Theorem 3.2. *Let A be a real *-algebra with unit 1 satisfying the following condition: for a in A, $a^*a = 0$ implies that $a = 0$. Suppose that A is also a real inner product space such that $\|1\| = 1$ and $\|a^*a\| \le \|a\|^2$ for all a in A. Then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.*

Proof. Let $\text{Sym}(A) := \{a \in A : a^* = a\}$, and let a belong to $\text{Sym}(A)$. Consider the subalgebra B of A that comprises all polynomials in a with real coefficients. Then B is contained in $\text{Sym}(A)$. Hence B satisfies hypotheses of Theorem 3.1. Since B is also commutative, B is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. Hence $a = \lambda 1$ for some real or complex number λ. This shows that $\text{Sym}(A)$ itself is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. Now the conclusion follows from Lemma 2.1 of [1]. ∎

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The Early History of the Ham Sandwich Theorem

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The following theorem is the well-known ham sandwich theorem: *for any three given sets in Euclidean space, each of finite outer Lebesgue measure, there exists a plane that bisects all three sets, i.e., separates each of the given sets into two sets of equal measure.* The early history of this result seems not to be well known. Stone and Tukey [2] attribute the theorem to Ulam. They say they got the information from a referee. Is this correct? The problem appears in *The Scottish Book* [1] as problem 123. The problem is posed by Steinhaus. A reference is made to the pre-World War II journal *Mathesis Polska* (Latin for "Polish Mathematics"). This journal is not easy to locate. It was finally located in the mathematics library of the University of Illinois, which seems to be the only library in the United States having the complete journal. One of the items

in the journal is a note on the ham sandwich theorem. This note clarifies the theorem's early history. Andrew Zardecki has translated the note from Polish into English for inclusion here.

The note can be divided into two parts. In the first part, the Ulam-Borsuk theorem is assumed to be correct and the ham sandwich is proved. This is a fairly easy proof. In the second part, a new proof of the Ulam-Borsuk theorem is given. That part is harder. The proof was found by Auerbach. It is claimed in the reference at the beginning of the translation to be easier than the proof given by Borsuk.

In the translation, the title "NOTES" is similar in use to that title in this MONTHLY. The title "From Topology" refers to a subdivision of NOTES that deals with topology. The translation follows.

<div align="center">

A Note on the Ham Sandwich Theorem
Hugo Steinhaus and others
From *Mathesis Polska* **XI**, (1938), pp. 26–28.

</div>

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NOTES
From Topology

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Several years ago, Mr. Ulam conjectured the following theorem: if a sphere is mapped continuously into a plane set, there is at least one pair of antipodal points having the same image; that is, they are mapped into the same point of the plane. This was proved by Mr. Borsuk in 1933 (*Fundamenta Mathematicae*, **XX**, p. 177), extending the theorem to n dimensions.

In the illustration of Mr. Steinhaus the Ulam-Borsuk theorem reads: at any moment, there are two antipodal points on the Earth's surface that have the same temperature and the same atmospheric pressure. In fact, if the temperature at point P is $x(P)$ and the pressure is $y(P)$, we have a continuous mapping of a sphere into the xy-plane. According to the theorem, there is a point P of the sphere such that its image and the image of its antipode P' are the same point of the plane; therefore, the temperature and the pressure at P and P' are the same. Of course, instead of temperature and pressure we could talk about average insolation and average rainfall, etc.

Is it always possible to bisect three solids, arbitrarily located, with the aid of an appropriate plane? This question, posed by Mr. Steinhaus, can be answered using Borsuk's theorem, as shown by Mr. Banach. [The question refers to problem 123, due to Steinhaus, in *The Scottish Book* [1].] It can be formulated as follows. Can we place a piece of ham under a meat cutter so that meat, bone, and fat are cut in halves?

For a sphere centered at S let us assign to each point P on the sphere a plane p perpendicular to SP. If we have three sets A, B, and C in space, which we are about to bisect, sets having finite (nonzero) volumetric measures a, b, and c, respectively, we consider a plane q parallel to p. We agree to call the upper side of q the side that corresponds to side P of the plane p; furthermore, we require that the measure of the part of set A located on the upper side of q be $a/2$. Such a location is always possible; if there is more than one plane q satisfying this condition, we choose the location that bisects the distance between the extreme locations. We now take the volume of the part of B on the upper side to be temperature at P, the volume of the part of C on the upper side of q to be the pressure at P. If we apply our "meteorological" theorem, we obtain two antipodal points P and P' with the same temperature and pressure. However, the same plane q_0 corresponds to the antipodal points; therefore, it bisects not only A, but also B and C, Q.E.D.

These applications (as well as the other known ones) make it interesting to obtain an easy proof of Borsuk's theorem, at least for an ordinary sphere. Such a proof was recently produced by Mr. Auerbach. Here we follow Mr. Steinhaus, who simplified the proof a little (by removing from it the general monodromy theorem).

We proceed by contradiction, assuming that the functions $x(P)$ and $y(P)$ that map a sphere into the xy-plane assign to P an image (x, y) different from the image (x', y') of its antipodal point P'. Therefore, the length of the vector whose origin is (x, y) and head is (x', y') attains a positive minimum m as P sweeps over the sphere. This vector forms an angle $f(P)$ with respect to the positive direction of the x-axis, $f(P)$ being a multivalued function of P. Different branches [in the original: determinations] of $f(P)$ differ by multiples of the full angle. If, for a given point P, we fix the angle $f(P)$ (for example, as the smallest nonnegative one), then the angle to a neighboring point Q is determined by the condition:

$$\left| f(P) - f(Q) \right| < \pi/2. \tag{1}$$

To this end [of obtaining a contradiction] we find a number d such that, for any points P and Q on the sphere whose distance is less than d, the values of functions $x(P)$ and $y(P)$ at P and Q differ by less than $m/3$. When PQ is smaller than d, (1) is satisfied for a certain (and unique) branch $f(Q)$. It is now easy to define $f(P)$ so as to be single-valued on the entire sphere. We define $f(P)$ for $P = P^0$ located at the north pole; then, using (1), we extend the definition to all points Q belonging to a zone encompassing the pole. We advance to a neighboring zone using the fact that for each point Q of the zone there exists a point P on the border of the first zone situated north of Q whose distance from Q is smaller than d, and so forth. Having defined a single-valued $f(P)$, we note that f is a continuous function, since in a neighborhood of any point there is one and only one continuous continuation of $f(P)$; namely, the continuation that satisfies equation (1); that is, the condition that we always meet. The function

$$r(P) = f(P) - f(P'), \tag{2}$$

where P' denotes the antipode of P, is also continuous. It is seen from the definition of $f(P)$, however, that $f(P')$ differs from $f(P)$ by an odd multiple of a half angle; that is,

$$r(P) = (2k_0 + 1)\pi, \tag{3}$$

where k_0 is an integer. If in formula (2) we substitute P' for P, we get $r(P') = -r(P)$. The same substitution in (3) leads to $r(P') = (2k_0 + 1)\pi$. We thus obtain $2k_0 + 1 = -2k_0 - 1$ or $k_0 = -1/2$, contradicting the fact that k_0 is an integer. In the case of a sphere or another convex solid, one can choose an interior point S and regard as antipodes the ends of diameters passing through S. The theorem will hold, which implies that, by connecting with straight lines the points having the same temperature and pressure, we fill the interior of a sphere (solid).

Mr. Auerbach noted that Mr. Borsuk's theorem has the following algebraic consequence. A system of n equation with n unknowns

$$R_1 = 0, R_2 = 0, \ldots, R_{n-1} = 0, \quad x_1^2 + x_2^2 + \cdots + x_n^2 = 1, \tag{4}$$

where the polynomials R_k have real coefficients and contain only odd powers of variables $x_1, x_2, \ldots, x_n$, has at least one real solution.

[This is the end of the translation.]

The conclusion from the foregoing is that Steinhaus conjectured the ham sandwich theorem and Banach gave the first proof, using the Ulam-Borsuk theorem. This shows that Stone and Tukey were not correct in attributing the ham sandwich theorem to Ulam. However, Ulam did make a fundamental contribution in proposing the antipodal map theorem.

Remarks. We first mention a recent application by Blair Swartz of ham sandwich theorems for fractions other than 1/2 to interface reconstruction in hydrodynamic calculations. See paragraph 20 of the web site:

http://www-troja.fjfi.cvut.cz/~liska/bbw/abs-list.html

There is a cautionary note stating that for some shapes or configurations of cells there exist n-tuples of mass fractions that cannot be simultaneously sliced from cells.

Finally, we note a paper by Steinhaus [3] that represents work Steinhaus did in Poland on the ham sandwich problem in World War II while hiding out with a Polish farm family.

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Roots Appear in Quanta

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We start with a special case. Consider an irreducible quintic polynomial

$$f(X) = X^5 + a_1 X^4 + a_2 X^3 + a_3 X^2 + a_4 X + a_5$$

with rational coefficients and with three real roots and one pair of complex conjugate roots. For example, $f(X)$ could be $X^5 - 10X + 5$.

Question. If α is a root of f, then how many roots of f lie in the field $\mathbf{Q}(\alpha)$?

The field $\mathbf{Q}(\alpha)$ is obtained by adjoining the root α to $\mathbf{Q}$. Thus $\mathbf{Q}(\alpha)$ contains at least one root of f, and of course it can contain at most five roots of f.

Answer. The number $r(f)$ of roots of f in $\mathbf{Q}(\alpha)$ is 1. We prove that, for an arbitrary irreducible polynomial f and root α, $r(f)$ divides the degree of f. For the quintic