local inertial reference frames, and by using local physical coordinates, students transfer essentially all they have learned in special relativity of the transformation properties of particles and fields to the broader context of general relativity (and they are not fooled into thinking that the more general coordinate markers allowed in general relativity in any way change the relations between physical quantities expressed in the Lorentz transformation).

Recently the physics community has witnessed the publication of a truly outstanding undergraduate level textbook on general relativity by James B. Hartle. The local physical coordinates discussed in this paper are components on a local ''orthonormal bases'' in Hartle's more elegant notation.[13] The utility of working with local physical coordinates (the subject of this paper) is emphasized, in Hartle's words, by the recommendation that we should ''*calculate* in coordinate bases and *interpret* the result in orthonormal bases,'' and *our* observation that transformations between local physical coordinate frames are Lorentz transformations is equivalent to Hartle's statement that transformations between orthonormal bases are Lorentz transformations. Finally, it is worth noting that components on an orthonormal basis have traditionally been called ''physical components.''

APPENDIX: DERIVATION OF THE SPATIAL METRIC

A formula for the Einstein length $d\ell$ of the coordinate displacement dx^i between fiducial observers A and B is easily derived from the general line element,

$$ds^2 = g_{00}(dx^0)^2 + 2g_{0i}\,dx^i\,dx^0, + g_{ij}\,dx^i\,dx^j \tag{A1}$$

as follows. (This derivation is an abbreviation of a proof found in Ref. 4, pp. 233–236.) Light propagating from A to B does so in coordinate time dx^0_{out} determined by the null condition $ds^2=0$:

$$g_{00}(dx^0_{\text{out}})^2 + 2g_{0i}\,dx^i\,dx^0_{\text{out}} + g_{ij}\,dx^i\,dx^j = 0. \tag{A2}$$

The solution of this quadratic equation is

$$dx^0_{\text{out}} = \frac{-g_{0i}\,dx^i - \sqrt{(g_{0i}g_{0j} - g_{00}g_{ij})dx^i\,dx^j}}{g_{00}}. \tag{A3}$$

On the return path light travels the displacement $-dx^i$ and takes coordinate time

$$dx^0_{\text{back}} = \frac{g_{0i}\,dx^i - \sqrt{(g_{0i}g_{0j} - g_{00}g_{ij})dx^i\,dx^j}}{g_{00}}. \tag{A4}$$

The total coordinate time out and back is $dx^0 = dx^0_{\text{out}} + dx^0_{\text{back}}$, and the proper time evolved on the clock at A in this time is

$$d\tau_A = \frac{\sqrt{-g_{00}}dx^0}{c} = \frac{2}{c}\sqrt{\left(g_{ij} - \frac{g_{0i}g_{oj}}{g_{00}}\right)dx^i\,dx^j}. \tag{A5}$$

Therefore, the local radar distance $d\ell = c\,d\tau_A/2$ is the radical in Eq. (A5), and the spatial metric reads

$$d\ell^2 = \gamma_{ij}\,dx^i\,dx^j, \tag{A6}$$

with spatial metric tensor

$$\gamma_{ij} = g_{ij} - \frac{g_{0i}g_{oj}}{g_{00}}. \tag{A7}$$

[1] The quote is taken from C. W. Misner, K. S. Thorn, and J. A. Wheeler, *Gravitation* (W. H. Freeman and Company, San Francisco, CA, 1973), p. 5. It was pieced together from ''Einstein's Autobiography,'' in *Albert Einstein Philosopher-Scientist*, edited by P. A. Schilpp (Library of Living Philosophers, Evanston, IL, 1949), pp. 65–67.

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