Symmetry improvement of 3-particle irreducible effective actions for $O(N)$ scalar field theories

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Introduction

Field theories are widely used in areas including high energy, cosmology and condensed matter physics. These applications are growing ever more demanding. However, a gap exists between the standard perturbation theory methods (Feynman diagrams) and large scale "black box" Monte Carlo simulations. *n*-particle irreducible effective actions (*n*PIEA) are powerful non-perturbative tools which fill the gap by summing certain subsets of Feynman diagrams to infinite order. Unfortunately, many important properties of field theories depend on order-by-order cancellations (e.g. symmetries, gauge invariances, supersymmetries, unitarity, etc.) which are "brittle" under approximations of *n*PI effective actions.

We have extended a method recently developed by Pilaftsis and Teresi[2] which is designed to restore all symmetries in the *n*PI formalism. Our extension sums more Feynman diagrams and enforces more of the symmetry, and suggests an extension to gauge theories.

The Model Theory

We consider a linear σ-model, a quantum field theory in 3+1 dimensional spacetime with N real scalar fields $\phi_a(x)$, $a = 1, \cdots, N$ and action

$$S = \int_x \frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a - \frac{\lambda}{4!}\left(\phi_a\phi_a - v^2\right)^2 .$$

The theory possesses an $O(N)$ symmetry

$$\phi_a \to \phi_a + i\epsilon_A T^A_{ab}\phi_b,$$

where T^A_{ab} are the generators of the symmetry group. At low temperatures the scalar field develops a symmetry breaking expectation value $(\langle\phi_1\rangle, \cdots, \langle\phi_N\rangle) = (0, \cdots, 0, v)$ and contains a massive σ or "Higgs" boson and $N-1$ massless Goldstone bosons. At high temperature the symmetry is restored by a second order phase transition.

The Effective Action

The 3PIEA for this theory is defined by considering the path integral in the presence of source terms:

$$\mathrm{e}^{iW[J,K^{(2)},K^{(3)}]} = \int \mathcal{D}[\phi]\mathrm{e}^{i\left(S - \int J\phi - \frac{1}{2}\int K^{(2)}\phi^2 - \frac{1}{6}\int K^{(3)}\phi^3\right)}$$

and performing Legendre transforms with respect to the sources:

$$\Gamma\left[\varphi_a, \Delta_{ab}, V^{(3)}_{abc}\right] = W - J_a\frac{\delta W}{\delta J_a} - K^{(2)}_{ab}\frac{\delta W}{\delta K^{(2)}_{ab}} - K^{(3)}_{abc}\frac{\delta W}{\delta K^{(3)}_{abc}},$$

where φ_a, Δ_{ab}, and $V^{(3)}_{abc}$ are the *exact* proper 1-, 2- and 3-point correlation functions. A diagrammatic expansion can be derived[1] and truncated at e.g. the three loop level. The truncated equations of motion are not compatible with the symmetry, however one can define a symmetry improved action

$$\tilde{\Gamma} = \Gamma - \lambda^\alpha f_\alpha(\{\mathcal{W}_i\}),$$

where $\mathcal{W}_i = 0$ are the Ward identities, λ^α are Lagrange multipliers and the f_α are constraint functionals. In the model at hand

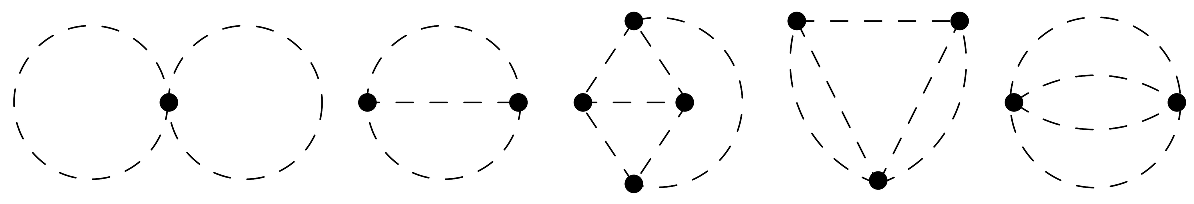
$$0 = \mathcal{W}_2 \equiv \int \Delta^{-1}_{ca} T^A_{ab}\varphi_b,$$

$$0 = \mathcal{W}_3 \equiv \int V^{(3)}_{dca}T^A_{ab}\varphi_b + \Delta^{-1}_{ca}T^A_{ad} + \Delta^{-1}_{da}T^A_{ac},$$

which express the masslessness of the Goldstone bosons and a relation between the masses and the Higgs-Goldstone interaction respectively.

Hartree-Fock Equations of Motion

We examine the unimproved 2PIEA, symmetry improved 2PIEA and symmetry improved 3PIEA. These possesses a diagrammatic expansion shown below up to three loop order. The difference between 2PIEA and 3PIEA is that the three point vertices are bare vertices in 2PIEA, but the resummed vertex functions $V^{(3)}_{abc}$ in the 3PIEA.



For numerical simplicity and to make contact with the literature we consider the Hartree-Fock approximation, which keeps only the first diagram, leading to the equations of motion

$$m_G^2 = \frac{\lambda}{6}\left(\varphi^2 - v^2\right) + \frac{\hbar\lambda}{6}(N+1)\,\mathcal{T}^{\text{fin}}_G + \frac{\hbar\lambda}{6}\mathcal{T}^{\text{fin}}_N,$$

$$m_N^2 = \frac{\lambda}{6}\left(3\varphi^2 - v^2\right) + a\frac{\hbar\lambda}{6}(N-1)\,\mathcal{T}^{\text{fin}}_G + b\frac{\hbar\lambda}{2}\mathcal{T}^{\text{fin}}_N,$$

where $\mathcal{T}^{\text{fin}}_{G/N}$ are Bose-Einstein integrals and $(a,b) = (1,1)$ in the 2PIEA and $\left(\frac{N+1}{N-1}, \frac{1}{3}\right)$ in the symmetry improved 3PIEA.

Hartree-Fock Solutions

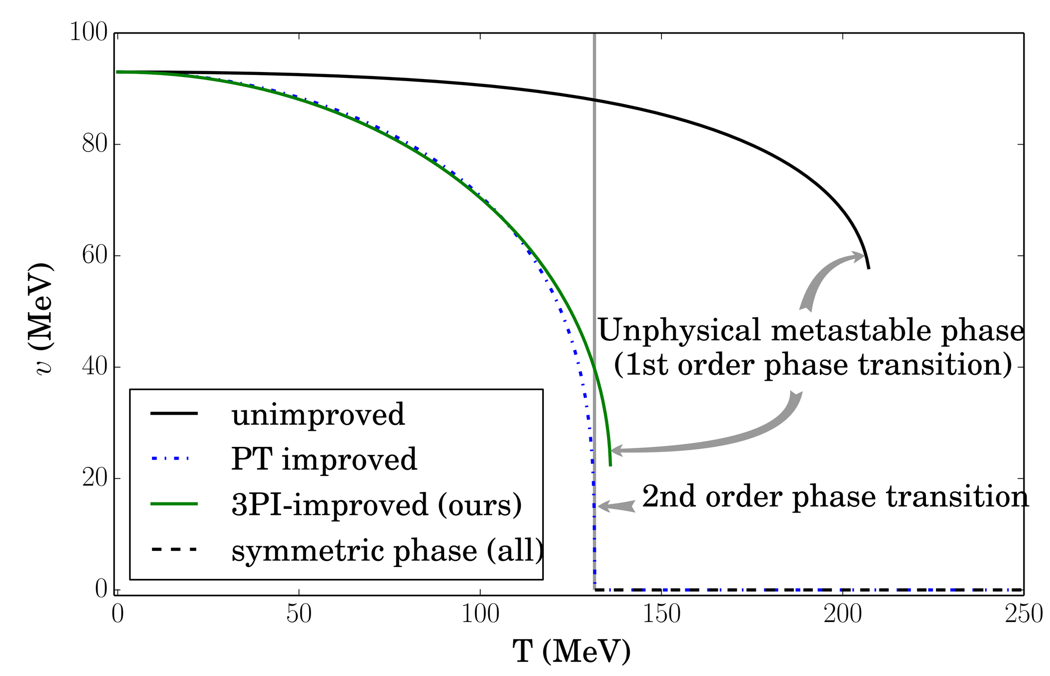


Figure : Vacuum expectation value of the scalar field φ as a function of temperature computed using three methods, all truncated at the Hartree-Fock level. Our method and the standard 2PIEA both give unphysical first order phase transitions, though ours is much closer to the qualitatively correct symmetry improved 2PIEA result. (We take $N = 4$, $\lambda = 87$, $v = 93$ MeV.)

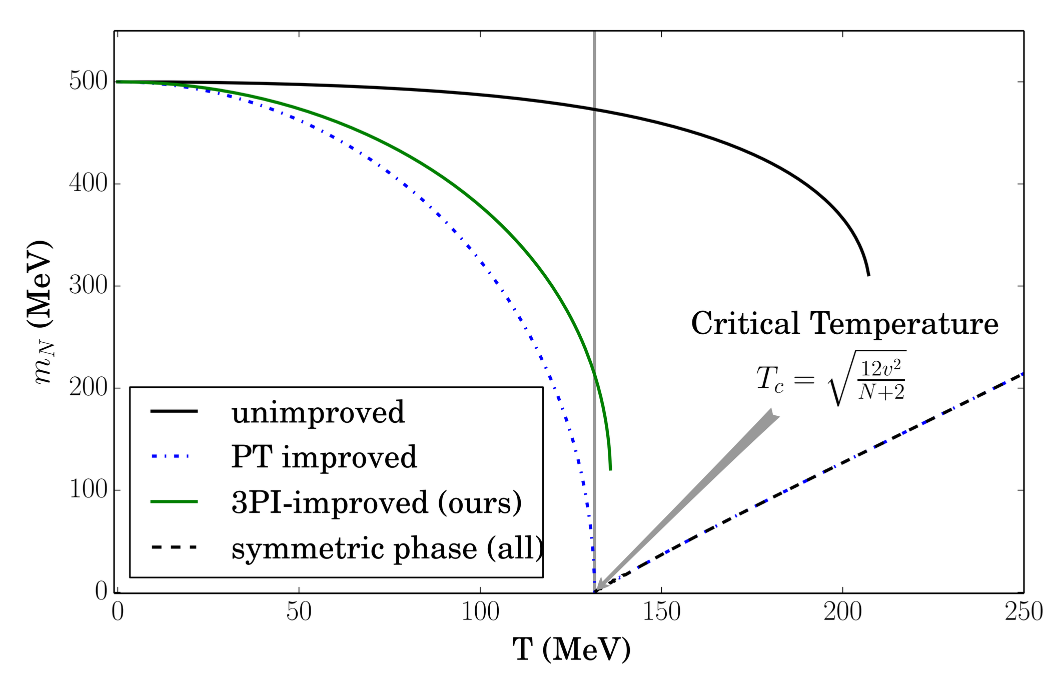


Figure : The "Higgs" mass m_N as a function of temperature (scare quotes because this is not a gauge theory). Unphysical metastable phases are again seen, though our method is much closer to the correct result.

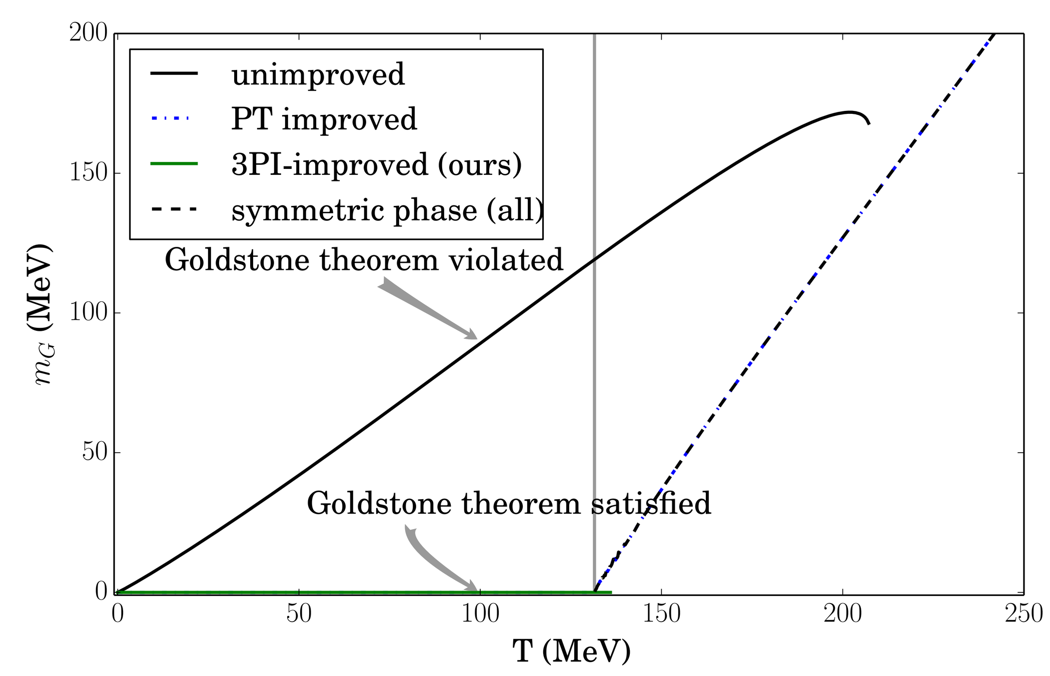


Figure : The Goldstone mass m_G as a function of temperature. The standard 2PIEA violates the Goldstone theorem, which is satisfied by the symmetry improved methods.

Hartree-Fock Effective Potential

The symmetry improved effective potential is defined via the Ward identity

$$\frac{\delta^2\Gamma[\varphi]}{\delta\varphi_c\delta\varphi_a}T^{(IJ)}_{ab}\varphi_b + \frac{\delta\Gamma[\varphi]}{\delta\varphi_a}T^{(IJ)}_{ac} = 0,$$

which simplifies in this model to

$$\frac{\partial \tilde{V}_{\text{eff}}(\varphi)}{\partial\varphi} = -\varphi\Delta_G^{-1}(k=0).$$

In the 1PI case $\tilde{V}_{\text{eff}}$ is numerically equal to the usual finite temperature effective potential, but this alternate definition generalises to symmetry improved *n*PIEA. $\tilde{V}_{\text{eff}}$ is shown below for both symmetry improvement methods for the same parameter values as before and at a range of temperatures straddling the critical temperature.

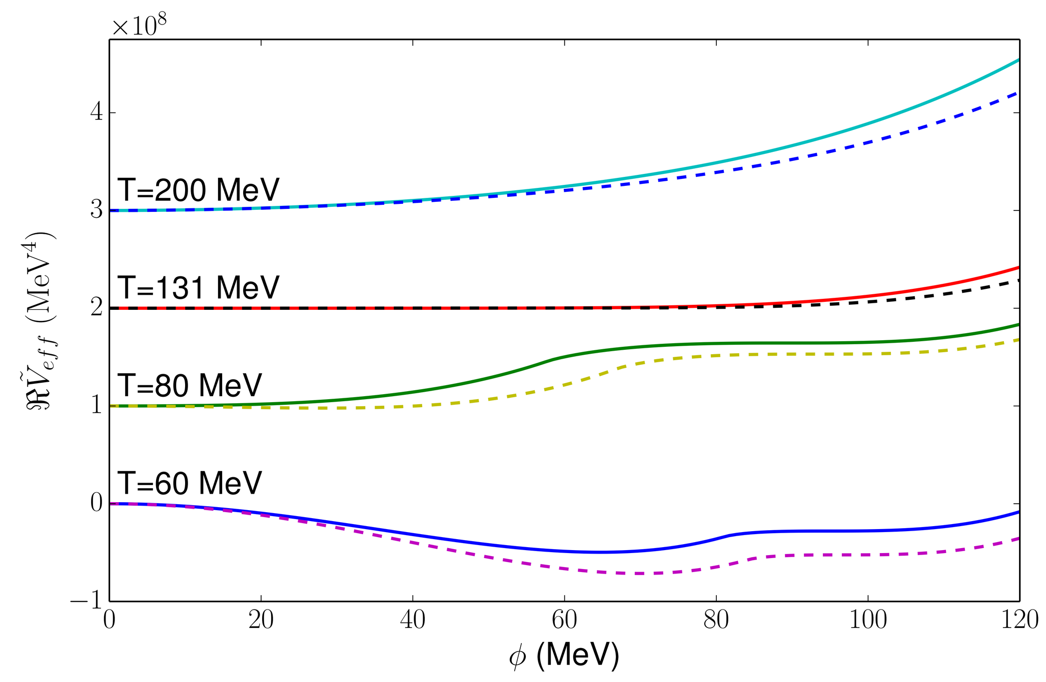


Figure : The symmetry improved effective potential from the 2PI (solid) and 3PI (dashed) solutions. Constant shifts have been added to aid visualisation. $T_c = 131$ MeV is the critical temperature.

Conclusions

- Symmetry improved *n*PI effective actions maintain manifest consistency with all symmetries of a theory while resumming many perturbative diagrams to infinite order.
- For *n*PIEA Ward identities must be enforced for each free correlation function.
- When truncated at the Hartree-Fock level, symmetry improved 3PIEA yields a more accurate phase transition than traditional techniques, but is worse than the symmetry improved 2PIEA.
- All methods investigated agree in the symmetric phase and the broken phase at zero temperature, but disagree in the broken phase at nonzero temperature.
- Our method is expected to be superior once vertex corrections are included. This is left to future work.

Acknowledgements

This work was supported by an Australian Postgraduate Award. We also thank D. Kosov and D. Teresi for helpful comments.

References

[1] J. Berges, Introduction to Nonequilibrium Quantum Field Theory. AIP Conference Proceedings, 739, 3-62 (2004). DOI:10.1063/1.1843591, arXiv:hep-ph/0409233

[2] A. Pilaftsis and D. Teresi, Symmetry-Improved CJT Effective Action. Nucl. Phys. B 874, 594 (2013). DOI:10.1016/j.nuclphysb.2013.06.004, arXiv:1305.3221

[3] M.J. Brown and I.B. Whittingham, Symmetry improvement of 3-particle irreducible effective actions for $O(N)$ scalar field theories, *manuscript in preparation*.

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