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Finite size scale invariance

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Abstract. We develop a new approach to scale symmetry, which takes into account the possible finite cut-offs of the fields or the parameters. This new symmetry, called finite size scale symmetry: i) includes the traditional self-similarity as a limiting case, when the cut-offs are set to infinity (infinite size-system); ii) is consistent with the traditional finite size scaling approach already used in critical phenomena; iii) enables the computation of some of the universal functions appearing in the finite size scaling formulation; iv) allows scale transformations leaving the cut-offs invariant, like in the traditional renormalization approach; v) can be formulated to allow for positive or negative fields and parameters; vi) leads to new predictions about the shape of some distributions in critical phenomena or turbulence which are in very good agreement with the experimental or numerical findings.

PACS. 05.65.+b Self-organized systems – 05.70.Jk Critical point phenomena – 47.27.Gs Isotropic turbulence; homogeneous turbulence

1 Introduction

Systems or phenomena involving no characteristic scales are usually called "scale invariant". This is the case of turbulent flows and of critical phenomena (hereafter CP), two examples will shall focus on in the present paper. One interesting consequence of scale invariance is the power law shape of the physical quantities describing the system: thermodynamic quantities behave like ξ^α, where α is a critical exponent and ξ is the correlation length, which diverges at a second order critical point; in turbulent flows, the nth moment of the distribution of velocity fluctuations over a distance ℓ scales like $\ell^{\zeta(n)}$, where $\zeta(n) = n/3$ in the Kolmogorov theory [1]. The concept of scale invariant system ceases of course to be valid when characteristic scales appear in the problem, such as upper or lower cut-offs, in finite size systems. The characterization of the influence of finite size effects in scale invariant systems has been the subject of a broad literature. In most cases, finite size effects are described using more or less elaborate version of a simple dimensional argument: if a new lengthscale L (*e.g.* the size of the system) appears in the problem, the original scale invariant law will be modified by a non dimensional factor built with this new scale. In CP, the thermodynamical quantities ϕ, near the critical point, in finite geometry, should behave like:

$$\phi = \xi^\alpha f(\xi/L), \tag{1}$$

where f is an unknown function, which could be universal [2]. This description of the crossover or rounding of the singularity in CP was first formulated by Fisher [3] and Fisher and Barber [4]. It is called Finite-Size Scaling and was intensively used to analyze numerical data on finite systems (for a review, see *e.g.* Barber [5]).

In turbulence, the modifications to the structure functions induced by finite size effects can be similarly described by introducing a family of unknown functions $f_n(\ell/L)$ modulating the usual power laws. In a suitable asymptotic limit $\ell/L \ll 1$, these function can behave themselves as a power law, and modify the apparent scaling exponents of the structure functions, thereby explaining the measured deviations to the linear law $\zeta(n) = n/3$ obtained in real experiments or numerical simulations [6]. This case was discussed at length in the book of Barenblatt [7] under the name of "similarity of second kind". More recently, Dubrulle and Graner [8,9] have developed a new formalism to study the scale symmetry breaking induced by finite size effect and allow a systematic comparison of the function f_n between different systems with different statistics. In this framework, they showed how to obtain the simplest (generic) symmetry breaking function in turbulent flows [10]. Their approach was inspired from an original attempt by Nottale [11–14], who tried for the first time to include physical cut-offs (like the Planck scale or the Cosmological scale) into a self-consistent formulation of scale invariance, named scale relativity. This theory introduces new scale transformations whose main characteristic is to leave invariant the physical

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cut-off of the theory. These scale transformations are not a symmetry of the equations of physics as presently written. Rather, they are a symmetry of new, effective equations of physics, which would arise if the existence of cut-offs (linked with the postulated non-differentiability of space-time in Nottale's theory) was included in the physics itself, and not *via* boundary conditions, as it is usually done.

Interestingly enough, the property of invariance of the scale transformation at the cut-off in Nottale's theory has some counterpart in the theory of critical phenomena. The analog of Nottale's set of scale transformations is, in CP, the renormalization group. This group is built so that the cut-off scale is invariant under any renormalization [15]. Also, in finite size renormalizations, the system size L is constrained to remain non-renormalized [16], which makes it another fixed point of the renormalization group. This remark motivated us to try to apply the strategy of Nottale to critical phenomena and to turbulence. Specifically, we shall introduce simple new scale transformations leaving cut-offs invariant and examine the generic properties of thermodynamical quantities or probability distributions in a system invariant under these transformations. Our approach is different from the traditional finite size scaling assumption because it is based on the existence of a new symmetry, generalizing the usual scale invariance. We shall refer to it as "finite-size scale invariance". As we shall see, it enables explicit derivation of the finite size scaling functions. The present approach is then difference in essence from the approach followed in [8–10]: Dubrulle and Graner develop a formalism to study scale symmetry breaking induced, *e.g.* by finite size effects – they are seen as boundary conditions; here, we include the finite size effects into a new definition of the symmetry, to study invariance properties *via* this new symmetry group. Interestingly enough, the scale symmetry breaking approach and the finite size scale symmetry approach lead to similar results in some special cases. We do not know if this coincidence is only fortuitous. If it were not, it might be a justification of the present new approach.

Also, we shall adopt a slightly more general point of view than Nottale, who only considered cut-offs in the scale space. In fact, in a power law like $\phi \sim \ell^\alpha$, the quantity ϕ is also unbounded: when ℓ goes from 0 to ∞, ϕ goes from 0 (resp. ∞) to ∞ (resp. 0) when α is positive (resp. negative). This shows that power laws are possible only in systems where both the scale and the field ϕ are unbounded. The finite-size generalization to scale invariance should be able to deal with cutoffs both in scale space and in the field space, at variance with Nottale's theory.

However, like in Nottale's theory, our finite size scale transformation shall not be a symmetry of the equations describing the system (like Navier-Stokes equation). The implicit requirement for applicability of our theory to real systems is that the system places itself in a state where effective laws of interaction appear, taking into account the cut-offs, and invariant under our finite-size transformations. At the present time, we have no mean to proving whether and when such hypothesis is valid, although we shall present some speculations about how it could happen in turbulence. Note that a similar kind of hypothesis about conformal invariance has been made by Polyakov [17] in the case of 2D turbulence: Navier-Stokes equations are not conformally invariant, but assumption that the structure function obey this symmetry leads to interesting predictions which are not in contradiction with present available data [18]. As we shall see, this is also the case if one instead assumes finite size scale invariance. Even if our main assumption turned out to be invalidated, we feel that the present theory could still be the basis of useful comparisons and help qualitative understanding of some finite size effects.

2 Finite size scale invariance

2.1 Useful preliminaries about scale invariance

Before introducing our new finite size scale operator, let us briefly recall a few salient features of the conventional scale symmetry. Consider a physical quantity ϕ depending on a parameter r like a power law:

$$\phi = \phi(r_0)\left(\frac{r}{r_0}\right)^\alpha, \tag{2}$$

where $\phi(r_0)$ and r_0 are reference field and parameter, and α is an exponent, characteristic of the field. For simplicity, we shall furthermore assume that both ϕ and r are positive quantities. The generalization to non-positive fields and parameters will be discussed in Section 4. In critical phenomena, we can think of ϕ as a thermodynamical quantity and r as the reduced order parameter (*e.g.* $r = (T - T_\mathrm{c})/T_\mathrm{c}$, where T and T_c are the temperature and the critical temperature); in turbulence, ϕ can be any order structure function and r the distance over which the velocity difference is taken. Notice that if we allow r to vary between 0 and ∞, ϕ can take any value between 0 and ∞, and *vice-versa*. This type of law does not seem to like finiteness of one of the parameter, if the other one is unbounded. Also, a striking feature of equation (2) is that the reference quantities appear rather arbitrary. One could as well rewrite (2) using another reference field ϕ_1 and another reference parameter r_1 provided they obey $\phi(r_0)/\phi_1 = (r_0/r_1)^\alpha$. This absence of characteristic reference is a well-known property of power laws, and makes them closely associated with scale invariance. In fact, it can be recast under a more formal invariance, reflecting the scale symmetry of the field:

$$\phi(\lambda r) = \lambda^\alpha \phi(r), \tag{3}$$

where λ is an arbitrary parameter. Equation (3) means that the power law field is invariant under the homotetical transformation $r \to \lambda r$; $\phi \to \lambda^{-\alpha}\phi$. Everybody working in critical phenomena or turbulence is familiar with this symmetry. Note that the invariance (3) explicitly introduces the scaling exponent α. This is annoying, because, sometimes, by looking at the equations of evolutions, and searching for rescaling (3) which leaves the equation invariant, you pick up an exponent (the dimensional exponent), which is not the one you observe! This means that there is probably another way of expressing the scale invariance, which does not involve explicitly the scaling exponent. Such a – less familiar – equivalent form of (3), which will be useful to introduce our finite size generalization of scale invariance, can be found by picking up an arbitrary reference field ϕ_0 and an arbitrary reference parameter r_0 and introducing the log variables:

$$\begin{aligned} U &= \ln(\phi/\phi_0) \\ s &= \ln(r/r_0) \\ U(0) &= \ln\big(\phi(r_0)/\phi_0\big), \\ U(s) - U(0) &= \alpha s. \end{aligned} \quad (4)$$

Note that everything is simpler if we pick up $\phi_0 = \phi(r_0)$, but this simplification is not needed here. We shall come back to this point later. Also, since both r and ϕ are allowed to vary between 0 and ∞, the log-variable describe the whole real interval between $-\infty$ and ∞.

In these variables, equation (3) can also be written:

$$\begin{aligned} U(s) &= U(s+\mu) + g(\mu), \\ g(\mu) &= U(0) - U(\mu), \end{aligned} \quad (5)$$

for any μ. Reciprocally, any regular function obeying (5) is necessarily of the shape $U(s) - U(0) = \alpha s$, where $\alpha = U(1) - U(0)$ is an arbitrary parameter selected by the initial conditions. So this new invariance is a property of any power law field. It is therefore a general scale symmetry and can be used whether the scaling of the field is dimensional or not!

The transformation associated with (5) is, in log-variable, $s \to s + \mu$; $U \to U + g(\mu)$, where $g(\mu)$ satisfies $g(\mu + \mu') = g(\mu) + g(\mu')$. This clearly defines an internal composition law for the transformations. The set of all such transformation defines a group, which corresponds to the traditional scale symmetry. We note that this scale symmetry has the "infinite limit" build-in because of the use of the addition as the composition law: by repeatedly adding any number to a given number, you can reach any arbitrary high value. So any field with this symmetry is necessarily unbounded. This remark is the key to our proposed generalization of (5) to finite systems: instead of the addition, we would like to pick up a composition law which operates only over a finite interval. Once the bounds of the interval have been defined, there is not so much freedom. Before discussing this, it is useful to underline a gauge invariance of our approach.

2.2 The gauge invariance

In the previous section, we have shown that by introducing simple log variable s and U defined by (4), one is able to reformulate the scale symmetry using the addition as an internal composition law. There is however a wider set of possibilities, linked with our freedom to non-dimensionalize the log-variables (see the detailed discussion about this point in [8]). Consider indeed a more general case where instead of non-dimensionalizing the field ϕ and the parameter r by two constants ϕ_0 and r_0, one chooses instead to pick up functions of $\ln\phi$ and $\ln r$, *i.e.* $\phi_0(\ln\phi)$ and $r_0(\ln r)$. This way, we now define two isomorphisms Φ and $\mathcal{R}$, linking our variables s and U:

$$s = \mathcal{R}(\ln r); \quad U = \Phi(\ln\phi). \quad (6)$$

In that case, the internal composition law associated with the scale symmetry will not be the addition law, but a law isomorph to it *via* the function Φ and $\mathcal{R}$:

$$\begin{aligned} s &\to s \perp \mu = \mathcal{R}\left(\mathcal{R}^{-1}(s) + \mathcal{R}^{-1}(\mu)\right), \\ U &\to U \mathbin{\tilde{\perp}} g(\mu) = \Phi\Big(\Phi^{-1}(U) + \Phi^{-1}\big(g(\mu)\big)\Big). \end{aligned} \quad (7)$$

This freedom to choose the variables associated with the definition of the scale symmetry is actually a sort of gauge invariance. An interesting but unsolved question would be to identify the physical field associated with this gauge[1]. In the sequel, we shall stick to simplicity and work under the simplest gauge and choose our reference quantities in log-variable so that the isomorphisms $\mathcal{R}$ and Φ are the identity.

2.3 Finite size composition law

In the most general case, composition laws over an interval $[s_-, s_+]$ are necessarily isomorph to the composition law [19]:

$$s \mathbin{\tilde{\otimes}} s' = \frac{s + s' - ss'(1/s_- + 1/s_+)}{1 - ss'/s_-s_+}\,. \quad (8)$$

This composition law admits $s = 0$ as a neutral element, and two absorbing fixed points s_- and s_+. Of course, once we have fixed the gauge by letting s follow (4), we have fixed the isomorphism to be the identity: the only law over an interval $[s_-, s_+]$ generalizing the addition (*i.e.* the scale symmetry) for $s = \ln r$ is the law (8)[2].

When r is the scale, and $s = \ln(\ell)$, the composition law (8) is almost identical to the "scale relativity" composition law adopted by Nottale [13]. We actually relaxed the constraint $s_- = -s_+$ adopted by Nottale. This last case corresponds to the situation where the composition law is Lorentzian (relativistic), hence the name used by Nottale. We would like actually to go one step beyond Nottale, because, as stressed in the Introduction, there is also the possibility of cut-offs in the field ϕ. By the same reasoning as for the parameter s, we therefore define a composition law for the "log-field" U as:

$$U \otimes U' = \frac{U + U' - UU'(1/U_- + 1/U_+)}{1 - UU'/U_-U_+}, \quad (9)$$

where U_- and U_+ are cut-offs for the log-field U. Note that our two composition laws tend towards the addition in two limits: when the cut-offs tend to infinity, and when the variable are small in front of the cut-offs.

[1] In Nottale's theory, for example, the gauge invariance stemming from his definition of scale invariance is associated with the electric charge.

[2] Indeed, the only isomorphism solution of $s + s' = \mathcal{R}\left(\mathcal{R}^{-1}(s) + \mathcal{R}^{-1}(s')\right)$ for any (s, s') is the identity.

2.4 The choice of the reference in the log-variables

To be coherent with our choice of the composition laws, we need now to define our reference quantities in the log-coordinates in a physically coherent way. Our compositions laws tend to the addition law when s and U are close to zero. This means that in that region of the parameter space, the properties of our physical quantities will be nearly indistinguishable from the "infinite scale invariance" behavior. In other words, we should choose r_0 and ϕ_0, the references in (4) as the values where the field behaves as a power law. Do such values exist and lead to a finite normalization for the log-variables? (Obviously, a normalization by $r_0 = 0$ is not very helpful!). Using a second-order development to a scale invariant equation, Nottale [14] shows that a finite scale naturally appears at the transition between the two limiting scales. In critical phenomena, there are heuristic arguments to prove that in a system of finite size L, the critical behavior occurs at a finite value r_L, shifted from 0 by a quantity $O(L^{-1/\nu})$, where ν is the exponent of the correlation length. Also, in turbulence, there are experimental evidence that the structure functions behaves like power laws around a typical length called the inertial length l_0. This length approximately increases with the Reynolds number. This means that we can here use the Reynolds number as a measure of finite size effects. In the sequel, the label L will therefore denote either the finite size in critical phenomena, or the finite Reynolds number in turbulence.

For the field ϕ, we can choose the correct normalization by exploiting the finite size scaling formula which should still hold for our theory to be a useful generalization of finite size scaling. Obviously, at the shifted critical point r_L, the correlation length verifies $\xi \sim L$, and so, from (1), we have: $\phi(r_L) \sim L^{-\alpha} \sim r_L^{-\alpha/\nu}$, where α is the critical exponent that ϕ would have in an infinite size system. We call $\phi_L = r_L^{-\alpha/\nu}$ and use this value to normalize our log-coordinate. Now, since ϕ_L necessarily lies in between the two cut-offs of the field, U_+ and U_- are necessarily of opposite sign. This avoids some pathological divergences found in [20,21].

2.5 The finite size scaling transformation

We can now summarize all our findings and define formally our finite size scaling transformations. For a given physical field ϕ of a parameter r, we introduce the log-variables:

$$\begin{aligned} U &\equiv \ln(\phi/\phi_L),\\ s &\equiv \ln(r/r_L). \end{aligned} \tag{10}$$

With our choice of references, we have exactly $U(0) = 0$. Now, we can introduce the physical cut-offs of the parameter $r_\pm$ of r and of the field $\phi_\pm$, with the convention that $r_- = 0$ (resp $\phi_- = 0$) if there is no lower cut-off for r (resp. for ϕ), and $r_+ = \infty$ (resp. $\phi_+ = \infty$) if there is no upper cut-off for r (resp. ϕ). In log-variables, these cut-offs translates into log-cut-offs noted $s_\pm$ and $U_\pm$. Our finite size scale transformation then can be written as:

$$\begin{aligned} S(\mu): s &\rightarrow s \tilde{\otimes} \mu,\\ U &\rightarrow U \otimes g(\mu), \end{aligned} \tag{11}$$

where $\tilde{\otimes}$ and $\otimes$ are the composition law given in (8) and (9), μ is the transformation parameter and $g(\mu)$ a function of μ which characterizes the transformation, and that we derive now.

2.6 Characterization of the function $\mathbf{g(\mu)}$

The function $g(\mu)$ can be derived by imposing that our scale transformations form a group. By composing two transformations of parameters μ and μ', and imposing that the resulting transformation itself belongs to the group, we obtain the condition:

$$g(\mu \tilde{\otimes} \mu') = g(\mu) \otimes g(\mu'). \tag{12}$$

This condition is sufficient to determine completely the shape of $g(\mu)$, as a function of the fixed points of the composition law $\tilde{\otimes}$ and $\otimes$. The proof is given in Appendix. For example, it is easy to check that for $s_\pm \rightarrow \infty$, and $U_\pm \rightarrow \infty$, where the composition laws tend to the addition, the rule (12) is only compatible with $g(\mu) = \alpha\mu$. This is the traditional self-similar case, studied in Section 2.1, (Eq. (5)). There are, however, other possibilities, arising in systems where at least one of the four parameter $s_\pm$ or $U_\pm$ is not infinite. The total number of possibilities is actually nine. The list of corresponding possible shapes for $g(\mu)$ is given in the Table 1.

Once the shape of $g(\mu)$ has been fixed according to the Table 1, it is easy to check that the set of transformations (11), labeled by μ and $g(\mu)$, has a (commutative) group structure, with neutral element corresponding to $\mu = g(\mu) = 0$. We have indeed constructed a new set of symmetry transformations, generalizing the traditional scale symmetry transformation (obtained when $s_\pm = U_\pm = \pm\infty$). We now explore some consequences of invariance with respect to this new symmetry.

3 Consequences

3.1 Symmetrical fields

3.1.1 Their shape

Invariant fields under traditional (infinite size) scale symmetry are power laws. How does this result generalize in the case of finite size scale symmetry? Invariant fields obey, in log-variables, the condition:

$$U(s) = g(\mu) \otimes U(s \tilde{\otimes} \mu), \quad \forall \mu. \tag{13}$$

Inverting this relation and setting $s = 0$, we get, since $U(0) = 0$:

$$U(\mu) \equiv g(\mu)^{[-1]} = -\frac{g(\mu)}{1 - g(\mu)\big(1/U_+ + 1/U_-\big)}, \tag{14}$$

Table 1. Possible shapes of $g(\mu)$ as a function of the nature of the fixed points of the composition law for s and for U. They are only nine possibilities. On the leftmost column, the only three possibilities for the fixed points s_+ and s_- of the composition law for s are indicated. On the first row, the only three possibilities for the fixed points U_+ and U_- of the composition law of U are indicated. The corresponding shape of g is at the intersection of the corresponding column and row. Here, α is a free parameter, which depends on the system and cannot be constrained by symmetry arguments.

.... $U_\pm$ $s_\pm$	$U_+=\infty$ $U_-=\infty$ $g(m)=m$	$U_+=\infty$ U_- finite $g(m)=U_-(1-\beta^m)$	$U_+\neq U_-$ finite U_- finite $g(m)=U_-U_+\dfrac{1-\beta^m}{U_+-U_-\beta^m}$
$s_+=\infty$ $s_-=\infty$ $m=\mu$	$g=\alpha\mu$	$\dfrac{g}{U_-}=1-\mathrm{e}^{\alpha\mu}$	$\dfrac{g}{U_-}=\dfrac{U_+-U_+\mathrm{e}^{\alpha\mu}}{U_+-U_-\mathrm{e}^{\alpha\mu}}$
$s_+=\infty$ s_- finite $m\propto\ln\left(1-\dfrac{\mu}{s_-}\right)$	$g=\alpha\ln\left(1-\dfrac{\mu}{s_-}\right)$	$\dfrac{g}{U_-}=1-\left(1-\dfrac{\mu}{s_-}\right)^\alpha$	$\dfrac{g}{U_-}=\dfrac{U_+-U_+\left(1-\frac{\mu}{s_-}\right)^\alpha}{U_+-U_-\left(1-\frac{\mu}{s_-}\right)^\alpha}$
$s_+\neq s_-$ finite s_- finite $m\propto\ln\left(\dfrac{1-\frac{\mu}{s_-}}{1-\frac{\mu}{s_+}}\right)$	$g=\alpha\ln\left(\dfrac{1-\frac{\mu}{s_-}}{1-\frac{\mu}{s_+}}\right)$	$\dfrac{g}{U_-}=1-\left(\dfrac{1-\frac{\mu}{s_-}}{1-\frac{\mu}{s_+}}\right)^\alpha$	$\dfrac{g}{U_-}=\dfrac{U_+-U_+\left(\frac{1-\frac{\mu}{s_-}}{1-\frac{\mu}{s_+}}\right)^\alpha}{U_+-U_-\left(\frac{1-\frac{\mu}{s_-}}{1-\frac{\mu}{s_+}}\right)^\alpha}$

where the superscript $[-1]$ means inverse of $g(\mu)$ *via* the composition law $\otimes$. So, the possible shape of U depends on the possible shape of $g(\mu)$, *i.e.*, on the nature (finite or infinite) of the fixed points of $\tilde{\otimes}$ and $\otimes$. Since there are nine possibilities, we shall not detail each case (see Tab. 1). Instead, we stress general features shared by all the cases. A first interesting feature of the symmetrical solution is that the field cut-offs are necessarily attained at the parameter cut-offs: $U(s_-)=U_+$ and $U(s_+)=U_-$. This is a necessary (but non sufficient) condition for a field to be invariant by the finite size scale invariance. A second interesting property is the disappearance of the classical power law behavior, replaced by an apparent scale dependence of the local scaling exponent:

$$\frac{\mathrm{d}U}{\mathrm{d}s}=\frac{\mathrm{d}g/\mathrm{d}s}{\big(1-g(s)(1/U_++1/U_-)\big)^2}\,. \tag{15}$$

This property had already been noted by Nottale [13] and used to characterize the classic/quantum transition. To illustrate this modification induced by finite size effects, we show in Figure 1 a typical finite size scale invariant function, compared with a self-similar function with same slope in $s=0$. The corresponding local exponent given by (15) is plotted in Figure 2 for comparison. In Figure 1, we observe only approximate self-similarity in the neighborhood of $s=0, r=r_L$. The influence of the two cut-offs becomes clearly visible, as the local exponent diverges or tends to zero at their location.

3.1.2 Compatibility with finite size scaling

The symmetrical fields, given by (14) do also obey the traditional Finite Size Scaling (1). By definition, $\phi=\phi_L\exp(U)$ and $s=\ln(r/r_L)=-\ln(\xi/L)/\nu$. So, since $\phi_L=r_L^{-\alpha/\nu}=L^\alpha$, we get:

$$\phi=\xi^\alpha\exp\left(-\frac{g(s)}{1-g(s)\,(1/U_++1/U_-)}-\alpha\ln(\xi/L)\right). \tag{16}$$

Since s is a function of ξ/L only, the function correcting the power law behavior is a function of ξ/L only. We have here finite size scaling, where the correction function can be computed explicitly using Table 1 once the cut-offs are known. Practical examples are worked out in Section 5. Since the shape of g only depends on the nature of the cut-offs, the correction function is "universal": it does not depend on the field itself directly, but only on the finiteness or nor of its cut-offs. We have in the present framework a direct proof of the universality conjecture of Fisher and Barber [4].

Formula (14) is also compatible with similarity of second kind. Indeed, in the neighborhood of $s=0$, the correction function behaves like $r^{\delta\alpha}$, where $\delta\alpha=\mathrm{d}g/\mathrm{d}s(s=0)-\alpha$ is a correction exponent. When this correction exponent is zero, we do not have similarity of second kind, the exponent measured in the finite size system near r_L is equal to the exponent measured in a infinite size system and the correction function behaves like a log-normal

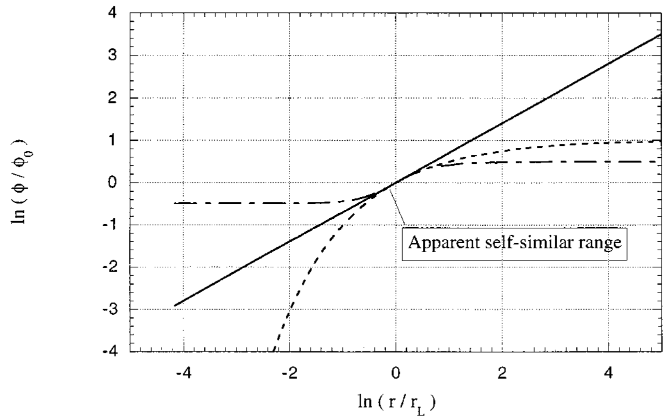

Fig. 1. Comparison between the self-similarity and the finite size similarity. For simplicity, we considered only the case where the field is bounded, while the parameter r can be unbounded. The purely self-similar function is given by the straight line, the two generic deviations to self-similarity in that case (see Tab. 1), are given by the dashed line, and the dashed dotted line. Note the apparent self-similar range appearing around $r = r_L$.

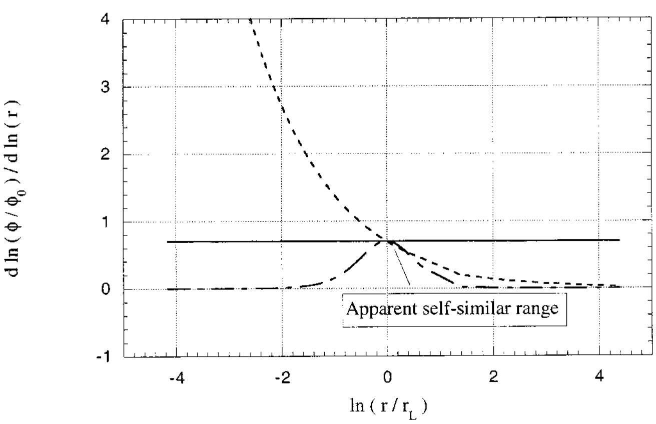

Fig. 2. Comparison between the local scaling exponents for the functions given in Figure 1. One sees that the three local exponents collapse at $r = r_L$ (in the middle of the apparent self-similar range), but that none of the finite size similar functions are truly ever self-similar, since the local exponent is never constant.

function $\mathrm{e}^{-(\ln(r))^2}$ in the neighborhood of $r = r_L$. When this correction exponent is non-zero, the correction function behaves like a power-law, with log-normal corrections.

4 Generalizations

4.1 The case of several field variable

The definition of the finite size scale symmetry can be easily generalized to function of several variables. One just needs to introduce one composition law of the type (8) per variable, each composition law depending of the cut offs of the corresponding variable. So, for a function $H(\phi_1, \phi_2, \ldots \phi_n)$, we introduce $n + 1$ log-variables: $h = \ln(H/H_0), U_1 = \ln(\phi_1/\phi_{1,0}), \ldots U_n = \ln(\phi_n/\phi_{n,0})$ where $H_0 \sim L^\beta, \phi_{1,0} \sim L^{\alpha_0}$..., and $n+1$ composition law $\otimes_0, \ldots \otimes_n$. The finite size scale symmetry for H then writes:

$$h(U_1, \ldots U_n) = \mu \otimes_0 h\big(U_1 \otimes_1 g_1(\mu_1), \ldots, U_n \otimes_n g_n(\mu)\big), \tag{17}$$

where $g_1, \ldots, g_n$ are functions of μ which can take only one of the nine generic shape given in Table 1.

This generalization to several variables can be used to define a finite size renormalization by taking one variable, say ϕ_n to be the scale ℓ tracking the renormalization procedure. Now, by iterating several finite scale renormalization, the scale ℓ necessarily converges to the fixed point, say L, and any function of several variable $h(U_1, \ldots, U_{n-1}, \ell)$ converges towards $h(U_1, \ldots, U_{n-1}, L)$, which is a fixed point of the renormalization procedure:

$$h(U_1, \ldots, U_{n-1}, L) = \mu \otimes_0 h\big(U_1 \otimes_1 g_1(\mu), \ldots U_{n-1} \otimes_{n-1} g_{n-1}(\mu), L\big). \tag{18}$$

As we show in Section 5.1, this property can be used to determine the finite size scaling properties of a function of several parameters.

4.2 Non positive fields and parameters

In general, fields and parameters can take positive and negative values. For example, the order parameter goes from positive to negative upon crossing the critical value. In turbulence, the velocity increments can take positive and negative values. So, it is of great interest to generalize the finite scale symmetry to non positive fields and parameters. This generalization sets only technical difficulties, not conceptual one. A non-positive field can indeed be considered as a two-dimensional field, with one coordinate representing its modulus, and one coordinate representing its phase. Then, one can upgrade the finite size composition law to two dimensions. This was done by [22]. The only technical difficulty is then the finding of the generic shapes of the symmetry parameters generalizing the one dimensional case (the results of Tab. 1). To understand that, let us consider only the case of one field variable, ϕ, depending on only one parameter r. As previously, we introduce the log-coordinates, except that now they are two-dimensional:

$$U = \ln(\phi/\phi_L), \quad V = \Theta - \Theta_L,$$
$$s = \ln(r/r_L), \quad t = \theta - \theta_L. \tag{19}$$

In these coordinates, Θ and θ are the phase of the field and of the parameter, and all the quantities labeled by L are references quantities. For a positive (resp. negative) field or parameter, the phase is equal to 0 (resp. π), but we allow a continuous variation of the phase through the finite scale symmetry by an analytic continuation in the complex plane. In two dimension, the composition law generalizing (8) is [22]:

$$(U'', V'') = (U', V') \otimes (U, V) \tag{20}$$

where $\otimes$ is the composition law:

$$\begin{cases} U'' &= \dfrac{U + U' - aUU' - dVV' - j(U'V + UV')}{1 - hUU' - kVV'} \\ V'' &= \dfrac{V + V' - cjUU' - bVV' - cd(U'V + UV')}{1 - hUU' - kVV'}. \end{cases} \tag{21}$$

Like in 1D, the composition law can be characterized by its fixed points, which depend on the parameters a, b, c, d, h, k, j. The coefficients a, b, c, d, h, k and j are parameters depending on the physical model. They satisfy

$$h = ck, \quad -k + ad + jb = j^2 + d^2 c. \tag{22}$$

We have therefore five free parameters, four to characterize the cut-offs of U and V, and one to characterize the coupling between phase and modulus. Moreover, since V is a phase, its two cut offs V_+ and V_- must satisfy $V_+ - V_- = 2\pi$. This sets an additional constraint on the parameters a, b, c, d, h, k, but it is not easy to express it, except in some simple cases (see *e.g.* Sect. 5.2.2).

With this two-dimensional composition law, we can now express the finite size scale transformation for any general field and parameter as:

$$\begin{aligned} S(\mu_1, \mu_2) : (s, r) &\to s \;\tilde{\otimes}\; (\mu_1, \mu_2), \\ (U, V) &\to (U, V) \otimes \big(g_1(\mu_1, \mu_2), g_2(\mu_1, \mu_2)\big), \end{aligned} \tag{23}$$

where $\tilde{\otimes}$ and $\otimes$ are the composition law given by formula like (21), (μ_1, μ_2) is the (2D) transformation parameter and $g_1(\mu_1, \mu_2), g_2(\mu_1, \mu_2)$ two functions which characterize the finite scale transformation. Because of the group structure, these function satisfy the condition generalizing the 1D case:

$$\begin{aligned} (g_1, g_2)\left((\mu_1, \mu_2) \;\tilde{\otimes}\; (\mu_1', \mu_2')\right) &= \\ (g_1, g_2)(\mu_1, \mu_2) &\otimes (g_1, g_2)(\mu_1', \mu_2'). \end{aligned} \tag{24}$$

Finding the functions g_1 and g_2 is in principle possible *via* an iteration formula, like in the 1D case: iterating m times equation (24) starting from $(1, 1)$, one obtains expression of $\mu_1(m)$, $\mu_2(m)$ and $g_1(m)$, $g_2(m)$ and so one gets an implicit representation of g_1 and g_2 as a function of μ_1 and μ_2. The technical difficulty here involves the finding of the explicit expression of $\mu_1(m), \ldots, g_2(m)$ as a function of the parameters of the composition law (21). This is however tractable in some simple examples. We explore one example in Section 5.2.2, as an application to turbulence.

5 Examples of application

To illustrate the interest of the concept of finite size scale invariance, we now present a few applications, borrowed from critical phenomena and turbulence.

5.1 Critical phenomena

5.1.1 The correlation function

The correlation function ξ_∞ is unambiguously defined in an infinite size system where the pair correlation function decays exponentially. Even when the exponential decay does not hold, it is generally assumed that some characteristic length ξ_∞ can be defined, such that it determines the scale variations of the correlation function, and that it diverges at the critical point: $\xi_\infty(t) \sim |t^{-\nu}|$, where $t = (T - T_\mathrm{c})/T_\mathrm{c}$, T_c being the critical temperature. The definition of a correlation length in a finite size system involves some subtilities (see *e.g.* [23]). Here, we assume that a correlation scale has been somehow defined to determine the scale variations of the correlation function, and we examine what are the possible shapes allowed by a finite size scale symmetry. For simplicity, we restrict our study to the case of positive variables, and we assume that there is only one cut-off in the scale space, determined by some larger scale L. Since the correlation scale is a scale itself, its composition law is determined by the existence of this single cut-off scale L. On the other hand, the temperature can take any value between $t = 0$ and $t = \infty$, so that the composition law for temperatures is just the addition. We note $t_L = O(L^{-1/\nu})$, the shifted temperature defining the critical finite size behavior, and $\xi_L = t_L^{-\nu}$ the corresponding value of ξ. This value is $O(L)$, we denote it bL, where $b < 1$ is a prefactor which cannot be computed from symmetry arguments only. The finite size scale symmetry then means that:

$$\ln\left(\frac{\xi}{bL}(\ln(t/t_L))\right) = g(\mu) \otimes \ln\left(\frac{\xi}{bL}(\ln(t/t_L) + \mu)\right), \tag{25}$$

where $U \otimes V = U + V + UV/\ln(b)$ and $g(\mu) = \ln(1/b)(1 - \mathrm{e}^{\alpha\mu})$, see Table 1. By applying (25) with $\mu = -\ln(t/t_L)$, we therefore get:

$$\ln\left(\frac{\xi}{bL}\right) = \ln(1/b)\left(1 - \left(\frac{t}{t_L}\right)^{-\alpha}\right). \tag{26}$$

As t goes from 0 to ∞, ξ goes from 0 to L. The correlation length can never exceed L, and so, never becomes infinite, unlike in the infinite size system case. There is approximate power law dependence of ξ near $t = t_L$. In that neighborhood, $\ln\xi \sim \alpha\ln(1/b)\ln(t)$. This defines an approximate scaling exponent which coincides with the infinite size case provided $\alpha = -\nu/\ln(1/b)$. Finally, equation (26) defines a finite size scaling function $f = \xi/t^{-\nu}$, which is

$$f(X) \propto \exp\left(1 - \mathrm{e}^{-\alpha\ln(X)} - \nu\ln(X)/\ln(1/b)\right), \quad X = tL^{1/\nu}. \tag{27}$$

This function is shown in Figure 3. in the cases $\alpha = -\nu/\ln(1/b)$ and $\alpha \neq -\nu/\ln(1/b)$.

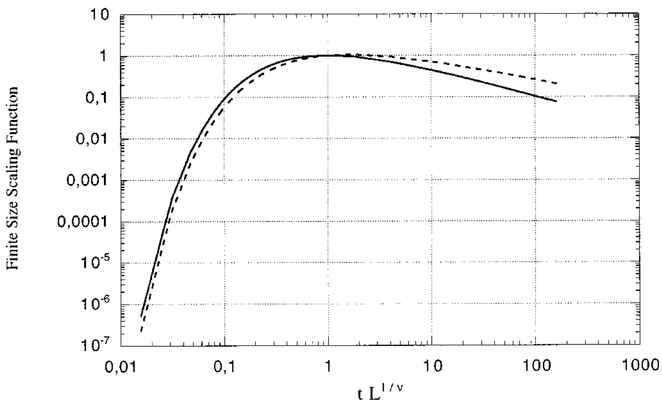

Fig. 3. Plot of the universal scaling function in the case $\alpha \ln(1/b) = -\nu$ (full line), and $\alpha \ln(1/b) \neq -\nu$ (dotted line). In the first case, the behavior of the function around $r = r_L$ is Gaussian, while in the second case, it is exponential.

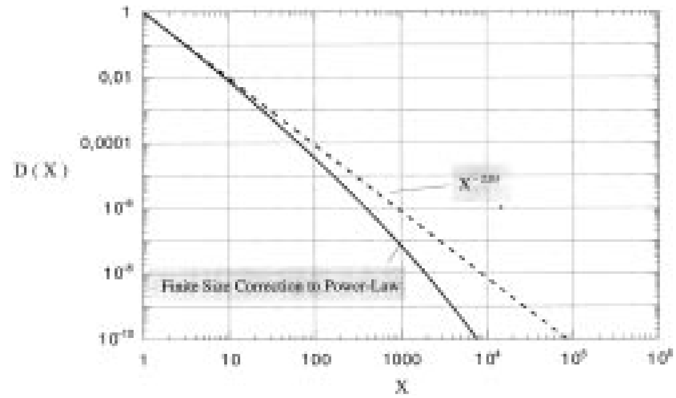

Fig. 4. Illustration of generic finite size corrections to a power-law distribution function. The cut-off here has been taken to 3×10^6. The power law is only valid near $X = 1$, which was taken as the reference.

5.1.2 An avalanche distribution function

To give a more elaborate application, let us now consider the case of the distribution function of some statistical quantity S, in a system with order parameter t. We denote $D(S,t)$ this distribution function. We assume that the finite size occurs because somehow the statistical quantity cannot take values larger than some value. We have in mind for example the case of random-field Ising model at zero temperature [24]. Near the critical point, the spin system goes into a series of avalanche, the distribution of which size is given by $D(S,t)$, where here t is the amplitude of the noise parameter. Obviously, on a finite size lattice, the number of possible spin is limited, so is the size of the largest avalanche. We denote this size by S_+. On the other hand, avalanche of size zero can be formally defined, so there is only one cut-off in the avalanche size. As for D and t, they can take *a priori* any value between 0 and ∞, so the finite scale invariance property of D is:

$$\ln D\big(\ln(S/S_L), \ln(t/t_L)\big) = \ln D\big(\ln(S/S_L) \otimes g(\mu); \ln(t/t_L) + \mu\big) + \gamma\mu, \quad (28)$$

where S_L and t_L are some reference quantities depending on the size of the system, $U \otimes V = U + V - UV/U_+$ and $g(\mu) = U_+(1 - e^{\alpha mu})$, $U_+ = \ln(S_+/S_L)$ see Table 1. A first interesting computation is to find the distribution $D_L(S)$ at the critical parameter $t = 0$. Since it is a fixed point of the transformation, it obeys:

$$\ln D_L\big(\ln(S/S_L)\big) = \ln D\big(\ln(S/S_L) \otimes g(\mu)\big) + \gamma\mu, \quad \forall\mu. \quad (29)$$

This equation has only two types of solutions: if $D(S) = 0$ for any S different from 0, D is simply proportional to a delta function $D(S) \sim \delta(S)$; if there is at least one S, say $S = 1$ for which D is non zero, we can write (29) with $S = 1$, and setting $X = S_L \exp(g)$, we find, since D is a distribution function and $D(\ln X) = XD(X)$:

$$D(X) = \frac{D(1)}{X}\big(1 - \ln(X)/U_+\big)^{-\alpha/\gamma}, \quad \forall X < S_+/S_L. \quad (30)$$

We note that in the limit $S_+ \to \infty$ (infinite size system), then $U_+ = \ln(S_+/S_L) \to \infty$, and this distribution tends to a power law $X^{-1+\alpha/(U_+\gamma)}$. This is the well-known scale invariant shape of many distribution near or at the critical point. In the presence of finite size effects, the power law shape is modified by a factor $O(S/S_+)$, which is only important near the cut-off. Far from the cut-off, the distribution still looks like a power law; near the cut-off, the distribution tends rapidly to zero, and deviates from the power law behavior (Fig. 4).

Now, we can also get the shape of the distribution function at any t by using the finite size scale symmetry (28) with $\mu = -\ln(t/t_L)$. A little algebra leads to:

$$D(S,t) = \frac{1}{S}\big(1 - \ln(S/S_L)/U_+\big)^{-\alpha/\gamma} F\left(e^{g(t)} S^{\beta(t)}\right), \quad (31)$$

where $g(t) = U_+\big(1 - (t/t_L)^{-\alpha}\big)$ and $\beta(t) = (t/t_L)^{-\alpha}$. This shape is different from the usual finite size scaling adopted *e.g.* by Sethna [24], which is:

$$D(S,t) = S^\tau F(tS^\sigma). \quad (32)$$

So, if one actually tries to obtain σ and τ by ordinary Widom collapse, one should observe a variation of σ and τ with t. Also, note that here, the parameter governing the prefactor of S in the universal function is e^g instead of t and varies like this: it goes from 0 at $t = 0$, up to a constant value e^{U_+} at larger values of t. This behavior is similar to the behavior of $(T - T_c)/T$, which goes from 0 to 1, as T varies. This suggest that the scaling should be better if instead of t one uses $(T - T_c)/T$ in the formula (32). These two features were actually noted in [24,25].

5.1.3 Illustration

To make the previous assertions more quantitative, we have conducted a finite size scaling analysis of the spin avalanche distribution along the lines discussed in the previous section. The data we used were kindly provided to us

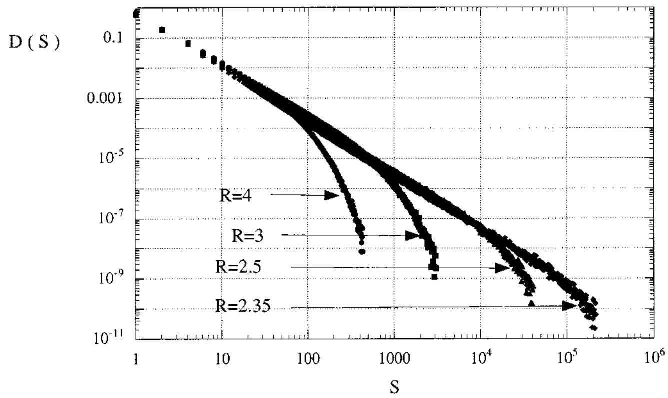

Fig. 5. Integrated (over a loop) size distribution of avalanches in a spin system forced by a random magnetic field. The distribution is shown for four values of the variance of the field, which is the order parameter in the system. As one approaches the critical point ($R_c \approx 2.07$, see text), one observes the development of a power-law behavior, characteristic of critical behavior. Data courtesy J. Sethna.

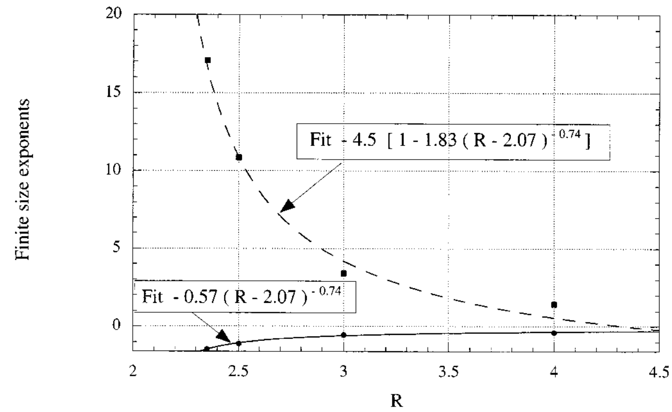

Fig. 6. Plot of the two characteristic finite size exponents in the spin system, $g(r)$ (squares) and $\beta(r)$ (circles). The lines are the fits using the theoretical predictions, dictated by finite size scale invariance. The fits suggest that $R_c = 2.07$.

by James Sethna. They come from numerical simulations of spins over an hyperlattice. The spin interact ferromagnetically and are subject to a uniform magnetic field H and to a random Gaussian field with width proportional to R. This quantity plays the role of the order parameter. The simulations and the properties of the system are discussed at length in [24,25]. For a critical value R_c of R and a critical value H_c of the uniform magnetic field, an infinite spin avalanche occurs and the magnetization curve shows a discontinuity. Near R_c and H_c, avalanches of all sizes occurs, and one finds critical scaling behaviors. We are here interested in the size distribution of all the avalanches that occur on a hysteresis loop, for H from $-\infty$ to ∞. We denote this distribution by $D(S, R)$. Figure 5 shows the typical distributions obtained at four different values of the parameter R. One sees that as R diminishes, one observes a more prominent power-law behavior over several decades. These data were recorded using a 320^3 lattice for $R = 2.5$ to $R = 4$, and a $1\,000^3$ lattice size for $R = 2.35$. This number defines the value of the maximal avalanche size S_+ which can occur in the system. For simplicity, we took $S_+ = 10^9$ for all four simulations.

As a first step, we tried to determine the shape of the distribution at criticality by trying to fit formula (30) onto the data at the smallest R. The best fit was obtained with $S_L = 0.25$ and $\alpha/\gamma = 0.79U_+$, leading to a power law behavior $D(S) = S^{-1.79}$ at criticality, for an infinite size system. This shape is a little bit different from the $S^{-2.03}$ shape found by [25] using traditional finite size scaling analysis. With this shape, we then try to get the finite size function $g(r)$ and $\beta(r)$ with $r = R - R_c$, using the collapse formula (31). For this, we divided all the distributions by their value at criticality, and located the value of the maximum $S_{\max}$ and of the cut-off $S_{\text{cut-off}}$ of the resulting curves. An easy algebra shows that if (31) is correct, $\beta(r)$ is proportional to $1/\ln(S_{\text{cut-off}}/S_{\max})$ and that $g(r)$ is equal to $-\ln(S_{\max})\beta(r)$ up to a multiplicative and an additive constants. The two corresponding functions are shown in Figure 6. According to the theoretical prediction, β should vary like a power law of $r = R - R_c$, and g like a constant plus a power-law, with the same exponent as β (note that the undeterminacy described above in the computation of β and g does not change this behaviors). Using these predictions, we performed a best fit over β and g to extract the value of the critical parameter, and the value of the exponent α. We found: $R_c = 2.07$ and $\alpha = 0.74$. Our value of the critical parameter is a little bit lower than the $R_c = 2.16$ value inferred by [25]. This discrepancy is not surprising regarding the differences of analysis. At this stage, we do not claim that it is significant, because our determination was made using only four distributions, and not eight or nine as was done in [25].

Note however the extremely good agreement between the "universal" theoretical shapes and the values we extracted. We shall see later using turbulent data an example where the same kind of agreement is obtained, using a much greater values of data points. The quality of our analysis can be checked *a posteriori* by plotting all the size distribution renormalized by the value at criticality, as a function of the universal variable $e^{g(r)}S^{\beta}$. This is done in Figure 7. One sees that the four curves collapse nicely, defining the scaling function F appearing in (31).

5.2 Turbulence

We now turn to turbulence. We assume here that the finite size effects – deviations from the Kolmogorov theory – are given by the finite size of the Reynolds number. We still label with a L any reference quantity representing the "almost scale invariant quantity" at that finite Reynolds number.

In turbulence, a traditional tool to explore the statistical properties of the velocity u as a function of the scale ℓ is the distribution function of velocity increments $\delta u(\ell)$ over a distance ℓ, defined as:

$$\delta u(\ell) = u(x + \ell) - u(x).$$

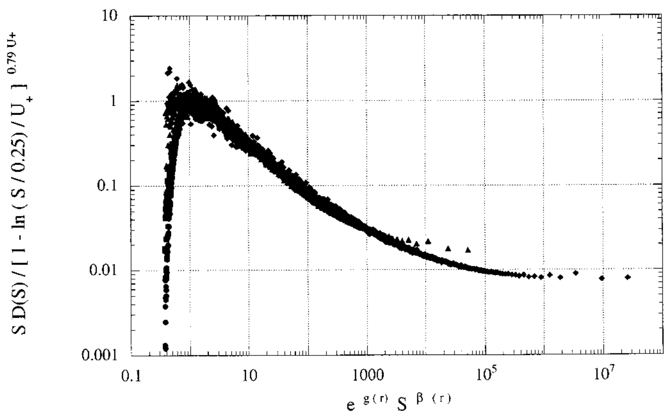

Fig. 7. Finite size collapse of the four distribution given in Figure 5, using the theoretical formula (31), and the values of β and g given in Figure 6. The collapsing curves defines a finite size scaling function, which takes into account the finite size of the system.

These quantities can be viewed as "poor man's" wavelet of the velocity field [26]. Similar but cleaner results can actually be obtained using real wavelet coefficients, as noted by [27,26]. Therefore, we shall here consider $\delta u(\ell)$ as a wavelet coefficient of u, without reference to the analyzing wavelet. The influence of its choice over the final result will be discussed later. Since the velocity field can be positive or negative, the wavelet coefficient $\delta u(\ell)$ can take any sign. We first discuss the case of absolute value (modulus) of the wavelet coefficient, to deal only with positive values. The extension to negative value will serve as an illustration of Section 4.2, and is done in Section 5.2.3.

5.2.1 About the characteristic length scale L

Before discussing distributions functions, it is interesting to discuss the meaning in turbulence of the length scale L which is used to in our theory to normalized the log-variable (10). As discussed in Section 3, this length scale defines the area of parameter were an apparent self-similar behavior is observed. We believe that this peculiar length scale has just been identified in turbulence by [28]. In turbulence, in the infinite Reynolds number, infinite size limit, a theoretical prediction due to Kolmogorov leads to $\langle(\delta u)^3\rangle = -\frac{4}{5}\epsilon\ell$, where ϵ measures the energy dissipation. In real turbulence at finite Reynolds number (finite viscosity) and finite size, [28] actually pointed out that the Kolmogorov function $K(\ell) = \langle\delta u)^3\rangle/\epsilon\ell$ is not constant, but instead form a bell like curve with an approximate plateau in the inertial range. As the Reynolds number is increased, the plateau tends to be flatter and wider. The interesting result found in [28] is that the location of the maximum of the bell curve (the center of the plateau) occurs at a specific length scale which varies like a power law of the Reynolds number. All these features make this newly discovered length scale the best candidate for our normalization length scale L.

5.2.2 Distribution function of $|\delta \mathrm{u}(\ell)|$

Any turbulent fluid is subject to a forcing. This forcing is usually random, therefore opening the possibility that the reference velocity fluctuation δu_L – around which the flow is approximately scale invariant – is also fluctuating. This means that the probability distribution of a wavelet coefficient $\delta u(\ell)$ can be written:

$$P(|\delta u|, \ell) = \int P_L\left(\frac{|\delta u|}{\sigma}\right) G_\ell(\sigma)\mathrm{d}\sigma, \qquad (33)$$

where P_L and G_ℓ are respectively the probability distribution functions of the reference velocity and of the ratio $|\delta u/\delta u_L|$. Such a shape was proposed several years ago by Castaing and collaborators [29–31], in connection with log-infinite divisible laws [32]. The function G is often referred to as the propagator. Let us compute for example this propagator in the case where there is not cut-off in the (log) scale space, and there is only a cut-off in the (log) velocity space. Physically, $\ln(|\delta u(\ell)/\delta u_L|)$ can be viewed as the depth of the cascade from ℓ to L. Our assumption therefore is equivalent to assume that this depth is limited by a maximum or minimum step. This case has already been discussed in the turbulent literature (see *e.g.* [33,32]), so it is interesting to see what outcome it given in the present framework.

This case is similar to the case discussed in Section 5.1.2, except that since the probability distribution functions must be normalizable to 1, $\gamma = 0$. Taking into account the normalization, we therefore write G_ℓ as

$$G_\ell(\sigma) = \frac{\beta(\ell)}{\sigma} F\left(\mathrm{e}^{g(\ell)}\sigma^{\beta(\ell)}\right), \qquad (34)$$

where F is some universal function, $g(\ell) = U_+(1 - (\ell/L)^{-\alpha})$ and $\beta(\ell) = (\ell/L)^{-\alpha}$. It can also be checked that the fixed point distributions obtained in that case (at $\ell = \infty$ and $\ell = 0$) are either a delta function, or the distribution

$$G(\sigma) \sim \frac{1}{\sigma\big(1 - \ln(\sigma)/U_+\big)} \cdot \qquad (35)$$

This means that, as ℓ varies, we obtain a continuous deformation of the propagator from the delta function to the distribution (35).

Several interesting properties can be derived from the shape of the propagator. First, we note that the shape (34) is a generalization of the self-similar case $\beta(\ell) = 1$ and $g(\ell) = \alpha\ln(\ell)$ pertaining the Kolmogorov theory. Despite its simplicity, (34) implies a non-trivial behavior of the probability distribution function. For example, if F contains an exponential part, the propagator G will be made of stretched exponentials, the amount of stretching depending on the scale. This feature is actually observed in turbulence data. It can also be checked that a general family of probability distribution function satisfying (34) are functions whose Fourier transform is of the shape $\exp(ak^\epsilon)$, *i.e.* are log-stable distributions. This includes of course the case of log-normal propagator proposed as early as 1962 by Kolmogorov and Obukhov [34], and by Castaing [29], or the case of log-Lévy propagators advocated by Schertzer and Lovejoy [35,36] for geophysical flows.

Let us now turn to the properties of the probability distribution function (PDF) of wavelet coefficients. The scale variation of the propagator means that as scale varies, the PDF varies from the PDF of the reference velocity, up to the convolution of this PDF by the distribution (35). This convolution creates a cusp at the center of the distribution, reminiscent of what is observed at small scales for the PDF's of velocity increments. It is also observed that at large scale, the PDF's go towards a quasi-normal distribution, which in our interpretation should correspond to the PDF of the reference velocity. The shape of the propagator (34) and the convolution imply an interesting behavior of the moments of the wavelet coefficient distribution. By simple changes of variables, these moments can be put under the shape:

$$\langle|\delta u(\ell)|^n\rangle = H_n \mathrm{e}^{ng(\ell)} \int \mathrm{e}^{nY/\beta(\ell)} F(Y)\mathrm{d}Y, \tag{36}$$

where the coefficients H_n only depend on the PDF at the reference scale *via* the Laplace transform of H. Now, if we allow the Reynolds number to tends to infinity (no finite size effect), U_+ tends to infinity, and $\beta(\ell) = 1$ and $g(\ell) = \alpha\ln(\ell)$. We recover the self-similar shape $\langle|\delta u(\ell)|^n\rangle \sim \ell^{\zeta(n)}$ with $\zeta_n = n\alpha$. In general, more complex behaviors can be expected. An interesting class of behavior is obtained when F is log-stable: in this case, the log of its Laplace transform is a power law n^ϵ, $0 \le \epsilon \le 2$ and the reduced moments obeys the simple law:

$$\frac{\langle|\delta u(\ell)|^n\rangle}{\mathrm{e}^{ng(\ell)}} \propto \left(\frac{\langle|\delta u(\ell)|^p\rangle}{\mathrm{e}^{pg(\ell)}}\right)^{(n/p)^\epsilon}. \tag{37}$$

In other words, reduced function of different order can be expressed as power-law of each other. This property was observed in turbulence by Benzi *et al.* and named General Scaling [37]. Finally, we stress that the shape of the functions $g(\ell)$ and $\beta(\ell)$ appearing in the propagator and in the moments is exactly that proposed by Castaing [29] using a hand-waving argument about the fractal dimension of the dissipation set. This shape was also derived by Dubrulle and Graner [10] using a scale symmetry breaking formalism, where the cut-offs are imposed *via* boundary conditions and not directly in the formalism as here. This may suggest that there is a deeper connection between the two approaches, although we were not able to establish it.

5.2.3 Distribution function of velocity increments

Let us now try to generalize the results obtained previously to the wavelet coefficients, which can be negative or positive. We can now generalize the convolution property (33) to take into account the phase as:

$$P(\delta u, \theta, \ell) = \int P_L\left(\frac{|\delta u|}{\sigma}\right) G_\ell(\sigma, \theta). \tag{38}$$

For simplicity, we have assumed that the reference phase quantity is non random and zero. The propagator G now describes how the phase and the modulus of the step of the cascade evolve with scale. The probability of positive increments is obtained from (38) by setting $\theta = 0$ (resp. $\theta = \pi$). We allow for a coupling between the phase and the modulus. A simple situation occurs when we allow only influence of phase onto the modulus, and not the opposite. In the sequel, we shall not try to give any physical justification of this simplifying assumption, nor on any subsequent ones. This section is indeed only provided for the sake of illustration of the method. As it turns out, the outcome of the simplification provides a result which is at least qualitatively in agreement with the observation. An illustration of the application of the method and comparison with turbulent data will be given in Section 4.2.4.

Upon this simplification, the relevant composition law for the log-modulus $U = \ln|\delta u/\delta u_L|$ and the phase $V = \Theta - \Theta_L$ is equation (21) with $c = 0$. As for the scale, we still consider it as a positive quantity, allowed to take any value between 0 and ∞, so its associated composition law is just the addition. A propagator $G(U, V)$ invariant under finite scale symmetry then satisfies:

$$\begin{aligned} &G\big(U, V, \ln(\ell/L)\big) = \\ &\qquad N(\ell) G\Big((U, V) \otimes \big(g_1(\mu), g_2(\mu)\big), \ln(\ell/L) + \mu\Big), \end{aligned} \tag{39}$$

where $N(\ell)$ is a factor to ensure the normalization of the propagator. Taking $\mu = -\ln(\ell/L)$, and assuming for simplicity that at $\ell = L$, the modulus and the phase decouple: $G(U, V, 0) = F_1(U)F_2(V)$, we get for the positive part and the negative part of the propagator, in normal (non log) variable:

$$\begin{aligned} G_+(\sigma, \ell) &\equiv G(\sigma, 0, \ell) = \frac{S_+(\ell)}{\sigma} F_1\left(\mathrm{e}^{g_+(\ell)}\sigma^{\beta_+(\ell)}\right), \\ G_-(\sigma, \ell) &\equiv G(\sigma, \pi, \ell) = \frac{S_-(\ell)}{\sigma} F_1\left(\mathrm{e}^{g_-(\ell)}\sigma^{\beta_-(\ell)}\right), \end{aligned} \tag{40}$$

where

$$\begin{aligned} S_+(\ell) &= \frac{F_2\Big(g_2\big(-\ln(\ell)\big)\Big)}{F_2\Big(g_2\big(-\ln(\ell)\big)\Big) + F_2\Big(\pi \tilde{\otimes} g_2\big(-\ln(\ell)\big)\Big)}, \\ S_-(\ell) &= \frac{F_2\Big(\pi \otimes g_2\big(-\ln(\ell)\big)\Big)}{F_2\Big(g_2\big(-\ln(\ell)\big)\Big) + F_2\Big(\pi \tilde{\otimes} g_2\big(-\ln(\ell)\big)\Big)}, \\ \beta_+(\ell) &= 1 - ag_1\big(-\ln(\ell)\big) - jg_2\big(-\ln(\ell)\big), \\ g_+(\ell) &= g_1\big(-\ln(\ell)\big), \\ \beta_-(\ell) &= \frac{1 - ag_1\big(-\ln(\ell)\big) - jg_2\big(-\ln(\ell)\big)}{1 - k\pi g_2\big(-\ln(\ell)\big)}, \\ g_-(\ell) &= \frac{g_1\big(-\ln(\ell)\big) - d\pi g_2\big(-\ln(\ell)\big) - j\pi g_1\big(-\ln(\ell)\big)}{1 - k\pi g_2\big(-\ln(\ell)\big)}. \end{aligned} \tag{41}$$

Obviously, equation (40) is a generalization of equation (34). The function g_1 and g_2 characterizing the scale behavior of the propagator can be computed by iteration following the procedure explained in Section 4.2. One finds:

$$g_2(t) = V_+ V_- \frac{1-\beta^t}{V_+ - V_- \beta^t},$$
$$g_1(t) = \mathrm{e}^{(1-a-j)\int \frac{\mathrm{d}x}{1-kg_2(x)}} \times \left(g_1(0) + \int \frac{1-(d+j)g_2(x)}{1-kg_2(x)} \times \mathrm{e}^{-(1-a-j)\int \frac{\mathrm{d}y}{1-kg_2(y)}}\,\mathrm{d}x\right), \quad (42)$$

where $V\pm$ are the fixed points of V, and β is a parameter. Summarizing, we find that when the sign of the wavelet coefficient is taken into account, one find a different propagator for the positive and negative part. The difference however only enters *via* some scale dependent functions $S_\pm, \beta_\pm, g_\pm$ entering the argument of an universal function F_1. This behavior is in qualitative agreement with the experimental data [38]. We now present a more detailed, quantitative comparison using real turbulence data.

5.2.4 Illustration

We provide here an illustration of several features of finite size scale invariance using data kindly provided by Maurice Meneguzzi. The data come from a direct simulation of Navier-Stokes equation with a spectral code at a resolution 512^3. Details of the numerical method and on the properties of the flow can be found in [39]. The Reynolds number (based on the Taylor microscale) of the simulation corresponds to $R_\lambda \approx 130$.

As a first step, we considered the structure functions $S_n(\ell)$ defined by:

$$S_n(\ell) = \langle (\delta u)^n \rangle, \quad (43)$$

where δ_u can be the velocity increment in the direction of the vector ℓ (longitudinal increments) or in the direction transverse to it (transverse increment). In the traditional Kolmogorov picture, these structure functions should behave like power laws $S_n(\ell) \sim \ell^{\zeta(n)}$, in the infinite Reynolds number limit. In finite size Reynolds number experiments or simulations, one in fact observes that these function are not power-law [40]. It is then legitimate to wonder whether these corrections to power-law scaling follow the finite size scale invariance predictions (Eq. (27)). To this end, we considered the second order and fourth order structure functions in both the longitudinal and transverse cases[3] and tried to fit them with the theoretical prediction (27). This prediction shows that the function $S_n/\ell^{\zeta(n)}$ reaches its maximum at the "normalizing" scale L discussed in Section 4.2.1. In the present case, it appears

[3] Odd order structure functions are zero for transverse velocity increments, and are very noisy for longitudinal increments, due to skewness effects.

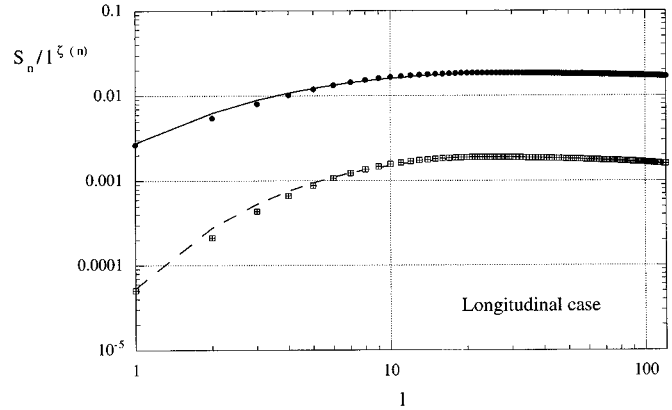

Fig. 8. Illustration of finite size corrections to power laws in turbulence, for two longitudinal structure functions S_2 (circles), and S_4 (squares with a cross). The scale units are in mesh size. The lines are the best fits using the theoretical formula (27), with $X = \ell/26$, and $\alpha = 0.44$ and $\nu/\ln(b) = 0.8$ for S_2, and $\alpha = 0.56$ and $\nu/\ln(b) = 1.28$ for S_4. Data courtesy M. Meneguzzi.

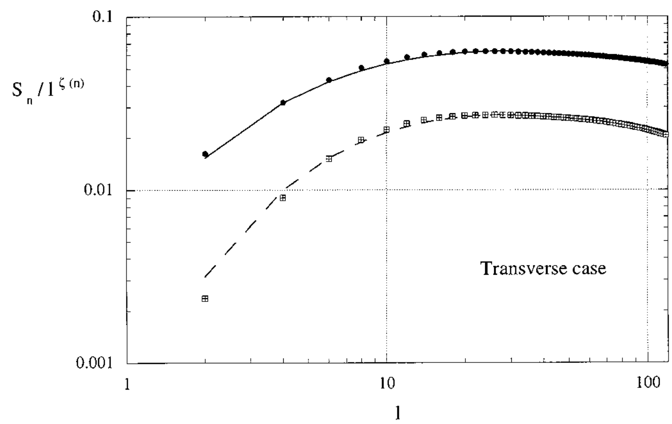

Fig. 9. Illustration of finite size corrections to power laws in turbulence, for two transverse structure functions S_2 (circles), and S_4 (squares with a cross). The scale units are in mesh size. The lines are the best fits using the theoretical formula (27), with $X = \ell/26$, and $\alpha = 0.49$ and $\nu/\ln(b) = 0.7$ for S_2, and $\alpha = 0.59$ and $\nu/\ln(b) = 1.24$ for S_4.

to be of the order $\ell = 26$ in mesh units. This property enables to find the scaling exponents by finding the value of $\zeta(n)$ for which $S_n(\ell)/\ell^{\zeta(n)}$ reaches a maximum. We found $\zeta(2) = 0.8$ and $\zeta(4) = 1.28$ for longitudinal exponents, and $\zeta(2) = 0.7$ and $\zeta(4) = 1.24$ for the transverse exponents. These values are consistent with the values directly measured by [39]. We next try to fit the resulting function $S_n/\ell^{\zeta(n)}$ using the theoretical formula (27). The result is shown in Figure 8 for the longitudinal structure functions, and in Figure 9 for the transverse structure functions. One sees that the fit is extremely good from the smallest computed scale $\ell = 1$, the mesh size, up to $\ell = 120$ which is about half the size of the computational domain (we did not have data for size largest than that). Another interesting observation is that the scaling function appears rather

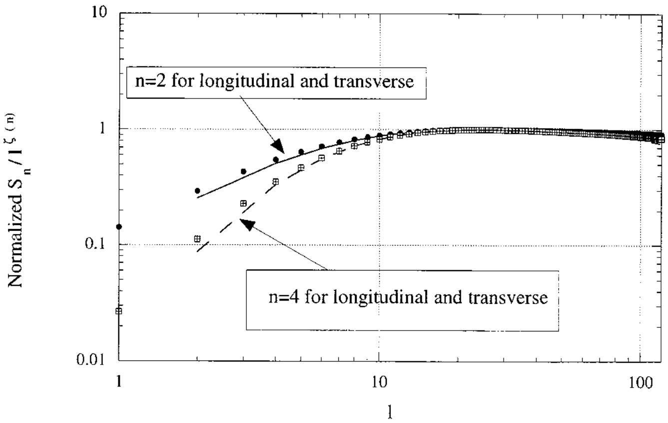

Fig. 10. Test of the universality of the scaling functions in turbulence. The finite size functions of the longitudinal structure function (circle for $n = 2$ and squares filled with cross for $n = 4$) appear very similar to the finite size function of the transverse structure function at same index (line for $n = 2$, and dotted line at $n = 4$).

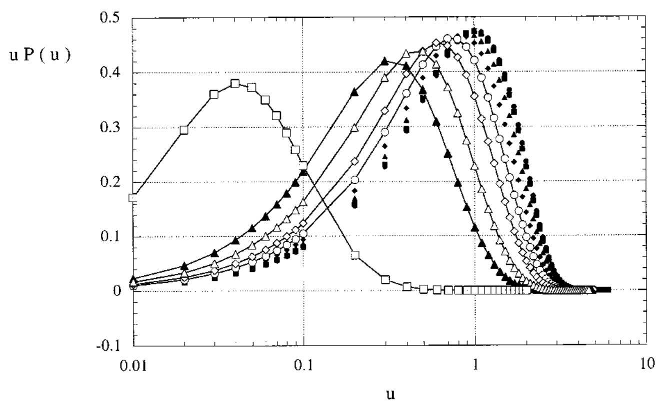

Fig. 11. Probability distribution function of the logarithm of the transverse velocity increments $\delta u P(\delta u)$ in turbulence at different scale separations $\ell = 1, 21, 41, 61, \ldots, 121, 129$. The broader distributions is for $\ell = 129$, and it tends to be shallower as ℓ decreases towards 1.

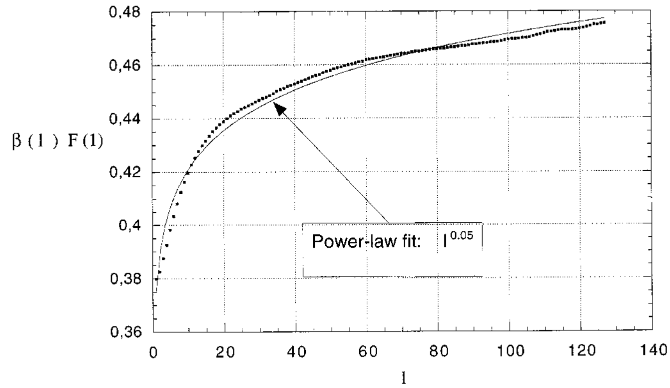

Fig. 12. Plot of the characteristic finite size exponents $\beta(\ell)$ for the transverse PDF's (squares). This function can only be determined up to a multiplicative constant. It was extracted using all PDF's between $\ell = 1$ up to $\ell = 129$ using a procedure described in the text. The line is the fit using the theoretical predictions, dictated by finite size scale invariance.

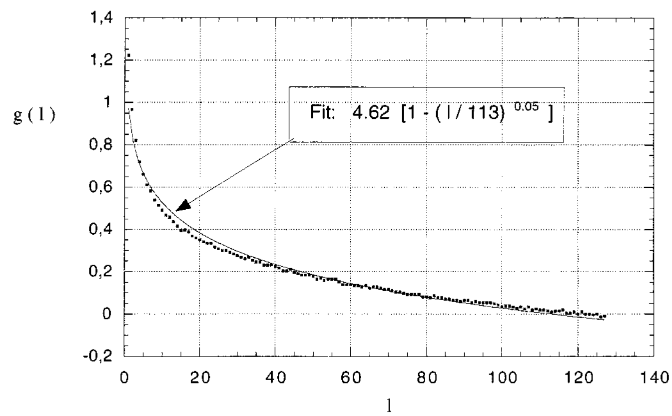

Fig. 13. Plot of the characteristic finite size exponents $g(\ell)$ for the transverse PDF's (squares). This function can only be determined up to a multiplicative and additive constant. It was extracted using all PDF's, for $\ell = 1$ to $\ell = 129$, using a procedure described in the text. The line is the fit using the theoretical predictions, dictated by finite size scale invariance, and the value of $\alpha = -0.05$ previously determined using β.

universal for a given order: in Figure 10, one can see a comparison between scaling functions for longitudinal and transverse, and for the two orders studied. They overlap pretty well at a given order, even up to the smallest scale of the simulation. Clearly, it would be interesting to try this approach on a larger number of structure functions to investigate more closely this issue.

In a second series of test, we tried to investigate the validity of our approach onto the PDF of the increments. As noted in Section 4.2.2, the scaling approach is formally valid only for the propagator, which must be extracted from the real PDF's using rather involved data analysis [26,38]. However, it was pointed to us by B. Andreotti (private communication) that in the case of the transverse increments, the propagator is close to a delta function and so the scaling should apply rather well directly on the PDF's. We thus consider only the PDF of the transverse velocity increments. These PDF are symmetrical, we may then consider only their symmetrical part by considering the absolute value of the velocity increments. Figure 11 shows several functions $|\delta u|P(|\delta u|)$ at different scale separations $\ell = 1,\ 21,\ 41, \ldots, 121, 129$. Using this figure, we recorded both the location of the maxima $u_{\max}$ and the value at the maxima $P_{\max}$. If the theoretical formulation (34) is valid, easy algebra shows that $\beta(\ell)$ is proportional to $P_{\max}$, and $g(\ell)$ is given by $-\ln(u_{\max}\beta)$ up to an additive and multiplicative constant. We computed these quantities for all values of ℓ between 1 and 129, resulting in the functions displayed in Figures 12 and 13. Once again, one sees that the theoretical (power law for β and power-law with a constant for g) works very well for both functions, and for all values of ℓ. The finite size collapse using these function is shown in Figure 14, and works extremely well, except maybe for $\ell = 1$ which display some significant deviations at large values of $e^g u^\beta$. This might

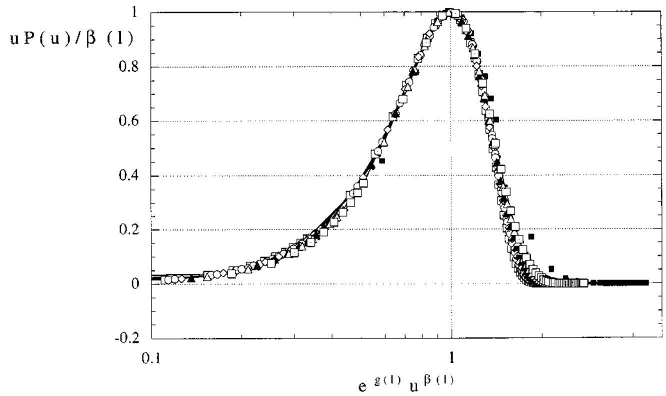

Fig. 14. Finite size collapse of the eight distributions given in Figure 11, using the theoretical formula (34), and the values of β and g given in Figures 12 and 13. The collapsing curves defines a finite size scaling function, which takes into account the finite size of the system.

be a resolution effect, since this scale only marginally resolved in the simulation (the viscous scale was a little bit smaller than the mesh size).

5.2.5 A mechanism for finite size scale invariance

We just have provided limited but strong evidence that turbulent data follow the finite size scale invariance symmetry. At this stage, we would like to propose a speculative mechanism allowing the arising of this invariance in a finite size turbulent system. It is easy to check that the finite size scale invariance is not formally satisfied by the Navier-Stokes equations. This can be traced back to to non-linear term, which severely limits the possible symmetry transformations of Navier-Stokes. In the context of 2D turbulence, however, we have shown [41] that the effective dynamics at small scales is governed by a linear equation, due to the non-locality of the interactions. For the time being, we have no theoretical or experimental proof that such situation also prevails in 3D turbulence. If we speculate that some sort of non-locality also exists in 3D turbulence, then we have a natural mechanism for the onset of finite size scale invariance: the effective dynamics will then be linear, and it will be naturally invariant by our linear finite size transformation. Note that the notion of non-locality of interaction (interaction between two scales of very different size), is intimately linked with finite size effects: in an infinite size system, there are no privileged scales, and the natural interactions can be only local.

6 Discussion

Let us summarize our main results. We have developed a new approach to scale symmetry, which takes into account the possible finite cut-offs of the fields or the parameters. This new symmetry, called finite size scale symmetry:

i) includes the usual self-similarity as a limiting case, when the cut-offs are set to infinity (infinite size-system);
ii) is consistent with the traditional finite size scaling approach already used in critical phenomena;
iii) enables the computation of some of the universal functions appearing in the finite size scaling formulation;
iv) allows scale transformations leaving the cut-offs invariant, like in the traditional renormalization approach;
v) can be formulated to allow for positive or negative fields and parameters;
vi) leads to new prediction about the shape of some distributions in critical phenomena or turbulence.

These findings were illustrated on several examples drawn from critical phenomena and turbulence. In all cases, we found extremely good agreement with the theoretical predictions arising from the finite size symmetry in a system with a largest scale. Systems with two bounding scales are actually not easy to find: a smallest characteristic scale usually implies some sort of discrete scale symmetry, linked with the possibility of complex scaling exponent [42]. It would be therefore interesting to investigate whether different shape of scaling function arise in this kind of systems. Sornette (private communication) actually pointed out to us that in the case of the rupture of a heterogeneous material, the scaling function seems to be well-approximated by an hyperbolic tangent, which is actually one generic form of a system with two limiting scales (see Tab. 1). Obviously, more work is needed to determine whether the finite size scale invariance is a useful concept and indeed leads to universal predictions.

If it is, it would mean that systems near their critical state, in a presence of finite size effect, place themselves in a state where effective law of interactions are invariant under finite size scale symmetry. This would be a surprising example of restoration of a lost symmetry – the scale invariance, broken by finite size effects. In any case, this new symmetry can be used for a better understanding of the influence of finite size cut-offs, which goes beyond the traditional dimensional analysis.

Appendix

In this appendix, we describe the nine possible generic shape for the function $g(\mu)$. This shape is fixed by iteration of the formula (12) starting from $\mu = 1$. It gives:

$$\mu(m) = 1 \tilde{\otimes} 1 \tilde{\otimes} \ldots \tilde{\otimes} 1 = 1^{\tilde{[m]}},$$
$$g(\mu(m)) = U(1) \otimes g(1) \otimes \ldots \otimes g(1) = g(1)^{[m]}. \qquad \text{(A.1)}$$

Here the notation $[m]$ (resp. $\tilde{[m]}$), stands for the mth iterate *via* $\otimes$ (resp. $\tilde{\otimes}$). The shape of these iterates depends on whether the fixed points of the composition laws are finite or infinite, equal or not. However, since by definition the fixed points are of opposite sign, (they are measured with respect to a reference which lies in between the minimum and the maximum cut-off), they can be only three generic cases per composition law: i) two infinite fixed points; ii) one finite, one infinite fixed point; iii) two finite fixed points. Overall, there are nine possibilities for the shape of g as a function of μ.

The classification was done in [20]. We can use their result to invert the function $\mu(m)$ and get m as a function of μ, and then $g(\mu)$ *via* (A.1). The possible cases can be summarized in the following table: one entry describes whether $s_{\pm}$, the fixed points of the composition law $\tilde{\otimes}$ are finite and what is the corresponding shape for $m(\mu)$. The second entry describes whether $U_{\pm}$, the fixed points of $\otimes$ are finite or infinite, and what is the corresponding shape for $g(m)$. The intersection gives $g(\mu)$ by inserting m as a function of μ. In the table, α is an arbitrary parameter, depending on the system, and we have not consider cases like U_- infinite, U_+ finite which can be obtained from the case U_- finite, U_+ infinite by a straight transformation of U_- into U_+. Note that since U_+ and U_- are of opposite sign, all the functions given in the table are regular over the interval $[s_-; s_+]$.

This work was initiated by a remark by J.-F. Muzy about the relevance of power law in a finite size system. I benefited from constant interaction with B. Andreotti whose suggestions had important impact on the development of the present work. The data used in my analysis were the courtesy of James Sethna and Maurice Meneguzzi. I thank them warmly. I have the pleasure to acknowledge useful comments of L. Nottale, D. Sornette and F. Graner. I have been partly supported by a NATO fellowship during my stay in Boulder.

References

1. A.N. Kolmogorov, Dokl. Akad. Nauk. SSSR **30**, 9 (1941).
2. V. Privman, M.E. Fisher, Phys. Rev. B **30**, 322 (1994).
3. M.E. Fisher, in *Critical Phenomena, Proc. 51st Enrico Fermi Summer School, Varenna*, edited M.S. Green (Academic Press, New York, 1972).
4. M.E. Fisher, M.N. Barber Phys. Rev. Lett. **28**, 1516 (1972).
5. M.N. Barber, in *Phase Transitions and Critical Phenomena, Vol. 8*, edited by C. Domb, J.L. Lebowitz (Academic Press, London, 1983).
6. U. Frisch, *Turbulence: The Legacy of A.N. Kolmogorov* (Cambridge University Press, 1995).
7. G.I. Barenblatt, *Similarity, Self-similarity and Intermediate Asymptotics* (Plenum, 1979).
8. F. Graner, B. Dubrulle, Phys. Rev. E. **56**, 6427 (1997).
9. B. Dubrulle, in *Scale invariance and beyond, Les Houches, March 10-14 1997*, edited by B. Dubrulle, F. Graner, D. Sornette (EDP Sciences, Springer, 1998), p. 275.
10. B. Dubrulle, F. Graner, Phys. Rev. E. **56**, 6435 (1997).
11. L. Nottale, *Fractal space-time and microphysics* (World Scientific, Singapour, 1993).
12. L. Nottale, Int. J. Mod. Phys. A **4**, 5047 (1989).
13. L. Nottale, Int. J. Mod. Phys. A **7**, 4899 (1992).
14. L. Nottale, in *Scale invariance and beyond, Les Houches, March 10-14 1997*, edited by B. Dubrulle, F. Graner, D. Sornette (EDP Sciences, Springer, 1998), p. 249.
15. S.K. Ma, Rev. Mod. Phys. **45**, 589 (1973).
16. E. Brezin, J. Zinn-Justin, Nucl. Phys. B **257**, 867 (1985).
17. A.M. Polyakov, Nucl. Phys. B **396**, 367 (1993).
18. R. Benzi, B. Legras, R. Scardovelli, Europhys. Lett. **29**, 203 (1995).
19. B. Dubrulle, F.-M. Breon, F. Graner, A. Pocheau, Eur. Phys. J. B **4**, 89 (1998).
20. B. Dubrulle, F. Graner, J. Phys. II France **6**, 797 (1996).
21. B. Dubrulle, F. Graner, J. Phys. II France **6**, 817 (1996).
22. B. Dubrulle, G. He, *Possible Statistics of Two coupled Fields* (submitted to Eur. Phys. J. B).
23. J.G. Brankov, D.M. Danchev, J. Math. Phys. **32**, 2543 (1991).
24. J. Sethna, in *Scale invariance and beyond, Les Houches, March 10-14 1997*, edited by B. Dubrulle, F. Graner, D. Sornette (EDP Sciences, Springer, 1998), p. 87.
25. O. Perkovic, K.A. Dahmen, J.P. Sethna, Phys. Rev. E **59**, 6106 (1999).
26. S. Roux, Thèse de l'Université de Marseille, (unpublished, 1997).
27. A. Arneodo, J.-F. Muzy, S. Roux, J. Phys. II France **7**, 363 (1997).
28. F. Moisy, P. Tabeling, H. Willaime, Phys. Rev. Lett. **82**, 3994 (1999).
29. B. Castaing , J. Phys. II France **50**, 147 (1989).
30. B. Castaing, Y. Gagne, M. Marchand, Physica D **68**, 387 (1993).
31. B. Chabaud, A. Naert, J. Peinke, F. Chilla, B. Castaing, B. Hébral, Phys. Rev. Lett. **73**, 3227 (1994).
32. B. Castaing, B. Dubrulle, J. Phys. II France **5**, 895 (1995).
33. Z.S. She, E.C. Waymire, Phys. Rev. Lett. **74**, 262 (1995).
34. A.N. Kolmogorov, J. Fluid Mech. **13**, 82 (1962).
35. D. Schertzer, S. Lovejoy, F. Schmitt, in *Small scale structures in 3D hydrodynamics and MHD turbulence*, edited by M. Meneguzzi, A. Pouquet (Springer-Verlag, 1995).
36. F. Schmitt, D. Lavallée, D. Schertzer, S. Lovejoy, Phys. Rev. Lett. **68**, 305 (1992).
37. R. Benzi, L. Biferale, S. Ciliberto, M. Struglia, R. Tripiccione, Physica D **1317**, 1 (1996).
38. B. Andreotti, Thèse de l'Université Paris VII, (unpublished, 1999).
39. A. Vincent, M. Meneguzzi, J. Fluid Mech. **225**, 1 (1991).
40. A. Arneodo *et al.*, Europhys. Lett. **34**, 411 (1996).
41. J.-P. Laval, B. Dubrulle, S. Nazarenko, Phys. Rev. Lett. **83**, 4061 (1999).
42. D. Sornette, in *Scale invariance and beyond, Les Houches, March 10-14 1997*, edited by B. Dubrulle, F. Graner, D. Sornette (EDP Sciences, Springer, 1998), p. 235.