AN ABSTRACT VIEW ON
# SYNTAX WITH SHARING

RICHARD GARNER

ABSTRACT. The notion of term graph encodes a refinement of inductively
generated syntax in which regard is paid to the the sharing and discard
of subterms. Inductively generated syntax has an abstract expression in
terms of *initial algebras* for certain endofunctors on the category of sets,
which permits one to go beyond the set-based case, and speak of inductively
generated syntax in other settings. In this paper we give a similar abstract
expression to the notion of term graph. Aspects of the concrete theory
are redeveloped in this setting, and applications beyond the realm of sets
discussed.

1. INTRODUCTION

A fundamental construction in both mathematics and computer science is
the one which to a signature of finitary operations Σ and a set A assigns the
collection $T_\Sigma(A)$ of terms over the signature with free variables in the set. It
provides both the raw syntax out of which semantic structures of all kinds are
constructed, and also the induction principle by which reasoning about such
structures may proceed. In most situations, the passage from the syntax to the
semantics is one of collapse, in which semantically interchangeable elements
of the sets $T_\Sigma(A)$ are identified under a suitable equivalence relation. How-
ever, for some applications we wish to move in the opposite direction, viewing
the sets $T_\Sigma(A)$ themselves as the extensional collapse of a more intensional
structure in which different traces of execution of the same term can be dif-
ferentiated from each other. The concern of this paper is with the particular
form of this differentiation in which regard is paid to the *sharing* and *discard*
of computational values. By this we mean the following: that for a syntactic
expression such as $(\alpha + \beta) * (\alpha + \beta)$, we wish to distinguish the evaluation
path which computes $(\alpha + \beta)$ just once and multiplies the result by itself, from
that which computes it twice, and multiplies the two results together, from
all of those which first go away and performs some irrelevant computation
before beginning the task at hand. Let us for the moment refer to any means
of encoding such distinctions as a *syntax with sharing*. Such syntaxes have
obvious applications to program optimisation; but beyond this, are important
in settings where the execution of functions can incur *side-effects*—changes
in the state of an external environment—which might render the result of a
computation dependent on its execution path.

Here we shall be concerned with a well-established syntax with sharing
based around the notion of *acyclic term graph* [2, 12, 15, 17]. Term graphs
generalise the familiar representation of elements of $T_\Sigma(A)$ as well-founded

trees, by dropping the requirement that the children of a particular node be distinct. This is perhaps most easily appreciated through an example. The term $(\alpha + \beta) * (\alpha + \beta)$ considered above is represented by the following well-founded tree:

$$\alpha \qquad \beta \qquad \alpha \qquad \beta \;.$$
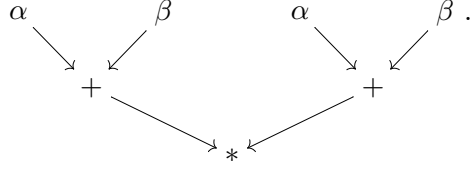
On the other hand, it is represented by any one of a number of different term graphs, each expressing a computation path with a differing degree of sharing. One of these term graphs is the same well-founded tree that was just displayed; and this corresponds to the execution path which computes $(\alpha + \beta)$ twice. On the other hand, the path which computes $(\alpha + \beta)$ only once is represented by a term graph

$$\alpha \qquad \beta$$
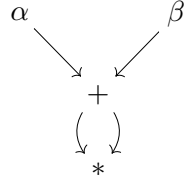
whilst the path which does this after first carrying out an irrelevant computation of $(\alpha + \alpha)$ is represented by

$$\alpha \qquad \beta \;.$$
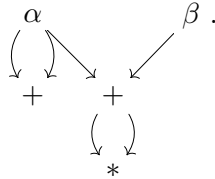
   Now, the construction with which we began this Introduction—that which to a signature Σ and a set A assigns the set of terms $T_\Sigma(A)$—has a well-known abstract characterisation, achieved by shifting the focus of our attention away from the signature Σ, and towards the corresponding *signature endofunctor*

$$(1) \qquad \begin{aligned} F_\Sigma \colon \mathbf{Set} &\to \mathbf{Set} \\ X &\mapsto \textstyle\sum_{\sigma \in \Sigma} X^{|\sigma|} \;, \end{aligned}$$

wherein we write $|{-}| \colon \Sigma \to \mathbb{N}$ for the function assigning arities to each element of the signature. At this level of generality, the set $T_\Sigma(A)$ may be characterised as an *initial algebra* for the endofunctor $X \mapsto A + F_\Sigma(X)$ (in a sense recalled in Definition 2.1 below). This abstract characterisation of $T_\Sigma(X)$ also captures its essential structural features—such as the inductive reasoning it supports—which justifies our interpreting, for an arbitrary endofunctor F of an arbitrary category $\mathscr{E}$, an initial algebra for $X \mapsto A + FX$ as being an "object of terms over F with free variables in A."

   The objective of this paper is to describe a similar abstraction of the notion of term graph. We will describe a construction on an endofunctor $F \colon \mathscr{E} \to \mathscr{E}$, which when applied to a signature endofunctor F_Σ on the category of sets, yields precisely the notion of term graph over Σ. Moving beyond this situation

allows us to recapture other kinds of term graph; thus taking $\mathscr{E} = \mathbf{Set}^S$ for some set S, we obtain *many-sorted* term graphs; taking $\mathscr{E} = \mathbf{Set}^{\mathbb{F}}$ (where $\mathbb{F}$ is the category of finite sets and bijections) allows us to describe term graphs over second-order syntax; whilst dualising our construction (in a sense to be made precise later) yields *cyclic* term graphs, allowing one to capture recursive computations. Of course, we may also leave the sphere of sets entirely, taking $\mathscr{E}$ to be a category of domains, or of complete metric spaces, or a topos, or the category of categories... The point is that we have a uniform construction providing us, in each context, with a workable notion of term graph.

We introduce our abstract notion of term graph in Section 2. From any suitable endofunctor F of a suitable category $\mathscr{E}$, we will construct a comonad L_F on the arrow category $\mathscr{E}^{\mathbf{2}}$, whose coalgebras we define to be abstract term graphs over the endofunctor F. We informally justify our definition by giving a worked example in the category of sets; and in Section 3, make this informal justification precise by proving that our abstract notion of term graph agrees with the established one for *any* signature endofunctor of the form (1). Section 4 then considers what the notion of abstract term graph gives us when we move beyond the motivating set-based case, whilst Section 5 shows how *cyclic* term graphs may be captured by dualising our construction in a particular manner. Finally, Sections 6 and 7 describe how further useful aspects of the set-based theory of term graphs may be recaptured in our abstract setting. In Section 6, we show how an abstract term graph may be interpreted in a suitable semantic domain, whilst in Section 7 we see how abstract term graphs may be composed into each other.

It is perhaps worth saying a few words about how our abstract treatment of term graphs is related to others in the literature. One particularly elegant approach is that described by Hasegawa in his Ph. D. thesis [12]; with an essentially equivalent one being given by Corradini and Gadducci in [4, 5]. The key idea there is to associate with each signature Σ a *classifying category* $\mathscr{S}[\Sigma]$, whose objects are the natural numbers, and whose morphisms $n \to m$ are term graphs over Σ with n free variables and m marked output nodes. The structure borne by this category encodes the various operations on term graphs: with more elaborate kinds of term graph giving rise to more elaborate kinds of structure on the classifying category. This approach generalises the notion of term graph in a different direction from ours; whilst still being tied to the category of sets, it allows one to impose equations on top of the raw theory of terms. By contrast, our approach allows one to move beyond the category of sets, but is, as yet, restricted to freely generated sharing syntaxes. We will see how the two approaches may be reconciled in Section 7.

A different abstract characterisation of term graphs is described by Hamana in [11]. In broad strokes the idea is to exploit the linear representation of term graphs using let syntax; which would, for example, represent the three term graphs displayed above as

$$(\alpha + \beta) * (\alpha + \beta),$$
$$\mathsf{let}\ z := (\alpha + \beta)\ \mathsf{in}\ z * z$$
$$\text{and}\quad \mathsf{let}\ w := (\alpha + \alpha)\ \mathsf{in}\ \mathsf{let}\ z := (\alpha + \beta)\ \mathsf{in}\ z * z\ .$$

One may exploit this to give a representation of term graphs based on the categorical higher-order syntax introduced by Fiore, Plotkin and Turi in [6]. This is not precisely what Hamana does, since he wishes to give an syntax in which inductively defined elements denote term graphs *uniquely*—something that is not the case for the let notation; but the general idea should be clear enough. Once again, this approach differs from ours in being still tied to the category of sets; on the other hand, it gives a representation of term graphs which is more suitable for direct implementation.

A third abstract treatment of term graphs, and the one closest in spirit to the present work, is given in [8]. The idea is to associate to any endofunctor F a monad S_F for which $S_\Sigma(A)$ is the set of all term graphs with free variables from A equipped with a marked node (specifying the output of the computation). However, whilst in principle this approach allows one to move beyond endofunctors of the category of sets, as the authors of [8] note in their Section 4.2, there are aspects of their development that rely on the use of elements, and so cannot be uniformly generalised beyond the set-based situation.
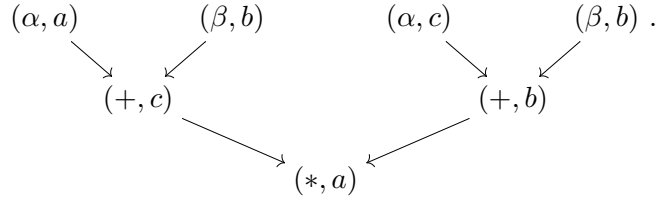
2. Abstract term graphs

This section describes the construction which underlies our abstract characterisation of term graphs. Given a suitable endofunctor F on a suitable category $\mathscr{E}$, it yields a comonad L_F on the arrow category $\mathscr{E}^{\mathbf{2}}$ whose coalgebras we shall define to be abstract term graphs over the signature F. We justify this definition in the next section, where we show that this is literally what they are in the case where F is the endofunctor on the category of sets associated to a signature Σ. First we recall the key abstract notion required for our construction.

2.1. **Definition.** For any category $\mathscr{E}$ and endofunctor $F\colon \mathscr{E} \to \mathscr{E}$, the category F-**Alg** of F-*algebras* has as objects, pairs $(X \in \mathscr{E}, x\colon FX \to X)$, and as maps $(X, x) \to (Y, y)$, those morphisms $f\colon X \to Y$ of $\mathscr{E}$ for which $y.Ff = f.x$. An *initial algebra* for F is an initial object of F-**Alg**. We denote the underlying object in $\mathscr{E}$ of such an initial algebra by $\mu X.FX$.

Before giving the construction, we motivate it with an example.

2.2. *Example.* Let Σ be the signature $\{\alpha, \beta, +, *\}$ in which $+$ and $*$ are binary operations and α and β are constants, and let $F_\Sigma(X) = X^2 + X^2 + 1 + 1$ be the corresponding endofunctor on the category of sets. This endofunctor has an initial algebra $\mu X.F_\Sigma X$ given by the set of closed terms over Σ. As in the Introduction, we may represent the elements of this set by well-founded trees, each of whose nodes is labelled with an element $\sigma \in \Sigma$, and where each such node has $|\sigma|$ children. We begin by showing how to generalise this description of closed *terms* to one of closed *term graphs*, given in terms of the coalgebras for a certain comonad Q on **Set**.

The action on objects of this comonad will be given by $QB = \mu X.B \times F_\Sigma X$. Thus QB is the set whose elements are well-founded trees over Σ in which every node has also been labelled with an element of B. The action of Q on a function $f\colon B \to B'$ is evident: given a tree $t \in QB$, we replace the label $b \in B$ at each node by $f(b) \in B'$ to obtain an element $(Qf)(t) \in QB'$. The counit map $\epsilon_B\colon QB \to B$ is equally straightforward: for each tree $t \in QB$, we take $\epsilon_B(t)$ to be the B-label of the root. The comultiplication map $\Delta_B\colon QB \to QQB$ is more subtle. Given a tree over Σ labelled in B, it must return a tree over Σ labelled in QB, and it does so by an operation which we might describe as "recursive copying of subtrees". It is best illustrated through an example. Let $B = \{a, b, c, d\}$ and consider $t \in QB$ given by

$$
\begin{array}{cccc}
(\alpha, a) & (\beta, b) & (\alpha, c) & (\beta, b) \;. \\
& \searrow \quad \swarrow & & \searrow \quad \swarrow \\
& (+, c) & & (+, b) \\
& & \searrow \qquad \swarrow & \\
& & (*, a) &
\end{array}
$$

The tree $\Delta_B(t) \in QQB$ will have the same underlying shape, but the B-label at each node will have been replaced by the QB-label given by the subtree of t lying above that node. In other words, $\Delta_B(t)$ is the following tree:

$$
\begin{array}{cccc}
(\alpha, (\alpha, a)) & (\beta, (\beta, b)) & (\alpha, (\alpha, c)) & (\beta, (\beta, b)) \;. \\
& \searrow \quad \swarrow & & \searrow \quad \swarrow \\
& (+, t_1) & & (+, t_2) \\
& & \searrow \qquad \swarrow & \\
& & (*, t) &
\end{array}
$$
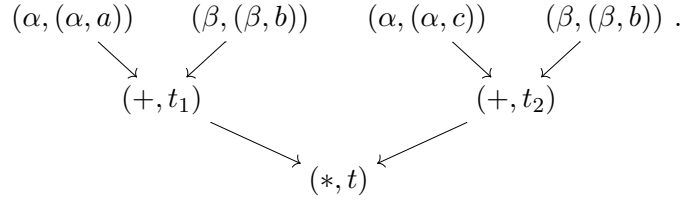
in which t_1 and t_2 are the respective elements

$$
\begin{array}{ccccc}
(\alpha, a) & (\beta, b) & & (\alpha, c) & (\beta, b) \\
\searrow & \swarrow & \text{and} & \searrow & \swarrow \\
(+, c) & & & (+, b) &
\end{array}
$$

of QB. Now a Q-coalgebra is given by a set B together with a map $s\colon B \to QB$ satisfying the two coalgebra axioms. The first of these says that $\epsilon_B(s(b)) = b$ for all $b \in B$: which is the requirement that the root of the tree $s(b)$ should be labelled by b. The second coalgebra axiom asks that $\Delta_B(s(b)) = (Qs)(s(b))$ for all $b \in B$: which says that if a node of $s(b)$ is labelled by $c \in B$, then the subtree of $s(b)$ lying above that node must coincide with $s(c)$. Our claim is that such coalgebras correspond with closed term graphs over Σ. To illustrate this, consider first a typical closed term graph:

$$
\begin{array}{ccc}
\alpha & & \beta \;. \\
\downarrow & \diagdown \diagup & \downarrow \\
+ & & * \\
& \diagdown \diagup & \\
& + &
\end{array}
$$
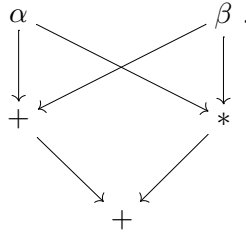
To obtain the corresponding Q-coalgebra, we choose a set of labels for the nodes of this term graph—say $B = \{1, 2, 3, 4, 5\}$ from top to bottom and left to right—and define a map $s \colon B \to QB$ by taking $s(1) = (\alpha, 1)$, $s(2) = (\beta, 2)$,

$$
s(3) \quad = \quad
\begin{array}{ccc}
(\alpha, 1) & & (\beta, 2) \\
& \searrow \quad \swarrow & \\
& (+, 3) &
\end{array}
\quad , \qquad
s(4) \quad = \quad
\begin{array}{ccc}
(\alpha, 1) & & (\beta, 2) \\
& \searrow \quad \swarrow & \\
& (*, 4) &
\end{array}
\quad ,
$$

and

$$
s(5) \quad = \quad
\begin{array}{ccccc}
(\alpha, 1) & & (\beta, 2) & & (\alpha, 1) & & (\beta, 2) \\
& \searrow \swarrow & & & & \searrow \swarrow & \\
& (+, 3) & & & & (*, 4) & \\
& & \searrow & & \swarrow & & \\
& & & (+, 5) & & &
\end{array}
\quad .
$$

This map evidently satisfies the coalgebra axioms. In general, given a closed term graph over Σ, we construct a Q-coalgebra structure on its set of nodes as follows. For each node b labelled with α or β we set $s(b) = (\alpha, b)$ or (β, b) as appropriate. Then for each node b labelled with $+$ or $*$, and with children b_1 and b_2, we set $s(b)$ to be the tree in which the root is $(+, b)$ or $(*, b)$ as appropriate, and the two subtrees of the root are $s(b_1)$ and $s(b_2)$. Conversely, given any Q-coalgebra $s \colon B \to QB$ we define a closed term graph as follows. Its nodes are the elements of B, with each such node b being labelled by that element of Σ which labels the root of $s(b)$, and having as children those elements of B which label the children of the root of $s(b)$.

In order to capture possibly open term graph over Σ, we now describe a more general comonad L; it resides not on **Set** but rather on the arrow category $\mathbf{Set}^{\mathbf{2}}$, the idea being that an L-coalgebra structure on an object $(f \colon A \to B)$ of $\mathbf{Set}^{\mathbf{2}}$ should correspond to a term graph with nodes labelled in B and free variables from the set A. In fact, for each set A, the comonad L will restrict and corestrict to the coslice category $A/\mathbf{Set}$, yielding a comonad whose coalgebras correspond to term graphs with free variables from A; in particular, taking $A = 0$ we will recover the earlier comonad Q. The underlying functor of the comonad L has its action on objects given by

$$
L(f \colon A \to B) = (\gamma_f \colon A \to \mu X.A + B \times F_\Sigma X) \ ,
$$

in which γ_f is defined as follows. Observe that the initial algebra $\mu X.A + B \times F_\Sigma X$ may be represented as the set of those well-founded trees built from either nodes labelled in $\Sigma \times B$ as before, or else leaves labelled only by an element of A: under which representation, the map $\gamma_f \colon A \to \mu X.A + B \times F_\Sigma X$ sends $a \in A$ to the tree consisting of the bare leaf a. The counit $L \Rightarrow 1$ of the comonad has as its f-component the morphism $\gamma_f \to f$ in $\mathbf{Set}^{\mathbf{2}}$ given by

$$
\begin{array}{ccc}
A & \xrightarrow{\ 1_A\ } & A \\
\gamma_f \downarrow & & \downarrow f \\
\mu X.A + B \times F_\Sigma X & \xrightarrow[\ \rho_f\ ]{} & B
\end{array}
$$

wherein ρ_f sends a bare leaf a to $f(a)$ and sends any other tree to the B-label of its root. The comultiplication of L is analogous to that of Q, and we shall not spell it out here. Now, to give an L-coalgebra is to give an object of $\mathbf{Set}^2$—which is a function $f: A \to B$—together with a map

$$
\begin{array}{ccc}
A & \xrightarrow{\;r\;} & A \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \gamma_f} \\
B & \xrightarrow{\;s\;} & \mu X.A + B \times F_\Sigma X
\end{array}
$$

in $\mathbf{Set}^2$ satisfying the two coalgebra axioms. It's easy to see that the counit axiom forces $r = 1_A$, whereupon commutativity of the preceding diagram says that for each $a \in A$, the tree $s(f(a))$ should be the bare leaf a. Note that this in turn forces f to be a monomorphism. Now the counit axiom says that $\rho_f(s(b)) = b$ for all $b \in B$. This is trivial for those b in the image of f, whilst for those those that are not, it says that $s(b)$ cannot be a bare leaf $a \in A$ (or else $b = \rho_f(s(b)) = \rho_f(a) = f(a)$ contradicting $b \notin \text{im } f$), and so must as before be a tree whose root is labelled by (σ, b) for some $\sigma \in \Sigma$. Finally, the comultiplication axiom says exactly what it did for Q: that if a node of $s(b)$ is labelled by $c \in B$, then the subtree of $s(b)$ lying above that node must coincide with $s(c)$. What we now claim is that an L-coalgebra such as we have just described corresponds to a term graph over Σ with free variables from A. We illustrate this only with a very simple example. Let $A = \{x, y\}$ and consider the term graph
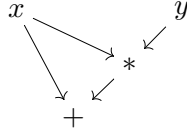
with free variables from A. One important point to observe is that in our framework all free variables are maximally shared: which is to say there must be *exactly one* node in the term graph corresponding to each free variable. To obtain the L-coalgebra corresponding to this term graph, we let $B = \{x, y, 1, 2\}$ be a labelling of its nodes (including those corresponding to free variables) and let $f: A \hookrightarrow B$ be the evident inclusion. To define an L-coalgebra structure on f we must give a map $s: B \to \mu X.A + B \times F_\Sigma X$ satisfying the appropriate axioms, which we do by setting $s(x) = x$, $s(y) = y$,

This completes our worked example; and we now provide the details of our construction in its general form.

2.3. **Definition.** Let there be given a category $\mathscr{E}$ with finite products and coproducts, and an endofunctor $F: \mathscr{E} \to \mathscr{E}$ such that for all $A, B \in \mathscr{E}$ the endofunctor $A + B \times F(-)$ has an initial algebra. We define the *term graph comonad* L_F associated to F as follows. Given an object $f: A \to B$ of $\mathscr{E}^2$, we

write Pf for the initial algebra of the endofunctor $A + B \times F(-)$, write

$$\iota_f = [\gamma_f, \theta_f] \colon A + B \times FPf \to Pf$$

for its algebra structure, and set $L_F(f \colon A \to B) := (\gamma_f \colon A \to Pf)$. To give the action of L_F on a morphism $(h, k) \colon f \to g$ of $\mathscr{E}^{\mathbf{2}}$, we set

$$
L_F \left(
\begin{array}{ccc}
A & \xrightarrow{\ h\ } & C \\
f \downarrow & & \downarrow g \\
B & \xrightarrow[\ k\ ]{} & D
\end{array}
\right)
\quad = \quad
\begin{array}{ccc}
A & \xrightarrow{\ h\ } & C \\
\gamma_f \downarrow & & \downarrow \gamma_g \\
Pf & \xrightarrow[P(h,k)]{} & Pg
\end{array}
$$

where $P(h, k)$ is defined by universality of Pf as the unique map making

$$
\begin{array}{ccc}
A + B \times FPf & \xrightarrow{A + B \times FP(h,k)} & A + B \times FPg \\
\iota_f \downarrow & & \downarrow \iota_g . (h + k \times FPg) \\
Pf & \xrightarrow[\qquad P(h,k) \qquad]{} & Pg
\end{array}
$$

commute. The counit and comultiplication natural transformations $L_F \Rightarrow 1$ and $L_F \Rightarrow L_F L_F$ have as their respective components at $f \in \mathscr{E}^{\mathbf{2}}$ the maps $\gamma_f \to f$ and $\gamma_f \to \gamma_{\gamma_f}$ of $\mathscr{E}^{\mathbf{2}}$ given by:

$$
\begin{array}{ccc}
A & \xrightarrow{\ 1_A\ } & A \\
\gamma_f \downarrow & & \downarrow f \\
Pf & \xrightarrow[\ \rho_f\ ]{} & B
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
A & \xrightarrow{\ 1_A\ } & A \\
\gamma_f \downarrow & & \downarrow \gamma_{\gamma_f} \\
Pf & \xrightarrow[\ \sigma_f\ ]{} & P\gamma_f \; ;
\end{array}
$$

here ρ_f is defined by universality of Pf as the unique map making

$$
\begin{array}{ccc}
A + B \times FPf & \xrightarrow{A + B \times F\rho_f} & A + B \times FB \\
\iota_f \downarrow & & \downarrow [f, \pi_1] \\
Pf & \xrightarrow[\qquad \rho_f \qquad]{} & B
\end{array}
$$

commute, whilst σ_f is defined by the same universal property as the unique map making

$$
\begin{array}{ccc}
A + B \times FPf & \xrightarrow{A + B \times F\sigma_f} & A + B \times FP\gamma_f \\
\iota_f \downarrow & & \downarrow [\gamma_{\gamma_f}, \kappa_f] \\
Pf & \xrightarrow[\qquad \sigma_f \qquad]{} & P\gamma_f .
\end{array}
$$

commute, where κ_f is defined as the composite

$$
B \times FP\gamma_f \xrightarrow{B \times (F\rho_{\gamma_f}, 1)} B \times FPf \times FP\gamma_f \xrightarrow{\theta_f \times 1} Pf \times FP\gamma_f \xrightarrow{\theta_{\gamma_f}} P\gamma_f .
$$

2.4. **Proposition.** *The above data determine a comonad L_F on $\mathscr{E}^{\mathbf{2}}$.*

Proof. Entirely routine using the unicity of maps out of an initial algebra. $\quad\square$

2.5. **Definition.** For a category $\mathscr{E}$ with finite products and coproducts, and an endofunctor $F\colon \mathscr{E} \to \mathscr{E}$ such that each $A + B \times F(-)$ has an initial algebra, we define the category $\mathbf{ATG}(F)$ of *abstract term graphs over F* to be the category of L_F-coalgebras. Explicitly, an abstract term graph over F is a pair of maps $(f\colon A \to B, s\colon B \to Pf)$ in $\mathscr{E}$ satisfying $\rho_f.s = 1_B$, $s.f = \gamma_f$, and $P(1_A, s).s = \sigma_f.s$; whilst a morphism of abstract term graphs $(f, s) \to (g, s')$ is a commutative square

$$
\begin{array}{ccc}
A & \xrightarrow{\ h\ } & C \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
B & \xrightarrow[\ k\ ]{} & D
\end{array}
$$

such that $P(h, k).s = s'.k$.

As in Example 2.2, the comonad L of our general construction restricts and corestricts to a comonad on each coslice category $A/\mathscr{E}$, whose coalgebras are the abstract term graphs over F whose underlying object in $\mathscr{E}^{\mathbf{2}}$ has domain A. In particular, on taking $A = 0$ we obtain a comonad $Q\colon \mathscr{E} \to \mathscr{E}$ with $QB = \mu X.B \times FX$; which, as in our example, we regard as the comonad for *closed* term graphs over F. The existence of the comonad Q was indicated in [9], though its meaning was not discussed; our comonad L may be be seen as a natural generalisation of it.

3. Concrete term graphs

In this Section, we show that, by specialising the abstract notion of term graph given in Definition 2.5 to the case of a signature endofunctor F_Σ on the category of sets, we recover the usual notion of acyclic term graph over Σ. First we give a formal definition of the latter.

3.1. **Definition.** A *concrete term graph T* over a signature Σ is given by:
- A set of *input nodes A*;
- A set of *internal nodes V*;
- A labelling function $\ell\colon V \to \Sigma$;
- For each $v \in V$ and $i \in 1, \dots, |\ell(v)|$ an element $\varphi_i(v) \in A + V$.

For such a term graph we define a binary relation on V by $w \lhd v$ iff $w = \varphi_k(v)$ for some k. We say that T is *acyclic* if the transitive closure of $\lhd$ is irreflexive, and *cyclic* otherwise.

Until further notice we will always interpret the unadorned phrase "term graph" as "acyclic term graph".

3.2. **Definition.** If T and T' are concrete term graphs over Σ, then a *morphism of term graphs $T \to T'$* comprises functions $f\colon A \to A'$ and $g\colon V \to V'$ such that for all $v \in V$ and for all i, we have $\ell'(g(v)) = \ell(v)$ and $(f + g)(\varphi_i(v)) = \varphi_i'(g(v))$. We write $\mathbf{CTG}(\Sigma)$ for the category of concrete term graphs over Σ.

The term graphs of Definition 3.1 do not, rightly said, represent terms so much as computations, since we do not indicate which nodes should be considered as return values. We may rectify this by adding a set of *output nodes B* and a labelling function $B \to A + V$ to the definition: and in Section 7

below, we shall. This will then allow us to compose term graphs by plugging the output nodes of one into the input nodes of another. However it is the more basic notion that is pertinent here, as it is the one needed to prove:

3.3. **Proposition.** *For any signature Σ, the categories of concrete term graphs over Σ and of abstract term graphs over F_Σ are equivalent.*

Proof. Recall that an abstract term graph over F_Σ is a coalgebra for the comonad $L := L_{F_\Sigma}$ of $\mathbf{Set}^\mathbf{2}$ obtained by the construction of Section 2. We begin by making explicit the structure of this comonad. On objects, L sends $f\colon A \to B$ to $\gamma_f\colon A \to Pf$, where Pf is the set defined by the following inductive clauses:

- $[a] \in Pf$ for all $a \in A$;
- $\alpha_b(z_1, \ldots, z_{|\alpha|}) \in Pf$ for all $b \in B$, $\alpha \in \Sigma$ and $z_1, \ldots, z_{|\alpha|} \in Pf$,

and where $\gamma_f(a) = [a]$. We introduce the notational convenience of abbreviating $z_1, \ldots, z_{|\alpha|}$ as $\vec{z}$, and—for any suitable function Γ—abbreviating $\Gamma(z_1), \ldots, \Gamma(z_{|\alpha|})$ as $\Gamma(\vec{z})$. With this notation, the action of L on morphisms

$$
\begin{array}{ccc}
A \xrightarrow{\ h\ } C & & A \xrightarrow{\ h\ } C \\
f\downarrow \qquad \downarrow g & \mapsto & \gamma_f\downarrow \qquad \downarrow \gamma_g \\
B \xrightarrow{\ k\ } D & & Pf \xrightarrow[P(h,k)]{} Pg
\end{array}
$$

may be recursively defined by

$$P(h,k)([a]) = [h(a)] \qquad \text{and} \qquad P(h,k)(\alpha_b(\vec{z})) = \alpha_{k(b)}(P(h,k)(\vec{z})) \ .$$

Now the map $\rho_f\colon Pf \to B$ giving the counit of L at f is defined by $\rho_f([a]) = f(a)$ and $\rho_f(\alpha_b(\vec{z})) = b$, whilst the map $\sigma_f\colon Pf \to P\gamma_f$ giving the comultiplication at f is defined recursively by

$$\sigma_f([a]) = [a] \qquad \text{and} \qquad \sigma_f(\alpha_b(\vec{z})) = \alpha_{\alpha_b(\vec{z})}(\sigma_f(\vec{z})) \ .$$

We will prove the result by defining a functor $F\colon \mathbf{CTG}(\Sigma) \to \mathbf{ATG}(F_\Sigma)$ and showing it to be an equivalence. On objects, the functor F assigns to each concrete term graph $T = (A, V, \ell, \varphi)$ the following L-coalgebra. Its underlying object in $\mathbf{Set}^\mathbf{2}$ is $\mathsf{inl}\colon A \to A + V$. According to Definition 2.5, its coalgebra structure is given by a map $s\colon A + V \to P(\mathsf{inl})$, which will be obtained as follows. For $a \in A$, we take $s(a) = [a]$. To define s on V, we first observe that since the term graph T is acyclic, the transitive closure $<$ of $\lhd$ is irreflexive, and hence a (strict) partial order on V. Moreover, for each $v \in V$, the set $\{w \mid w \lhd v\}$ is finite, and hence $\{w \mid w < v\}$ is too; we denote its cardinality by $c(v)$. Now given $v \in V$, suppose we have recursively defined $s(w)$ for all $w \in V$ with $c(w) < c(v)$. By irreflexivity of $\lhd$, this means in particular that we have defined $s(w)$ for all $w \lhd v$; and so may validly define

$$(2) \qquad s(v) = \ell(v)_v\big(s(\varphi_1(v)), \ldots, s(\varphi_n(v))\big) \qquad (\text{where } n = |\ell(v)|).$$

By recursion, this defines s at every $v \in V$. It remains to verify the coalgebra axioms. It's easy to show that $s.\mathsf{inl} = \gamma_{\mathsf{inl}}$ and that $\rho_{\mathsf{inl}}.s = 1_{A+V}$; so it remains to show that $P(1_A, s).s = \sigma_f.s$. This is trivial on elements of $A \subseteq A+V$; whilst to show it on elements of $V \subseteq A + V$ we proceed by induction. Suppose that

$v \in V$ and that we have verified the equality for all w with $c(w) < c(v)$. Writing $\vec{\varphi}$ as an abbreviation for $s(\varphi_1(v)), \ldots, s(\varphi_n(v))$, we have

$$
\begin{aligned}
\sigma_f(s(v)) &= \sigma_f(\ell(v)_v(s(\vec{\varphi}))) \\
&= \ell(v)_{s(v)}(\sigma_f(s(\vec{\varphi}))) \\
&= \ell(v)_{s(v)}(P(1,s)(s(\vec{\varphi}))) \\
&= P(1,s)(\ell(v)_v(s(\vec{\varphi}))) \\
&= P(1,s)(s(v))
\end{aligned}
$$

by the recursive definitions of σ_f, $P(1,s)$ and s and the inductive hypothesis. Hence by induction we have $P(1_A, s).s = \sigma_f.s$ as required. This completes the definition of the functor $F \colon \mathbf{CTG}(\Sigma) \to \mathbf{ATG}(F_\Sigma)$ on objects. To define it on morphisms, let $T' = (A', V', \ell', \varphi')$ be another concrete term graph, and $(f, g) \colon T \to T'$ a morphism between them. We shall take $F(f, g) \colon FT \to FT'$ to be given by

(3)

$$
\begin{array}{ccc}
A & \xrightarrow{\;\;f\;\;} & A' \\
{\scriptstyle \mathrm{inl}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{inl}} \\
A + V & \xrightarrow[\;f+g\;]{} & A' + V' \;.
\end{array}
$$

For this to be well defined, we must show that (3) is a map of L-coalgebras $FT \to FT'$, i.e., that $P(f, f+g).s = s'.(f+g)$ holds. This is straightforward on elements of $A \subseteq A+V$; whilst for elements of $V \subseteq A+V$, we proceed once more by induction. Suppose that $v \in V$ and that we have verified the equality for all $w \in V$ with $c(w) < c(v)$. Then we have that

$$
\begin{aligned}
P(f, f+g)(s(v)) &= P(f, f+g)(\ell(v)_v(s(\varphi_1(v))), \ldots, s(\varphi_n(v)))) \\
&= \ell(v)_{gv}(P(f, f+g)(s(\varphi_1(v))), \ldots, P(f, f+g)(s(\varphi_1(v)))) \\
\text{(4)} \qquad &= \ell(v)_{gv}(s'(f+g)(\varphi_1(v)), \ldots, s'(f+g)(\varphi_n(v))) \;,
\end{aligned}
$$

whilst

(5) $\qquad s'(f+g)(v) = s'(gv) = \ell'(gv)_{gv}(s'(\varphi'_1(gv)), \ldots, s'(\varphi'_n(gv))) \;.$

But since (f, g) is a map of term graphs, (4) and (5) are equal; and so by induction we conclude that $P(f, f+g).s = s'.(f+g)$ as required. This completes the definition of the functor F; we next show that it is fully faithful. For this, let T and T' be concrete term graphs as before, and suppose that

$$
\begin{array}{ccc}
A & \xrightarrow{\;\;f\;\;} & A' \\
{\scriptstyle \mathrm{inl}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{inl}} \\
A + V & \xrightarrow[\;\;h\;\;]{} & A' + V'
\end{array}
$$

is a map of L-coalgebras $FT \to FT'$. We claim first that $h = f + g$ for some $g \colon V \to V'$; for which it is clearly enough to show that $h(V) \subseteq V'$. But were this not so, we would have $h(v) = a'$ for some $v \in V$ and $a' \in A'$, whence $[a'] = s'(h(v)) = P(f, h)(s(v))$, which is impossible by the definitions of $P(f, h)$ and s. Consequently, every map of L-coalgebras $FT \to FT'$ is of the form (3); and so we will be done if we can show that for every such map,

the pair (f, g) is a map of concrete term graphs $T \to T'$. But since for every $v \in V$ we have $P(f, f + g)(s(v)) = s'(f + g)(v)$, equating (4) and (5) shows that that $\ell(v) = \ell'(g(v))$ and that $s'((f + g)(\varphi_i(v))) = s'(\varphi_i'(g(v)))$ for each $1 \leqslant i \leqslant n$; which since s' is injective, implies that $(f + g)(\varphi_i(v)) = \varphi_i'(g(v))$ for each i, so that (f, g) is a map of term graphs as desired.

Thus F is a fully faithful functor; to complete the proof, we must show that it is also essentially surjective. So for each L-coalgebra ℓ we must find a concrete term graph T and an isomorphism $FT \cong \ell$. From Definition 2.5, to give ℓ is to give maps $f \colon A \to B$ and $s \colon B \to Pf$ in $\mathscr{E}$ satisfying three axioms. The first is that $s.f = \gamma_f$, which says that $s(f(a)) = [a]$ for each $a \in A$. Note that this forces f to be injective, so that taking $V = B \setminus \operatorname{im} f$, we have a bijection $B \cong A + V$ under which f is identified with the coproduct injection $A \hookrightarrow A + V$. The next coalgebra axiom is that $\rho_f.s = 1_B$, which by case analysis says that

$$s(b) = [a] \;\Rightarrow\; b = f(a) \qquad \text{and} \qquad s(b) = \alpha_{b'}(\vec{z}) \;\Rightarrow\; b = b' \;;$$

whence $b = f(a)$ if and only if $s(b) = [a]$, so that for $b \in B \setminus \operatorname{im} f = V$, we must have $s(b) = \alpha_b(\vec{z})$ for some $\alpha \in \Sigma$ and $z_1, \ldots, z_{|\alpha|} \in Pf$. We claim that these z_i in fact satisfy $z_i = s(\rho_f(z_i))$. Indeed, either $z_i = [a]$ for some $a \in A$, in which case $s(\rho_f(z_i)) = s(f(a)) = [a] = z_i$; or $z_i = \beta_c(\vec{w})$ for some β, c and $\vec{w}$: in which case by the third coalgebra axiom $P(1, s).s = \sigma_f.s$ we have

$$\alpha_{s(b)}(P(1, s)(\vec{z})) = P(1, s)(s(b)) = \sigma_f(s(b)) = \alpha_{s(b)}(\sigma_f(\vec{z}))$$

whence $P(1, s)(z_i) = \sigma_f(z_i)$, which implies that

$$\beta_{s(c)}(P(1, s)(\vec{w})) = P(1, s)(z_i) = \sigma_f(z_i) = \beta_{z_i}(\sigma_f(\vec{w}))$$

so that in particular $z_i = s(c) = s(\rho_f(z_i))$ as claimed. Consequently, we uniquely determine a function $\ell \colon V \to \Sigma$ and an assignation to each $v \in V$ of elements $\psi_1(v), \ldots, \psi_{|\ell(v)|}(v) \in B$ by the requirement that for all $v \in V$,

$$(6) \qquad s(v) = \ell(v)_v(s(\psi_1(v)), \ldots, s(\psi_n(v))) \qquad (\text{where } n = |\ell(v)|),$$

since if $s(v) = \alpha_v(\vec{z})$, we necessarily have $\ell(v) = \alpha$ and $\psi_i(v) = \rho_f(z_i)$ (the second being forced by injectivity of s).

We now have a term graph $T = (A, V, \ell, \varphi)$, where φ is obtained by composing the ψ above with the canonical isomorphism $B \cong A + V$. It is clear by comparing (2) and (6) that FT is isomorphic to the L-coalgebra we started with; and so we will be done as soon as we have checked that T is acyclic. To do this we consider the function $d \colon Pf \to \mathbb{N}$ defined by $d([a]) = 0$ and $d(\alpha_b(\vec{z})) = \max(d(z_1), \ldots, d(z_n)) + 1$. Recall that for $v, w \in V$, the relation $v \lhd w$ holds just when $v = \varphi_k(w)$ for some k; but from above, this happens just when $s(w) = \alpha_b(\vec{z})$ and $v = \rho_f(z_k)$ for some k, which is equally well when $s(v) = z_k$ for some k. But this implies that $d(v) < d(w)$ in $\mathbb{N}$, so that the transitive closure of $\lhd$ must be acyclic as required. $\qquad\square$

4. Generalisations

Having established the correspondence between concrete term graphs over a signature and abstract term graphs over the corresponding endofunctor, let us now see how our abstract notion extends beyond that case. Note that for

the moment, all of our generalisations will remain in the acyclic world; we shall consider cyclicity in some detail in the following section.

Operations with unordered inputs. To a finitary signature $|-| \colon \Sigma \to \mathbb{N}$ we can associate an endofunctor F'_{Σ} of **Set**, different from that of (1), by the formula:

$$(7) \qquad\qquad F'_{\Sigma}(X) = \textstyle\sum_{\sigma \in \Sigma}(X^{|\sigma|}/S_{|\sigma|}) \; ;$$

here the set $X^{|\sigma|}$ is being quotiented by that action of the symmetric group on $|\sigma|$ letters which permutes the order of the factors. The abstract term graphs generated by such endofunctors are like concrete term graphs in which the *ordering* of the input variables to an operation is considered irrelevant.

Infinitary operations. Staying in the category of sets, we can clearly lift the restriction that the operations in our signature be finitary: indeed, a signature with infinitary operations still gives rise to a signature endofunctor by the formula (1) (or (7) for that matter).

Typed operations. A *many-sorted signature* is given by a set S of sorts, a set Σ of operations, and typing functions for input $i \colon \Sigma \to \mathbb{N}^S$ and output $o \colon \Sigma \to S$. Any such signature generates an endofunctor F_{Σ} of **Set**^{S} by the formula

$$F_{\Sigma}(\, X_s \mid s \in S \,) = \left(\sum_{\substack{\sigma \in \Sigma \\ o(\sigma)=t}} \prod_{s \in S} (X_s)^{i(\sigma,s)} \,\middle|\, t \in S \right).$$

The corresponding notion of abstract term graph corresponds exactly to the notion of many-sorted concrete term graph (as defined in [12], for example).

Higher-order syntax with sharing. In [6], Fiore, Plotkin and Turi describe a categorical framework for the study of second-order syntax. The key idea is to replace the category of sets with the presheaf category **Set**$^{\mathbb{F}}$, where $\mathbb{F}$ denotes the category of finite cardinals. In this category, initial algebras for endofunctors can be seen as the collection of terms inductively generated by a second-order signature, with the value of such an initial algebra at some $n \in \mathbb{F}$ being the set of all terms with n free variables over the signature.

This framework was later extended by Tanaka and Power [18–20] to deal with more general second-order syntaxes, in which, for example, the second-order entities may be constrained to bind their first-order arguments in a linear fashion. In this more general setting one still works with a presheaf category, and still describes terms over a second-order signature in terms of initial algebras for suitable endofunctors.

In both situations our construction applies, and so we obtain corresponding notions of second-order syntax with sharing. A thorough investigation of this will be a paper in itself but we hope to convey at least some of what is involved through a simple example. Consider the following sequent calculus

for polynomials over $\mathbb{N}$:

$$\frac{}{x_1, \ldots, x_n \vdash x_i} \ (1 \leqslant i \leqslant n) \qquad\qquad \frac{}{x_1, \ldots, x_n \vdash 0}$$

$$\frac{x_1, \ldots, x_n \vdash p \qquad x_1, \ldots, x_n \vdash q}{x_1, \ldots, x_n \vdash (p + q)} \qquad \frac{x_1, \ldots, x_n \vdash p \qquad x_1, \ldots, x_n \vdash q}{x_1, \ldots, x_n \vdash (p \cdot q)}$$

$$\frac{x_1, \ldots, x_n, x_{n+1} \vdash p}{x_1, \ldots, x_n \vdash p[a/x_{n+1}]} \ (a \in \mathbb{N})$$

We can organise the terms of this sequent calculus into an object $P \in \mathbf{Set}^{\mathbb{F}}$ in which Pn is the set of all derivable judgements $x_1, \ldots, x_n \vdash p$ and the reindexing function $Pf \colon Pn \to Pm$ is defined by induction in the obvious way. We may characterise the object P as the initial algebra for the endofunctor F of $\mathbf{Set}^{\mathbb{F}}$ given by:

$$(FX)(n) = n + 1 + (Xn \times Xn) + (Xn \times Xn) + (\mathbb{N} \times X(n+1))$$

with each term in this sum corresponding to one of the deduction rules listed above. So we think of elements of P as being closed terms over the signature F (where here closed is meant in the sense of having no second-order variables). Applying the constructions of Section 2 now yields a corresponding notion of term *graph*. Without wishing to enter into any detailed calculations, let us at least give an example of what such a term graph will look like. Consider the object $y_3 + y_2 + y_2 + y_3 + y_1$ of $\mathbf{Set}^{\mathbb{F}}$, where $y \colon \mathbb{F}^{\mathrm{op}} \to \mathbf{Set}^{\mathbb{F}}$ is the Yoneda embedding. There is a closed term graph structure on this which can represented in let notation as:

$$
\begin{aligned}
x \vdash \ &\mathsf{let} \ p(x, y, z) = y \ \mathsf{in} \\
&\mathsf{let} \ q(x, y) = y \ \mathsf{in} \\
&\mathsf{let} \ r(x, y) = p(x, y, x) + q(y, x) \ \mathsf{in} \\
&\mathsf{let} \ s(x, y, z) = r(x, y) \times r(y, z) \ \mathsf{in} \\
&s(x, x, 48) \ .
\end{aligned}
$$

It should be clear from this that what such term graphs share are *second-order* terms: namely the polynomials p, q, r and s. Likewise, when we move from closed term graphs to arbitrary ones, what we are adding are second-order variables. By way of illustration, we could in the previous example turn p and q into variables, obtaining a term which in let notation would be written as

$$
\begin{aligned}
p(-, -, -), q(-, -), x \vdash \ &\mathsf{let} \ r(x, y) = p(x, y, x) + q(y, x) \ \mathsf{in} \\
&\mathsf{let} \ s(x, y, z) = r(x, y) \times r(y, z) \ \mathsf{in} \\
&s(x, x, 48) \ .
\end{aligned}
$$

This corresponds to an abstract term graph whose underlying object in $(\mathbf{Set}^{\mathbb{F}})^{\mathbf{2}}$ is given by $y_3 + y_2 \to y_3 + y_2 + y_2 + y_3 + y_1$.

Proof theory. Our final generalisation is very much in the same spirit as the previous one, and so we do not dwell on the details but merely indicate a potential application. The categorical proof theory of classical logic is famously thorny and the hope is that the notion of abstract term graph may provide an

elegant way of encoding some of the computational structure of classical proofs. The thought is as follows. Starting from some set V of primitive propositions, we may form $\mathbb{F}(V)$, the free category with strictly associative finite products on V. Its objects are finite lists $A := (A_1, \ldots, A_n)$ of elements of V and its morphisms $(A_1, \ldots, A_n) \to (B_1, \ldots, B_m)$ are functions $n \to m$ such that $B_{f(i)} = A_i$ for $1 \leqslant i \leqslant m$. Now we can express the collection of classical proof-trees over the basic propositions in V as an initial algebra for an endofunctor on the category $\mathbf{Set}^{\mathbb{F}V \times V}$ (for a one-sided sequent calculus) or on the category $\mathbf{Set}^{\mathbb{F}V \times (\mathbb{F}V)^{\mathrm{op}}}$ (for a two-sided one). Passing to the corresponding notion of term graph we obtain structures which should allow a smooth representation of the duplication and discard of sub-proofs which is central to classical cut-elimination.

5. Duality and cyclicity

In this section, we describe how *cyclic* term graphs over an endofunctor $F \colon \mathscr{E} \to \mathscr{E}$ may be captured in our framework. They will also arise as the coalgebras for a comonad on $\mathscr{E}^{\mathbf{2}}$; but this comonad will no longer be obtained by our original construction, but rather by its dual in the following sense. The endofunctor F is equally well an endofunctor $F^{\mathrm{op}} \colon \mathscr{E}^{\mathrm{op}} \to \mathscr{E}^{\mathrm{op}}$; and if when we regard it in this way, the hypotheses of Definition 2.3 are still satisfied (which amounts to the existence of certain *final coalgebras* in $\mathscr{E}$) then we may apply our construction in $\mathscr{E}^{\mathrm{op}}$ and regard the result as structure back in $\mathscr{E}$. Prima facie there is a serious problem with this, since on the first dualisation we obtain a comonad $L_{F^{\mathrm{op}}}$ on $(\mathscr{E}^{\mathrm{op}})^{\mathbf{2}}$, which on the second dualisation becomes a *monad* and not a comonad on $\mathscr{E}^{\mathbf{2}}$. We could overcome this if we were to know that the construction of Definition 2.3 produced not just a comonad, but also at the same time a monad on $\mathscr{E}^{\mathbf{2}}$; for then the same would be the true when we passed to the dual. Remarkably, this is the case; and we now describe this monad explicitly.

5.1. **Definition.** Let there be given a category $\mathscr{E}$ with finite products and coproducts, and an endofunctor $F \colon \mathscr{E} \to \mathscr{E}$ such that for all $A, B \in \mathscr{E}$ the endofunctor $A + B \times F(-)$ has an initial algebra. We define the *term graph monad* R_F associated to F as follows. The underlying functor is given on objects by $R_F(f \colon A \to B) = (\rho_f \colon Pf \to B)$, and on morphisms by:

$$
R\left(
\begin{array}{ccc}
A & \xrightarrow{\ h\ } & C \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
B & \xrightarrow[\ k\ ]{} & D
\end{array}
\right)
\quad = \quad
\begin{array}{ccc}
Pf & \xrightarrow{P(h,k)} & Pg \\
{\scriptstyle \rho_f}\downarrow & & \downarrow{\scriptstyle \rho_g} \\
C & \xrightarrow[\ k\ ]{} & D
\end{array} \ .
$$

Here, ρ and P are given as in Definition 2.3. The transformations $1 \Rightarrow R_F$ and $R_F R_F \Rightarrow R_F$ making R_F into a monad have their components $f \to \rho_f$

and $\rho_{\rho_f} \to \rho_f$ at some $f \in \mathscr{E}^{\mathbf{2}}$ given by

$$
\begin{array}{ccc}
A \xrightarrow{\gamma_f} Pf & & P\rho_f \xrightarrow{\pi_f} Pf \\
{\scriptstyle f}\downarrow \quad \quad \downarrow{\scriptstyle \rho_f} & \text{and} & {\scriptstyle \rho_{\rho_f}}\downarrow \quad \quad \downarrow{\scriptstyle \rho_f} \\
B \xrightarrow[1_B]{} B & & B \xrightarrow[1_B]{} B
\end{array}
$$

respectively. Now γ is also as in Definition 2.3, and the only new datum is the morphism $\pi_f \colon P\rho_f \to \rho_f$, which we define by the universality of Pf as the unique map rendering commutative the diagram:

$$
\begin{array}{ccc}
Pf + B \times FP\rho_f & \xrightarrow{Pf + B \times F\pi_f} & Pf + B \times FPf \\
{\scriptstyle \iota_{\rho_f}}\downarrow & & \downarrow{\scriptstyle [1,\theta_f]} \\
P\rho_f & \xrightarrow{\pi_f} & Pf \ .
\end{array}
$$

5.2. Proposition. *The above data determine a monad R_F on $\mathscr{E}^{\mathbf{2}}$.*

Proof. Again, all of these are entirely routine calculations with the universal property of an initial algebra. $\square$

Unwinding the definitions show that to give an R_F-algebra structure on a map $f \colon A \to B$ is to give a morphism $p \colon Pf \to A$ satisfying the three equations $p.\gamma_f = 1_A$, $f.p = \rho_f$ and $p.P(p, 1_B) = p.\pi_f$. In fact, since in giving p we are mapping *out* of an initial algebra, this description simplifies further.

5.3. Proposition. *To give an R_F-algebra structure on some $(f \colon A \to B) \in \mathscr{E}^{\mathbf{2}}$ is equally well to give a map $\phi \colon B \times FA \to A$ satisfying $f.\phi = \pi_1$; and in these terms, a morphism $(h, k) \colon f \to g$ of $\mathscr{E}^{\mathbf{2}}$ is a map of R_F-algebras $(f, \phi) \to (g, \phi')$ just when the equation $h.\phi = \phi'.(k \times Fh)$ is validated.*

Proof. For an R_F-algebra $p \colon Pf \to A$, the corresponding map $\phi \colon B \times FA \to A$ over B is given by the composite

$$
B \times FA \xrightarrow{B \times F\gamma_f} B \times FPf \xrightarrow{\theta_f} Pf \xrightarrow{p} A \ .
$$

Conversely, for a map $\phi \colon B \times FA \to A$ over B, the corresponding R_F-algebra structure $p \colon Pf \to A$ is obtained as the unique map making the square

$$
\begin{array}{ccc}
A + B \times FPf & \xrightarrow{A + B \times Fp} & A + B \times FA \\
{\scriptstyle \iota_f}\downarrow & & \downarrow{\scriptstyle [1_A, \phi]} \\
Pf & \xrightarrow{p} & A
\end{array}
$$

commute. The remaining verifications are straightforward. $\square$

Just as the comonad L_F induces a comonad on each coslice category $A/\mathscr{E}$, so R_F induces a monad on each slice category $\mathscr{E}/B$. In particular, when $B = 1$, we obtain the monad on $\mathscr{E}$ whose underlying assignation on objects is given by $A \mapsto \mu X.A + FX$. We recognise this as the *free monad* on the endofunctor F, which is characterised by the property that its category of algebras is canonically isomorphic to the category of F-algebras in the sense

of Definition 2.1. The monad R_F may be seen as a generalisation of this, with the preceding Proposition being the corresponding generalisation of the universal property of the free monad.

We now give the promised dualisation of the constructions of Definitions 2.3 and 5.1.

5.4. **Definition.** Let $\mathscr{E}$ be a category with finite products and coproducts, and $F\colon \mathscr{E} \to \mathscr{E}$ an endofunctor such that for all $A, B \in \mathscr{E}$, the endofunctor $B \times (A + F-)$ admits a final coalgebra. Then we define the *cyclic term graph comonad* to be $\bar{L}_F := (R_{F^{\mathrm{op}}})^{\mathrm{op}}$, and the *cyclic term graph monad* to be to be $\bar{R}_F := (L_{F^{\mathrm{op}}})^{\mathrm{op}}$.

Before going on, let us extract an explicit description of the comonad $\bar{L}_F$. Given an object $f\colon A \to B$ of $\mathscr{E}^{\mathbf{2}}$, we write $\bar{P}f$ for the final coalgebra of $X \mapsto B \times (A + FX)$, write

$$\bar{\iota}_f = (\bar{\rho}_f, \bar{s}_f)\colon \bar{P}f \to B \times (A + F\bar{P}f)$$

for its coalgebra structure, and write $\bar{\gamma}_f\colon A \to \bar{P}f$ for the unique map making the square

$$
\begin{array}{ccc}
A & \xrightarrow{\ \ \bar{\gamma}_f\ \ } & \bar{P}f \\
{\scriptstyle (f,\mathrm{inl})}\downarrow & & \downarrow{\scriptstyle \bar{\iota}_f} \\
B \times (A + FA) & \xrightarrow[B\times(A+F\bar{\gamma}_f)]{} & B \times (A + F\bar{P}f)
\end{array}
$$

commute. We now define $\bar{L}_F$ on objects by $\bar{L}_F(f\colon A \to B) := (\bar{\gamma}_f\colon A \to Pf)$. We define its action on morphisms $(h, k)\colon f \to g$ of $\mathscr{E}^{\mathbf{2}}$ by

$$
\bar{L}_F \left(
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
f\downarrow & & \downarrow g \\
B & \xrightarrow{k} & D
\end{array}
\right)
\quad = \quad
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
\gamma_f\downarrow & & \downarrow \gamma_g \\
\bar{P}f & \xrightarrow[\bar{P}(h,k)]{} & \bar{P}g
\end{array}
$$

where $\bar{P}(h, k)$ is defined by universality of $\bar{P}g$ as the unique map making

$$
\begin{array}{ccc}
\bar{P}f & \xrightarrow{\ \ \bar{P}(h,k)\ \ } & \bar{P}g \\
{\scriptstyle (k\times(h+F\bar{P}f)).\bar{\iota}_f}\downarrow & & \downarrow{\scriptstyle \bar{\iota}_g} \\
D \times (C + F\bar{P}f) & \xrightarrow[D\times(C\times F\bar{P}(h,k))]{} & D \times (C + F\bar{P}g)
\end{array}
$$

commute. The natural transformations $\bar{L}_F \Rightarrow 1$ and $\bar{L}_F \Rightarrow \bar{L}_F\bar{L}_F$ providing the comonad structure have respective f-components given by maps

$$
\begin{array}{ccc}
A & \xrightarrow{1_A} & A \\
\bar{\gamma}_f\downarrow & & \downarrow f \\
\bar{P}f & \xrightarrow[\bar{\rho}_f]{} & B
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
A & \xrightarrow{1_A} & A \\
\bar{\gamma}_f\downarrow & & \downarrow \bar{\gamma}_{\bar{\gamma}_f} \\
\bar{P}f & \xrightarrow[\bar{\sigma}_f]{} & \bar{P}\bar{\gamma}_f
\end{array}
$$

in $\mathscr{E}^{\mathbf{2}}$, where $\bar{\rho}_f$ is defined as above, and $\bar{\sigma}_f$ is defined by universality of $\bar{P}\bar{\gamma}_f$ as the unique map rendering commutative the square:

$$
\begin{array}{ccc}
\bar{P}f & \xrightarrow{\;\;\bar{\sigma}_f\;\;} & \bar{P}\bar{\gamma}_f \\
{\scriptstyle(1,\bar{\theta}_f)}\downarrow & & \downarrow{\scriptstyle\bar{\iota}_{\bar{\gamma}_f}} \\
\bar{P}f \times (A + F\bar{P}f) & \xrightarrow[\bar{P}f\times(A+F\bar{\sigma}_f)]{} & \bar{P}f \times (A + F\bar{P}\bar{\gamma}_f) \ .
\end{array}
$$

5.5. **Definition.** For a category $\mathscr{E}$ with finite products and coproducts, and an endofunctor $F\colon \mathscr{E} \to \mathscr{E}$ such that each $B \times (A + F(-))$ has a final coalgebra, we define the category $\mathbf{ATG}_\infty(F)$ of *cyclic abstract term graphs over F* to be the category of $\bar{L}_F$-coalgebras.

By the dual of Proposition 5.3, the category $\mathbf{ATG}_\infty(F)$ is isomorphic to the category whose objects are pairs $(f\colon A \to B,\; s\colon B \to A + FB)$ for which $s.f = \mathsf{inl}$, and whose morphisms $(f,s) \to (g,s')$ are maps $(h,k)\colon f \to g$ in $\mathscr{E}^{\mathbf{2}}$ for which $s'.k = (h + Fk).s$; which is almost precisely the definition of cyclic term graph given in [2]. Note that if we also took this as our definition of cyclic abstract term graphs, then it would make sense under much weaker hypotheses than those of Definition 5.5: it is enough that $\mathscr{E}$ should have binary coproducts. Although this extra generality is certainly useful, for the present paper we shall retain the narrower definition, and this for two reasons: firstly, to highlight the duality between the cyclic and the acyclic cases; and secondly, so that later on, when we consider further aspects of the theory, we can treat these two cases in a uniform manner.

Let us now show that abstract cyclic term graphs are a faithful generalisation of the concrete ones.

5.6. **Proposition.** *For any signature Σ, the categories of cyclic concrete term graphs over Σ and of cyclic abstract term graphs over F_Σ are equivalent.*

Proof. The method of proof is the same as Proposition 3.3: we define a functor $F\colon \mathbf{CTG}_\infty(\Sigma) \to \mathbf{ATG}_\infty(F_\Sigma)$ and show it to be an equivalence. On objects, given a cyclic concrete term graph $T = (A, V, \ell, \varphi)$, we observe that ℓ and φ together determine a morphism $\ell\colon V \to F_\Sigma(A)$; so that we may take $F(T)$ to be the $\bar{L}_F$-coalgebra whose underlying object in $\mathbf{Set}^{\mathbf{2}}$ is $\mathsf{inl}\colon A \to A + V$, and whose coalgebra structure corresponds under the isomorphism of Proposition 5.3 to the map $A + \ell\colon A + V \to A + F_\Sigma(A + V)$. The remaining details are entirely analogous to Proposition 3.3 (though simpler) and hence omitted.

(Observe that when we apply Proposition 5.3 here, we are really doing something quite familiar. Turning the map $A + \ell$ into an $\bar{L}_{F_\Sigma}$-coalgebra structure on $\mathsf{inl}\colon A \to A + V$ corresponds to taking the concrete cyclic term graph T and unfolding it into a possibly-infinite labelled tree.) $\qquad\square$

It is quite straightforward to see that for each of the more general examples discussed in Section 4, applying the dual construction yields an appropriate notion of cyclic term graph.

6. Interpretation

Now that we have good abstract notions of both acyclic and cyclic term graph, we wish to develop further aspects of their theory. In this section, we

discuss how to interpret term graphs in a suitable semantic domain; whilst in the next, we shall discuss how abstract term graphs may be composed. In order to give a uniform treatment of both kinds of term graph, we shall describe a general structure of which both are particular instances; so that by framing subsequent results in terms of this general structure, we may deal with both cases simultaneously. The structure in question is not an *ad hoc* one, but one of importance in abstract homotopy theory and category theory.

6.1. **Definition.** A *natural weak factorisation system* [7, 10] on a category $\mathscr{E}$ is given by the assignation of a factorisation

$$(8) \qquad A \xrightarrow{f} B \qquad \mapsto \qquad A \xrightarrow{\gamma_f} Pf \xrightarrow{\rho_f} B$$

to every morphism of $\mathscr{E}$; a factorisation

$$(9) \qquad
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
h \downarrow & & \downarrow k \\
C & \xrightarrow{g} & D
\end{array}
\qquad \mapsto \qquad
\begin{array}{ccccc}
A & \xrightarrow{\gamma_f} & Pf & \xrightarrow{\rho_f} & B \\
h \downarrow & & \downarrow P(h,k) & & \downarrow k \\
C & \xrightarrow{\gamma_g} & Pg & \xrightarrow{\rho_g} & D
\end{array}$$

to every commutative square of $\mathscr{E}$, functorial in (h, k); and for each $f\colon A \to B$ in $\mathscr{E}$, choices of maps $\sigma_f\colon Pf \to P\gamma_f$ $\pi_f\colon P\rho_f \to Pf$ such that:

- There is a comonad (L, ϵ, Δ) on $\mathscr{E}^{\mathbf{2}}$ with $Lf = \gamma_f$, with $\epsilon_f = (1, \rho_f)\colon \gamma_f \to f$ and with $\Delta_f = (1, \sigma_f)\colon \gamma_f \to \gamma_{\gamma_f}$.
- There is a monad (R, η, μ) on $\mathscr{E}^{\mathbf{2}}$ with $Rf = \rho_f$, with $\eta_f = (\gamma_f, 1)\colon f \to \rho_f$ and with $\mu_f = (\pi_f, 1)\colon \rho_{\rho_f} \to \rho_f$.
- There is a distributive law $\gamma\colon LR \Rightarrow RL$ whose component at f is given by $\gamma_f = (\sigma_f, \pi_f)\colon \gamma_{\rho_f} \to \rho_{\gamma_f}$.

The notion of natural weak factorisation system is a strengthening of Quillen's notion of *weak factorisation system* [16], which has found use in computer science in the open map approach to bisimulation of [3, 13].

6.2. **Proposition.** *For any category $\mathscr{E}$ with finite products and coproducts, and any endofunctor $F\colon \mathscr{E} \to \mathscr{E}$ for which each initial algebra $\mu X.A + B \times FX$ exists, the monad-comonad pair (L_F, R_F) on $\mathscr{E}^{\mathbf{2}}$ is a natural weak factorisation system.*

Proof. All that remains is to exhibit the required distributive law γ. Since we already have the necessary data, we need only check the corresponding axioms, which we may do through a further straightforward manipulation using the universal property of an initial algebra. $\qquad\square$

By duality, we immediately obtain:

6.3. **Corollary.** *For any category $\mathscr{E}$ with finite products and coproducts, and any endofunctor $F\colon \mathscr{E} \to \mathscr{E}$ for which each final coalgebra $\nu X.B \times (A + FX)$ exists, the monad-comonad pair $(\bar{L}_F, \bar{R}_F)$ on $\mathscr{E}^{\mathbf{2}}$ is a natural weak factorisation system.*

The two preceding results are more than a convenient framing device for our subsequent development: they actually guide that development, by allowing

us to apply aspects of the theory of natural weak factorisation systems to the study of term graphs. For our first such application, we derive a notion of *interpretation* for abstract term graphs, using the following basic result from the theory of natural weak factorisation systems:

6.4. **Proposition** ("Lifting"). *If (L, R) is a natural weak factorisation system on a category $\mathscr{E}$, then for any commutative square*

$$
\begin{array}{ccc}
A & \xrightarrow{\ h\ } & C \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
B & \xrightarrow[\ k\ ]{} & D
\end{array}
$$

in $\mathscr{E}$, any L-coalgebra structure on f and any R-algebra structure on g, there is a canonical choice of morphism $j\colon B \to C$ such that $gj = k$ and $jf = h$.

Proof. The L-coalgebra structure on f is given by a morphism $s\colon B \to Pf$ satisfying axioms; likewise, the R-algebra structure on g is given by a map $p\colon Pg \to C$. We may therefore take j to be the composite

$$(10) \qquad\qquad B \xrightarrow{\ s\ } Pf \xrightarrow{\ P(h,k)\ } Pg \xrightarrow{\ p\ } C \ . \qquad\qquad \square$$

Let us see how this pertains to term graphs. We shall specialise Proposition 6.4 to the particular case where $D = 1$, and apply it first to the acyclic situation of Proposition 6.2, and then to the cyclic situation of Corollary 6.3. In the former case, the basic data we have is a diagram

$$(11) \qquad\qquad
\begin{array}{ccc}
A & \xrightarrow{\ h\ } & C \\
{\scriptstyle f}\downarrow & & \\
B & &
\end{array}
$$

where f is an L_F-coalgebra—hence an acyclic term graph over F—and the unique map $C \to 1$ is an R_F-algebra; which by the discussion following Proposition 5.3, is equally well to say that C bears an F-algebra structure $c\colon FC \to C$. Now the object A is the object of free variables of the acyclic term graph f; and so to give the map h is to give an interpretation of these variables in the F-algebra C. The canonical map $j\colon B \to C$ whose existence is assured by Proposition 6.4 extends this to an interpretation of *all* nodes of the given term graph in C, and does so using the F-algebra structure in the obvious manner. In fact, by unwinding the definitions in (10), we find that the map j specified there is obtained by composing the coalgebra map $s\colon B \to Pf$ with the map $\mathsf{ev}\colon Pf \to C$ obtained by universality in

$$
\begin{array}{ccc}
A + B \times FPf & \xrightarrow{\ A + B \times F(\mathsf{ev})\ } & A + B \times FC \\
{\scriptstyle \iota_f}\downarrow & & \downarrow{\scriptstyle [h, c\pi_2]} \\
Pf & \xrightarrow[\ \mathsf{ev}\ ]{} & C \ ;
\end{array}
$$

which is easily seen to agree with the natural notion of interpretation we would give in the concrete situation. Let us now consider the cyclic case. This time, our basic data are a diagram of the form (11), an $\bar{L}_F$-coalgebra structure on

f—making it into a cyclic term graph—and an $\bar{R}_F$-algebra structure on $C \to 1$. Now, to give the latter is equally well to equip C with an algebra structure for the monad obtained on $\mathscr{E} \cong \mathscr{E}/1$ by restricting and corestricting R_F to those objects of $\mathscr{E}^2$ with codomain 1. The monad in question is the one whose underlying assignation on objects is given by

$$A \mapsto \nu X.A + FX \; ;$$

it has been studied carefully in [1, 14], where it is called the *free completely iterative monad* on F. Its algebras are called *completely iterative F-algebras*, and are characterised as being those F-algebras with the property that every system of guarded recursive equations over F has a solution. Without going into the details of this let us merely say that this is precisely what is captured by our notion of interpretation. We may regard the cyclic term graph $f\colon A \to B$ as a system of guarded recursive equations over F with constants in the set A. The map $h\colon A \to C$ indicates how to interpret the constants of the recursive equations in the completely iterative F-algebra C; whilst the extension to a map $j\colon B \to C$ provides the corresponding solution.

7. Composition

In this final section, we shall show that our abstract term graphs admit an operation of *composition*, which chains the results of computation from one term graph to another. In order to perform such a chaining, an extra datum is required, indicating how the free variables of the second term graph should be filled by values from the computation of the first. As presaged in the discussion following Definition 3.2, we shall determine this extra datum by considering term graphs equipped with a distinguished collection of "output nodes".

7.1. **Definition.** Let $F\colon \mathscr{E} \to \mathscr{E}$ be an endofunctor to which the construction of Section 2 (respectively, its dual) applies. An *acyclic* (respectively, *cyclic*) *term graph from A to B* over F is a cospan

$$
\begin{array}{ccc}
A & & B \\
& \searrow f \quad g \swarrow & \\
& X &
\end{array}
$$

together with an L_F- (respectively, $\bar{L}_F$-) coalgebra structure on f.

In the concrete case, we see by Proposition 3.3 that acyclic or cyclic term graphs from A to B over a signature endofunctor F_Σ correspond to concrete term graphs $T = (A, V, \ell, \varphi)$ equipped with a function $g\colon B \to A + V$. In that case, we may compose such a pair $(T, g)\colon A \to B$ with another pair $(T', g')\colon B \to C$, to obtain the pair $(T' \circ T, h)\colon A \to C$ given as follows.

- The set of input nodes of $T' \circ T$ is A (as it must be);
- The set of internal nodes is $V + V'$;
- The labelling function is $[\ell, \ell']\colon V + V' \to \Sigma$;
- The children of an element $v \in V$ are given by $\varphi_i(v)$;
- The children of an element $v' \in V'$ are given by

$$
\psi_i(v') = \begin{cases} \varphi'_i(v) & \text{if } \varphi'_i(v) \in V' \\ g(\varphi'_i(v)) & \text{if } \varphi'_i(v) \in B \end{cases}
$$

• The function $h\colon C \to A + V + V'$ is given by

$$h(c) = \begin{cases} g'(c) & \text{if } g'(c) \in V' \\ g(g'(c)) & \text{if } g'(c) \in B \ . \end{cases}$$

In the acyclic case, the required acyclicity of the composite follows from that of the two parts and a case analysis. What we shall now do is provide an abstract analogue of this composition.

7.2. **Proposition.** *Let $F\colon \mathscr{E} \to \mathscr{E}$ be an endofunctor to which the construction of Section 2 (respectively, its dual) applies, and let $\mathscr{E}$ have pushouts. Under these hypotheses there is a category $\mathscr{S}[F]$ (respectively $\mathscr{S}_\infty[F]$) whose objects are those of $\mathscr{E}$ and whose morphisms $A \to B$ are equivalence classes of acyclic (respectively cyclic) term graphs from A to B.*

The notion of equivalence we use in this Proposition identifies two term graphs $A \to B$ just when there is an isomorphism $\lambda\colon X \to X'$ making

commute, and making the left-hand triangle a map of coalgebras for the appropriate comonad. The reason for quotienting in this way is that we intend to define the composition of two cospans $A \to X \leftarrow B$ and $B \to Y \leftarrow C$ by taking it to be the outer edge of the diagram

(12)

wherein the bottom square is a pushout. As it stands, this composition is only associative up to isomorphism: and to rectify this, we must quotient out as above. This could be avoided if we were to make **Cospan**(F) into a *bicategory* rather than a category, but for our purposes, passing to the quotient seems to be the simplest way to proceed.

The other obstacle to defining the composition as in (12) lies in showing that the given coalgebra structures on f and h induce one on pf. We shall do this in two stages: first we show that "a pushout of an coalgebra is a coalgebra"—which gives us a coalgebra structure on p from the one on h—and then we show that "the composite of two coalgebras is a coalgebra"—which gives us the one on pf from those on p and on f. We may prove these results for the acyclic and the cyclic cases simultaneously, as they are completely general facts about natural weak factorisation systems.

7.3. **Proposition** ("L-coalgebras push out"). *Let (L, R) be a natural weak factorisation system on a category $\mathscr{E}$. For any pushout square*

$$
\begin{array}{ccc}
A & \xrightarrow{\ h\ } & C \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
B & \xrightarrow[\ k\ ]{} & D
\end{array}
$$

and any L-coalgebra structure on f, there is a unique L-coalgebra structure on g making the square a map of L-coalgebras.

Proof. To give an L-coalgebra structure on f is to give a map $s\colon B \to Pf$ satisfying $\rho_f.s = 1_B$, $s.f = \gamma_f$ and $P(1_A, s).s = \sigma_f.s$. We induce a corresponding map $t\colon D \to Pg$ for g by applying the universal property of pushout to the square

$$
\begin{array}{ccc}
A & \xrightarrow{\ \ \ h\ \ \ } & C \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \gamma_g} \\
B & \xrightarrow[\ P(h,k).s\ ]{} & Pg \ .
\end{array}
$$

Thus t is the unique map $D \to Pg$ satisfying $t.g = \gamma_g$ and $t.k = P(h,k).s$, and so will be the unique L-coalgebra structure on g making (h, k) into a map of L-coalgebras as soon as we have verified the other two L-coalgebra axioms: which is easy by the universal property of pushout. $\qquad\square$

7.4. **Proposition** ("L-coalgebras compose"). *Let (L, R) be a natural weak factorisation system on a category $\mathscr{E}$ and let $f\colon A \to B$, $g\colon B \to C$ in $\mathscr{E}$. For every choice of L-coalgebra structures on f and g, there is a unique compatible L-coalgebra structure on gf.*

By a *compatible* L-coalgebra structure on gf, we mean the following. By virtue of the given coalgebra structures on f and g, we have for any square

$$
\begin{array}{ccc}
A & \xrightarrow{\ h\ } & D \\
{\scriptstyle gf}\downarrow & & \downarrow{\scriptstyle p} \\
C & \xrightarrow[\ k\ ]{} & E
\end{array}
$$

and any R-algebra structure on p, a choice of filler $j\colon C \to D$ obtained by applying Proposition 6.4 twice: first with f on the left, and then with g. An L-coalgebra structure on gf is compatible if the preceding choices of fillers agree with those obtained by applying Proposition 6.4 once to gf.

Proof. For uniqueness, we observe that any given L-coalgebra structure on gf may be recovered by applying Proposition 6.4 to the square

$$
\begin{array}{ccc}
A & \xrightarrow{\ \gamma_{gf}\ } & P(gf) \\
{\scriptstyle gf}\downarrow & & \downarrow{\scriptstyle \rho_{gf}} \\
C & \xrightarrow[\ 1_C\ ]{} & C
\end{array}
$$

where ρ_{gf} is given its free R-algebra structure. Thus there can be at most one compatible L-coalgebra structure on gf, and we can calculate what it must be by applying Proposition 6.4 twice to the above square, first with f along the left and then with g. Let the L-coalgebra structures on f and g be given by $s\colon B \to Pf$ and $t\colon C \to Pg$ respectively. Then direct calculation shows that the induced L-coalgebra structure on gf is given as follows. First form the composite

$$\xi = B \xrightarrow{s} Pf \xrightarrow{P(1,g)} P(gf) \ .$$

Now the induced L-coalgebra structure $C \to P(gf)$ is given by the composite

$$C \xrightarrow{t} Pg \xrightarrow{P(\xi,1_C)} P\rho_{gf} \xrightarrow{\pi_{gf}} P(gf) \ .$$

That this is indeed an L-coalgebra structure, and a compatible one, is straightforward calculation. $\qquad\square$

Applying the preceding two results to the natural weak factorisation systems of Proposition 6.2 and Corollary 6.3, we obtain:

Proof of Proposition 7.2. As anticipated, we define composition in $\mathscr{S}[F]$ and $\mathscr{S}_\infty[F]$ by pushouts of the form (12), using the preceding two Propositions to induce the required coalgebra structure on the composite left leg from the coalgebra structures on the constituents. We give the identity map at A by the cospan

$$
\begin{array}{ccc}
A & & A \\
& \searrow^{1_A} \quad \swarrow_{1_A} & \\
& A &
\end{array}
$$

where the left leg is equipped with its unique possible coalgebra structure. We must check that this composition is associative and unital. Because we have passed to equivalence classes, we have this at the level of underlying cospans; it remains to verify that the induced coalgebra structures on the composites are likewise well-behaved. But this follows easily from the universal properties ascribed to the constructions of Propositions 7.3 and 7.4. $\qquad\square$

Using the results of Proposition 3.3 we may now show that the composition of abstract term graphs, when specialised to a signature endofunctor on **Set**, agrees with the composition described after Definition 7.1. Note that this in particular provides an abstract reason why this latter composition should be associative, a fact which would otherwise have required a direct calculation.

Let us remark that the categories $\mathscr{S}[F]$ and $\mathscr{S}_\infty[F]$ will not typically be locally small, even if $\mathscr{E}$ is so. The reason is essentially that a term graph from A to B may choose to do an arbitrary amount of irrelevant computation which is invisible from the perspective of the output nodes in B. Thus in practice it may be convenient to consider a suitable full small subcategory $\mathscr{A} \subset \mathscr{E}$, and to cut down from $\mathscr{S}[F]$ to the subcategory whose objects are those lying in $\mathscr{A}$, and whose morphisms $A \to X \leftarrow B$ are those cospans where A, X and B all lie in $\mathscr{A}$ (where for this definition to work we must assume that $\mathscr{A}$ is closed under the appropriate pushouts in $\mathscr{E}$). Thus when $\mathscr{E} = \mathbf{Set}$ and F is the endofunctor associated to a signature Σ, we may take $\mathscr{A}$ to be the category of finite cardinals and so obtain the category of *finite* term graphs; which in

the terminology of [12], is the classifying category of the pure sharing theory over the signature Σ. Likewise, when $\mathscr{E} = \mathbf{Set}^{\mathbb{F}}$ and F is an endofunctor of the kind considered in Section 4, a sensible choice for $\mathscr{A}$ would be the full subcategory of $\mathbf{Set}^{\mathbb{F}}$ comprised of the finite coproducts of representables.

We now show that the categories $\mathscr{S}[F]$ and $\mathscr{S}_{\infty}[F]$ defined above play well with the notion of interpretation described in Section 6. Consider the acyclic case first. Given any F-algebra $c\colon FC \to C$, we obtain maps

$$(13) \qquad \mathscr{S}[F](A,B) \times \mathscr{E}(A,C) \to \mathscr{E}(B,C)$$

as follows. Given a cospan $A \xrightarrow{f} X \xleftarrow{g} B$ in $\mathscr{S}[F]$ and a map $h\colon A \to C$, we may apply Proposition 6.4 to obtain an extension $j\colon X \to C$; and composing this with g yields the required map $jg\colon B \to C$. The point is that this process is well-behaved with respect to composition of term graphs.

7.5. **Proposition.** *To any F-algebra $c\colon FC \to C$, we may associate a functor $\mathscr{S}[F] \to \mathbf{Set}$ given on objects by $A \mapsto \mathscr{E}(A,C)$ and on morphisms by (13).*

Proof. We need only show functoriality; for which we apply the universal properties of the two constructions given in Proposition 7.3 and 7.4. $\qquad\square$

Transposing these results into the cyclic case we obtain:

7.6. **Corollary.** *To any completely iterative F-algebra $c\colon FC \to C$, we may associate a functor $\mathscr{S}_{\infty}[F] \to \mathbf{Set}$ given on objects by $A \mapsto \mathscr{E}(A,C)$ and on morphisms by the cyclic analogue of (13).*

Proof. By duality. $\qquad\square$

Let us conclude by briefly considering the extra structure carried by the category $\mathscr{S}[F]$ and $\mathscr{S}_{\infty}[F]$; this is very much in the spirit of [12] and one can envisage further development along those lines.

7.7. **Proposition.** *The category $\mathscr{S}[F]$ admits a symmetric monoidal structure and an identity-on-objects strict symmetric monoidal embedding $\mathscr{E}^{\mathrm{op}} \to \mathscr{S}[F]$, where $\mathscr{E}^{\mathrm{op}}$ is equipped with its cartesian monoidal structure.*

Proof. The unit of the monoidal structure on $\mathscr{S}[F]$ is the object 0 of $\mathscr{E}$. The tensor product is given on objects by $A \otimes A' = A + A'$ and on morphisms by $(A \to X \leftarrow B) \otimes (A' \to X' \leftarrow B') = (A + A' \to X + X' \leftarrow B + B')$; the coalgebra structure on the left leg of this tensor product being the coproduct in the category of L_F-algebras. The embedding $\mathscr{E}^{\mathrm{op}} \to \mathscr{S}[F]$ is the identity on objects, and on morphisms sends $f\colon A \to B$ to the term graph

from B to A, where 1_B is seen as equipped with its unique L_F-coalgebra structure. $\qquad\square$

REFERENCES

[1] ACZEL, P., ADÁMEK, J., MILIUS, S., AND VELEBIL, J. Infinite trees and completely iterative theories: a coalgebraic view. *Theoretical Computer Science 300* (2003), 1–45.

[2] BARENDREGT, H., VAN EEKELEN, M., GLAUERT, J., KENNAWAY, J., M., P., AND SLEEP, M. Term graph rewriting. In *Parallel Architectures and Languages Europe (Eindhoven, 1987)*, vol. 259 of *Lecture Notes in Computer Science*. Springer, 1987, pp. 141–158.

[3] CATTANI, G. L., AND WINSKEL, G. Profunctors, open maps and bisimulation. *Mathematical Structures in Computer Science 15*, 3 (2005), 553–614.

[4] CORRADINI, A., AND GADDUCCI, F. A 2-categorical presentation of term graph rewriting. In *Category Theory and Computer Science VII (Santa Margherita Ligure, 1997)*, vol. 1290 of *Lecture Notes in Computer Science*. Springer, 1997, pp. 87–105.

[5] CORRADINI, A., AND GADDUCCI, F. An algebraic presentation of term graphs, via gs-monoidal categories. *Applied Categorical Structures 7*, 4 (1999), 299–331.

[6] FIORE, M., PLOTKIN, G., AND TURI, D. Abstract syntax and variable binding. In *Logic in Computer Science 14 (Trento, 1999)*. IEEE Computer Society Press, 1999, pp. 193–202.

[7] GARNER, R. Understanding the small object argument. *Applied Categorical Structures 17*, 3 (2009), 247–285.

[8] GHANI, N., LÜTH, C., AND DE MARCHI, F. Monads of coalgebras: Rational terms and term graphs. *Mathematical Structures in Computer Science 15* (2005), 433–451.

[9] GHANI, N., LUTH, C., DE MARCHI, F., AND POWER, J. Algebras, coalgebras, monads and comonads. *Electronic Notes in Theoretical Computer Science 44*, 1 (2001), 1–18.

[10] GRANDIS, M., AND THOLEN, W. Natural weak factorization systems. *Archivum Mathematicum 42*, 4 (2006), 397–408.

[11] HAMANA, M. Initial algebra semantics for cyclic sharing structures. *Logical Methods in Computer Science 6*, 3:15 (2010), 1–23.

[12] HASEGAWA, M. *Models of Sharing Graphs (A Categorical Semantics of Let and Letrec)*. PhD thesis, University of Edinburgh, 1997.

[13] JOYAL, A., NIELSEN, M., AND WINSKEL, G. Bisimulation from open maps. *Information and Computation 127*, 2 (1996), 164–185.

[14] MILIUS, S. Completely iterative algebras and completely iterative monads. *Information and Computation 196*, 1 (2005), 1–41.

[15] PLUMP, D. Term graph rewriting. In *Handbook of Graph Grammars and Computing by Graph Transformation*. World Scientific, 1999, pp. 3–61.

[16] QUILLEN, D. G. *Homotopical algebra*, vol. 43 of *Lecture Notes in Mathematics*. Springer, 1967.

[17] SLEEP, M., PLASMEIJER, M., AND VAN EEKELEN, M., Eds. *Term Graph Rewriting: Theory and Practice*. John Wiley & Sons, 1993.

[18] TANAKA, M. Abstract syntax and variable binding for linear binders. In *Mathematical Foundations of Computer Science (Bratislava, 2000)*, vol. 1893 of *Lecture Notes in Computer Science*. Springer, 2000, pp. 670–679.

[19] TANAKA, M. *Pseudo-Distributive Laws and a Unified Framework for Variable Binding*. PhD thesis, University of Edinburgh, 2005.

[20] TANAKA, M., AND POWER, J. A unified category-theoretic semantics for binding signatures in substructural logics. *Journal of Logic and Computation 16*, 1 (2006), 5–25.

DEPARTMENT OF COMPUTING, MACQUARIE UNIVERSITY, NSW 2109, AUSTRALIA
E-mail address: `richard.garner@mq.edu.au`