This, in turn, implies inequality (1.4) and thus completes the proof.

Conclusion

In this note, we have proved two inequalities which involve the singular values of matrices. Proposition 2.1 has particular implication in the H^∞ approach to control system design. The modification of the proofs to the case where $Z = [X \vdots Y]$ and m denotes the number of rows is straightforward by considering ZZ^*.

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Necessary and Sufficient Conditions for Balancing Unstable Systems

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Abstract—**Necessary and sufficient conditions are given for the existence of balancing transformations for minimal state-space realizations (A, B, C) where A may be unstable. These conditions are expressed in terms of the real diagonalizability of the product of the reachability Gramian and the observability Gramian. For symmetric realizations these conditions can be reformulated in terms of the real diagonalizability of the cross Gramian, and we show that minimal symmetric systems can be internally balanced if the associated Hankel matrix is positive semidefinite. Examples are given of minimal systems, including symmetric systems, which cannot be balanced.**

Introduction

In this note we present simple necessary and sufficient conditions for the existence of balancing transformations for minimal state-space realizations (A, B, C) where A may be unstable. In particular, we show that a balancing transformation exists if and only if the product $W_r W_o$ of the reachability Gramian W_r and the observability Gramian W_o is similar to a real diagonal matrix. We further show that a minimal system can be internally balanced if and only if the product $W_r W_o$ is similar to a positive diagonal matrix. These results are easily proved by applying established congruence theorems for symmetric matrices [10], and a simple example is given of a minimal system which cannot be balanced.

The above condition on $W_r W_o$ is related to the work of Fernando and Nicholson [2]–[4] and Laub *et al.* [8] on symmetric realizations where it is shown that the cross Gramian W_{ro} satisfies $W_{ro}^2 = W_r W_o$. From this we conclude that the Jordan structure of $W_r W_o$ is determined by the Jordan structure of W_{ro}, and therefore a minimal symmetric realization can be internally balanced if and only if the cross Gramian is similar to a real diagonal matrix. An example is given which shows that there are minimal symmetric systems which cannot be balanced. Finally, we show that any minimal symmetric system which has a positive semidefinite Hankel matrix can be internally balanced.

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Balancing

Let (A, B, C) be a minimal (that is observable and controllable) state-space realization with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{r\times n}$ for the linear time-invariant system

$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Cx.$$

Although we allow A to be unstable, we shall assume throughout that $\lambda + \mu \neq 0$ for any eigenvalues λ, μ of A [i.e., $\lambda, \mu \in \sigma(A)$]. This assumption is needed to ensure the existence and uniqueness of the reachability Gramian W_r and the observability Gramian W_o which are defined implicitly by the equations

$$AW_r + W_r A^T = -BB^T \tag{3}$$

$$A^T W_o + W_o A = -C^T C. \tag{4}$$

The goal of balancing is to find a coordinate transformation such that in the new coordinate system the reachability Gramian and the observability Gramian are both diagonal and, if possible, equal. More specifically, if we transform the state coordinates $x = T\hat{x}$ where T is a nonsingular real matrix, then with respect to the new coordinates $\hat{x}$, we have

$$\hat{A} = T^{-1}AT,\ \hat{B} = T^{-1}B,\ \hat{C} = CT \tag{5}$$

and

$$\hat{W}_r = T^{-1}W_r T^{-T},\ \hat{W}_o = T^T W_o T. \tag{6}$$

If $\hat{W}_r$ and $\hat{W}_o$ are both diagonal, then we say that the system (A, B, C) is *balanced* by T, and that T is a *contragredient* transformation for W_r and W_o. If $\hat{W}_r$ and $\hat{W}_o$ are both diagonal and equal, then we say that the system (A, B, C) is *internally* balanced by T. One important aspect of balancing lies in its relationship to model reduction (see [9], [5], [6]).

It is well known that (A, B, C) can be internally balanced when A is stable and (A, B, C) is observable and controllable [6]. However, for many state-space realizations the matrix A is not stable. In the next section we characterize those (A, B, C) which can be balanced for unstable A.

Necessary and Sufficient Conditions for Balancing

Definitions: Two matrices M_1, M_2 in $\mathbb{R}^{n\times n}$ are *similar* if there exists a nonsingular real matrix X such that

$$M_1 = X^{-1}M_2X. \tag{7}$$

Two matrices M_1, M_2 in $\mathbb{R}^{n\times n}$ are *congruent* if there exists a nonsingular real matrix Y such that

$$M_1 = Y^T M_2 Y. \tag{8}$$

We need the following result which can be found in [10].

Theorem 1: Let S_1 and S_2 be nonsingular real symmetric matrices. Then the following are equivalent.

1) S_1 and S_2 can be simultaneously diagonalized by a real congruence transformation.

2) $S_1^{-1}S_2$ is similar to a real diagonal matrix.

Using Theorem 1 we obtain the following.

Theorem 2: Assume that the state-space realization (A, B, C) is observable and controllable and that $\lambda + \mu \neq 0$ for any $\lambda, \mu \in \sigma(A)$. Then the following are equivalent.

1) There exists a balancing transformation T for (A, B, C).

2) W_rW_o is similar to a real diagonal matrix.

Proof: Since $\mu + \lambda \neq 0$ for any $\mu, \lambda \in \sigma(A)$, we have that there exist unique solutions W_r and W_o to (3) and (4), respectively, and these solutions are symmetric (see [7]). By the assumptions of observability and controllability we also have that W_r and W_o are nonsingular (see [1, Prop. 4, p. 65]). Now assume 1). Then there exists a real nonsingular matrix T such that (6) holds with $\hat{W}_r$, $\hat{W}_o$ both diagonal and necessarily real since T, W_o, W_r are real. This means that

$$T^{-1}W_rW_oT=\hat{W}_r\hat{W}_o \tag{9}$$

so that W_rW_o is similar to a real diagonal matrix. Thus, 1) $\Rightarrow$ 2).

Now assume 2). Then $W_rW_o = (W_r^{-1})^{-1}W_o$ is similar to a real diagonal matrix. By Theorem 1 this means that W_r^{-1} and W_o are simultaneously diagonalized by a real congruence transformation T:

$$T^TW_r^{-1}T=D_r \tag{10}$$

$$T^TW_oT=D_o \tag{11}$$

where D_r and D_o are real and diagonal. Since W_r and T are nonsingular, we see that D_r is nonsingular and

$$T^{-1}W_rT^{-T}=D_r^{-1}. \tag{12}$$

Since D_r^{-1} is diagonal, we have that T is a contragredient transformation for W_r and W_o and that (A, B, C) is balanced by T. Thus, 2) $\Rightarrow$ 1). This completes the proof of Theorem 2. ■

The following example illustrates that there are observable and controllable unstable systems (A, B, C) which cannot be balanced.

Example 1: Let

$$A=\begin{bmatrix}-1 & 1\\ 0 & 2\end{bmatrix}, B=C=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

Then

$$W_r=\begin{bmatrix}3/4 & 1/4\\ 1/4 & -1/4\end{bmatrix} W_o=\begin{bmatrix}1/2 & -1/2\\ -1/2 & 0\end{bmatrix}$$

$$\text{and } W_rW_o=\begin{bmatrix}2/8 & -3/8\\ 2/8 & -1/8\end{bmatrix}.$$

Since W_rW_o has eigenvalues $\lambda_\pm = 1/16 \pm \sqrt{15/16}i$, it cannot be similar to a real diagonal matrix and hence, by Theorem 2, the system (A, B, C) cannot be balanced.

The next theorem is the analog of Theorem 2 for internally balanced systems.

Theorem 3: Assume that the state-space realization (A, B, C) is observable and controllable and that $\lambda + \mu \neq 0$ for any $\lambda, \mu \in \sigma(A)$. Then the following are equivalent.

1) There exists an internal balancing transformation T for (A, B, C).

2) W_rW_o is similar to a positive diagonal matrix.

Proof: As in the proof of Theorem 2 there exist real symmetric nonsingular solutions W_r and W_o to (3) and (4).

Now assume 1). Then there is a real nonsingular matrix T such that (6) holds with $\hat{W}_r = \hat{W}_o \equiv D$ diagonal. Since W_r, W_o and T are nonsingular, we must have that D is nonsingular, and hence

$$T^{-1}W_rW_oT=\hat{W}_r\hat{W}_o=D^2>0, \tag{13}$$

that is W_rW_o is similar to a positive diagonal matrix and so 1) $\Rightarrow$ 2).

Now assume 2). Then by Theorem 1 there exists a nonsingular real matrix T such that

$$T^TW_r^{-1}T=\hat{D}_1,\ T^TW_oT=D_2 \tag{14}$$

where $\hat{D}_1$ and D_2 are diagonal. Now since W_r and T are invertible, so is $\hat{D}_1$. Thus,

$$T^{-1}W_rT^{-T}=\hat{D}_1^{-1} \equiv D_1, \tag{15}$$

and so

$$D_1D_2=T^{-1}W_rW_oT. \tag{16}$$

That is, W_rW_o is similar to the diagonal product D_1D_2. But by assumption W_rW_o is similar to a positive diagonal matrix. Thus, the main diagonal entries of D_1D_2 are positive

$$(D_1D_2)_{ii}>0 \qquad \text{for } i=1 \text{ to } n. \tag{17}$$

Now let

$$\hat{T} \equiv T(D_1D_2)^{1/4}|D_2|^{-1/2} \tag{18}$$

where

$$(D_1D_2)^{1/4}=\operatorname{diag}((D_1D_2)_{ii}^{1/4}) \tag{19}$$

and

$$|D_2|^{-1/2}=\operatorname{diag}(|D_{2ii}|^{-1/2}) \tag{20}$$

and all roots are positive.

Then

$$\begin{aligned}\hat{T}^TW_0\hat{T}&=|D_2|^{-1/2}(D_1D_2)^{1/4}D_2(D_1D_2)^{1/4}|D_2|^{-1/2}\\ &=(D_1D_2)^{1/2}D_2|D_2|^{-1}\end{aligned} \tag{21}$$

where we have used the fact that diagonal matrices commute.

We also have

$$\begin{aligned}\hat{T}^{-1}W_r\hat{T}^{-T}&=|D_2|^{1/2}(D_1D_2)^{-1/4}D_1(D_1D_2)^{-1/4}|D_2|^{1/2}\\ &=(D_1D_2)^{-1/2}D_1|D_2|\\ &=(D_1D_2)^{-1/2}D_1D_2D_2^{-1}|D2|\\ &=(D_1D_2)^{1/2}D_2^{-1}|D_2|.\end{aligned} \tag{22}$$

But

$$\begin{aligned}D_2|D_2|^{-1}&=\operatorname{diag}(D_{2ii}/|D_{2ii}|)\\ &=\operatorname{diag}(|D_{2ii}|/D_{2ii})=D_2^{-1}|D_2|.\end{aligned} \tag{23}$$

Thus, by (23) we see that the right-hand sides of (21) and (22) are equal, so that $\hat{T}$ is an internal balancing transformation for (A, B, C):

$$\hat{T}^TW_o\hat{T}=\hat{T}^{-1}W_r\hat{T}^{-T}$$

is diagonal. Thus 2), $\rightarrow$ 1) and the proof is complete. ■

Symmetric Realizations

A state-space realization (A, B, C) with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ is symmetric if the transfer function $G(s) \equiv C(sI - A)^{-1}B$ is symmetric for all complex s which are not eigenvalues of A. A simple geometric series argument shows that if (A, B, C) is symmetric, then CA^KB is symmetric for $K = 0, 1, \cdots$. Symmetric systems arise naturally in circuit theory [11] and any SISO system is trivially symmetric.

For symmetric systems we can define the cross Gramian W_{ro} as the solution to

$$AW_{ro}+W_{ro}A=-BC \tag{24}$$

where we assume that $\mu + \lambda \neq 0$ for any $\mu, \lambda \in \sigma(A)$. The next theorem provides an elegant connection between the reachability, observability, and cross Gramians.

Theorem 4: Assume that (A, B, C) is symmetric and that $\mu + \lambda \neq 0$ for any $\mu, \lambda \in \sigma(A)$. Then $W_{ro}^2 = W_rW_o$.

Proof: This result was first proved in [2] for the SISO case, extended by [8] to the stable case, and proved in general in [4].

One of the first uses of the cross Gramian involved characterizing the minimality of (A, B, C). By Theorem 4, (A, B, C) is minimal if and only if W_{ro} is nonsingular (see [2], [8], [4]). Then in [3] it was shown that W_{ro} determines the Cauchy index of the system (A, B, C) and places restrictions on possible model reduced systems. The next theorem shows that the Jordan structure of W_{ro} determines whether the system (A, B, C) can be internally balanced.

Theorem 5: Assume that (A, B, C) is an observable and controllable symmetric state-space realization with $\mu + \lambda \neq 0$ for any $\mu, \lambda \in \sigma(A)$. Then the following are equivalent.

1) The cross Gramian W_{ro} is similar to a real diagonal matrix.

2) There exists an internal balancing transformation T for (A, B, C).

Proof: As in the proof of Theorem 4 W_r, W_o, W_{ro} exist and are uniquely defined by (3), (4), and (24). Moreover, by Theorem 4 we have $W_rW_o = W_{ro}W_{ro}$. As in the proof of Theorem 2 we have that W_r, W_o are nonsingular, and hence so is W_{ro}.

Now assume 1). That is, there exists a nonsingular real matrix X such that

$$X^{-1}W_{ro}X = D_{ro} \tag{25}$$

where D_{ro} is a real nonsingular diagonal matrix. This means that

$$W_rW_o = XD_{ro}X^{-1}XD_{ro}X^{-1} = XD_{ro}^2X^{-1} \tag{26}$$

so that W_rW_o is similar to a positive diagonal matrix. Thus, by Theorem 3 there exists an internal balancing transformation T for (A, B, C). That is 1) $\Rightarrow$ 2).

Now assume 2). Then by Theorem 3 the matrix W_rW_o is similar to a positive diagonal matrix. This means that W_{ro}^2 is similar to a positive diagonal matrix. Now suppose for the sake of contradiction that W_{ro} is not similar to a diagonal matrix. Then

$$W_{ro} = Y^{-1}JY \text{ where } J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_K \end{bmatrix} \tag{27}$$

where the J_i are $n_i \times n_i$ Jordan blocks of the form

$$J_i = [\lambda_i] \quad \text{for } n_i = 1$$

$$J_i = \lambda_i 1 \quad \text{for } n_i > 1,$$

$$\begin{bmatrix} \lambda_i 1 & & & \\ & \lambda_i 1 & & \\ & & \ddots & \\ & & & \lambda_i \end{bmatrix}$$

where $\lambda_i \neq 0$ for any i since W_{ro} is nonsingular. We may assume without loss of generality that the first block J_1 has $n_1 > 1$. Now

$$W_{ro}W_{ro} = Y^{-1}J^2Y = Y^{-1}\begin{bmatrix} J_1^2 & & & \\ & J_2^2 & & \\ & & \ddots & \\ & & & J_K^2 \end{bmatrix} Y \tag{28}$$

so that J^2 is similar to a positive diagonal matrix, and hence J_1^2 is similar to a positive diagonal matrix. But

$$J_1^2 = \begin{bmatrix} \lambda_1^2 & 2\lambda_1 & 1 & & \\ & \lambda_1^2 & 2\lambda_1 & 1 & \\ & & \ddots & & \ddots \\ & & & \lambda_1^2 & 2\lambda_1 \end{bmatrix} \tag{27}$$

and in particular J_1^2 has an eigenvector $V_1 = (1, 0, \cdots, 0)^T$ and a generalized eigenvector $V_2 = (0, 1, 0, \cdots, 0)^T$. This means that J_1^2 is not similar to a diagonal matrix and we have a contradiction. Thus, W_{ro} is similar to a diagonal matrix. Moreover, since W_{ro}^2 is similar to a positive diagonal matrix we must have that W_{ro} is similar to a real diagonal matrix. That is 2) $\Rightarrow$ 1) and the proof is complete. ■

We now give an example of a minimal symmetric realization which cannot be balanced.

Example 2: Let

$$A = \begin{bmatrix} 1 & -1/4 \\ 1/4 & -1/2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = [1 \quad -2].$$

Then

$$W_r = \begin{bmatrix} -4/5 & -2/5 \\ -2/5 & 4/5 \end{bmatrix}, \quad W_o = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}, \quad W_{ro} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

and

$$W_rW_o = W_{ro}^2 = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}.$$

Since W_{ro} has eigenvalues $-1 \pm i$, we see by Theorem 5 that (A, B, C) cannot be internally balanced. Moreover, since W_rW_o has eigenvalues $\pm 2i$, we see by Theorem 2 that (A, B, C) cannot be balanced at all.

We now show that there is a class of symmetric systems which can always be internally balanced; namely those systems with positive semidefinite Hankel matrices.

For a state-space realization (A, B, C) define the Hankel matrix H by

$$H = \begin{bmatrix} C \\ CA \\ CA^{n-1} \end{bmatrix} [B \quad AB \cdots A^{n-1}B]. \tag{30}$$

If (A, B, C) is also symmetric with $\mu + \lambda \neq 0$ for any $\lambda, \mu \in \sigma(A)$, define the "cross-Hankel" matrix H_{ro} by

$$H_{ro} = \begin{bmatrix} C \\ CA \\ CA^{n-1} \end{bmatrix} W_{ro}[B \quad AB \cdots A^{n-1}B]. \tag{31}$$

We need the following preliminary result.

Lemma: Let (A, B, C) be a symmetric realization with $\mu + \lambda \neq 0$ for any $\mu, \lambda \in \sigma(A)$. Then the Hankel matrix H and the cross-Hankel matrix H_{ro} are both symmetric.

Proof: Symmetry implies that CA^KB is symmetric for $K = 0, 1, \cdots$, thus H is symmetric. In order to show that H_{ro} is symmetric we need only show that

$$(CA^iW_{ro}A^jB)^T = CA^jW_{ro}A^iB \tag{32}$$

for any i, j between 0 and $n - 1$. To do this we need the fact that for $|\lambda| > \rho(A) \equiv \max |\lambda(A)|$

$$(CA^K(\lambda - A)^{-1}B)^T = CA^K(\lambda - A)^{-1}B \tag{33}$$

which follows from the geometric expansion

$$(\lambda - A)^{-1} = \frac{1}{\lambda}\left(1 + \frac{A}{\lambda} + \frac{A^2}{\lambda^2} + \cdots\right) \tag{34}$$

and the symmetry assumption.

Now let Γ_1 and Γ_2 be two circles in the complex plane centered at the origin of radii $\rho_1 \neq \rho_2$ with $\rho_1, \rho_2 > \rho(A)$. We may then represent W_{ro} in two ways [7]

$$W_{ro} = \frac{1}{4\Pi^2}\int_{\Gamma_1}\int_{\Gamma_2} \frac{(\lambda_1 - A)^{-1}BC(\lambda_2 - A)^{-1}}{\lambda_1 + \lambda_2}\, d\lambda_2\, d\lambda_1, \tag{35}$$

$$W_{ro} = \frac{1}{4\Pi^2}\int_{\Gamma_1}\int_{\Gamma_2} \frac{(\lambda_2 - A)^{-1}BC(\lambda_1 - A)^{-1}}{\lambda_1 + \lambda_2}\, d\lambda_2\, d\lambda_1, \tag{36}$$

thus

$$CA^iW_{ro}A^jB=\frac{1}{4\Pi^2}\int_{\Gamma_1}\int_{\Gamma_2}\frac{CA^i(\lambda_1-A)^{-1}BC(\lambda_2-A)^{-1}A^jB}{\lambda_1+\lambda_2}\,d\lambda_2\,d\lambda_1, \tag{37}$$

$$\begin{aligned}(CA^iW_{ro}A^jB)^T&=\frac{1}{4\Pi^2}\int_{\Gamma_1}\int_{\Gamma_2}\frac{C(\lambda_2-A)^{-1}A^jBCA^i(\lambda_1-A)^{-1}B}{\lambda_1+\lambda_2}\,d\lambda_2\,d\lambda_1\\&=\frac{1}{4\Pi^2}\int_{\Gamma_1}\int_{\Gamma_2}\frac{CA^j(\lambda_2-A)^{-1}BC(\lambda_1-A)^{-1}A^iB}{\lambda_1+\lambda_2}\,d\lambda_2\,d\lambda_1\\&=CA^jW_{ro}A^iB.\end{aligned} \tag{38}$$

Thus, $H_{ro}^T = H_{ro}$ and the proof is complete. ∎

We now have the following.

Theorem 6: Let (A, B, C) be a minimal symmetric realization with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$. If the associated Hankel matrix is positive semidefinite, then (A, B, C) can be internally balanced.

Proof: We will show that there exists matrices $Q \in \mathbb{R}^{n\times mn}$ and $\tilde{B} \in \mathbb{R}^{n\times n}$ such that

$$QH_{ro}Q^T=\tilde{B}^{-1}W_{ro}\tilde{B}. \tag{39}$$

This means that W_{ro} is similar to a symmetric matrix since H_{ro} is symmetric, and hence W_{ro} is similar to a real diagonal matrix. Then, by Theorem 5, (A, B, C) can be internally balanced.

In order to construct Q and $\tilde{B}$, we note that since H is positive semidefinite and (A, B, C) is minimal, then H can be written as

$$H=U\Sigma^2U^T \tag{40}$$

where $U \in \mathbb{R}^{mn\times n}$ has orthonormal columns and

$$\Sigma=\text{diag}\,(h_1, h_2, \cdots h_n) \tag{41}$$

with $h_1 \geq h_2 \geq \cdots \geq h_n > 0$.

Now let

$$Q=\Sigma^{-1}U^T \tag{42}$$

and define $C, B \in \mathbb{R}^{n\times n}$ by

$$\tilde{C}=Q\begin{bmatrix}C\\CA\\CA^{n-1}\end{bmatrix},\ \tilde{B}=[BAB\cdots A^{n-1}B]Q^T. \tag{43}$$

Then $\tilde{C} = \tilde{B}^{-1}$ because

$$\tilde{C}\tilde{B}=\Sigma^{-1}U^TU\Sigma^2U^TU\Sigma^{-1}=I. \tag{44}$$

Thus

$$\begin{aligned}QH_{ro}Q^T&=Q\begin{bmatrix}C\\CA\\CA^{n-1}\end{bmatrix}W_{ro}[B\ AB\cdots A^{n-1}B]Q^T\\&=CW_{ro}\tilde{B}=\tilde{B}^{-1}W_{ro}\tilde{B}.\end{aligned} \tag{45}$$

and the proof is complete. ∎

Remark: Theorem 6 is still true if H is negative semidefinite rather than positive semidefinite. In this case we would write $-H = U\Sigma^2U^T$ and proceed as above. The next example illustrates Theorem 6.

Example 3: Let A, B be as in Example 2, but change C to $C = B^T = [1\ 1]$. Then

$$H=\begin{bmatrix}C\\CA\end{bmatrix}[B,\ AB]=\begin{bmatrix}2&0\\0&10\end{bmatrix}>0$$

so (A, B, C) can be internally balanced.

Finally we note that the idea of symmetry can be extended to systems (A, B, C) for which $r \neq m$ where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{r\times n}$, in such a way that the previous results are retained. Specifically we say that (A, B, C) has extended symmetry if there exists matrices $U \in \mathbb{R}^{n\times r}$, $V \in \mathbb{R}^{m\times n}$ such that U and V have orthonormal columns and rows, respectively,

$$U^TU=I \in \mathbb{R}^{r\times r}$$

$$VV^T=I \in \mathbb{R}^{m\times m}$$

and

$$UC(sI-A)^{-1}BV=(UC(sI-A)^{-1}BV)^T$$

for any $s \notin \sigma(A)$. For U and V as above, define W_{ro} to be the solution to

$$AW_{ro}+W_{ro}A=-BVUC$$

where we assume that $\mu + \lambda \neq 0$ for any $\mu, \lambda \in \sigma(A)$. Then we have by the same proof techniques.

Theorem 4: Assume that (A, B, C) has extended symmetry as above and that $\mu + \lambda \neq 0$ for any $\mu, \lambda \in \sigma(A)$. Then $W_{ro}^2 = W_rW_o$.

Theorem 5: Assume that (A, B, C) is an observable and controllable extended symmetric realization with $\mu + \lambda \neq 0$ for any $\mu, \lambda \in \sigma(A)$. Then the following are equivalent.

1) The cross Gramian W_{ro} is similar to a real diagonal matrix.

2) There exists an internal balancing transformation T for (A, B, C).

Theorem 6: Let (A, B, C) be as in Theorem 5. If the Hankel matrix H defined by

$$H=\begin{bmatrix}UC\\UCA\\UCA^{n-1}\end{bmatrix}[BV\quad ABV\cdots A^{n-1}BV]$$

is positive semidefinite, then (A, B, C) can be internally balanced.

Conclusion

We have presented simple necessary and sufficient conditions for the existence of balancing transformations for minimal realizations (A, B, C). These conditions do not hold in some cases as seen by example. For symmetric realizations we have shown that the real diagonalizability of the cross Gramian determines whether (A, B, C) can be internally balanced. Lastly, we have shown that any minimal symmetric realization with a positive semidefinite Hankel matrix can be internally balanced.

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