# On the Parity of Planar Covers

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**ABSTRACT**

A *covering* is a graph map $\phi$: $G \to H$ that is an isomorphism when restricted to the star of any vertex of $G$. If $H$ is connected then $|\phi^{-1}(v)|$ is constant. This constant is called the *fold number*. In this paper we prove that if $G$ is a planar graph that covers a nonplanar $H$, then the fold number must be even.

In this paper we consider only finite graphs. All graphs are connected unless specifically stated otherwise. For notational convenience we do not allow loops or multiple adjacencies, although the results of this paper extend easily to include these cases. A *graph map* is a function of $\phi$: $V(G) \to V(H)$ such that for each edge $uv$ in $G$, $\phi(u)\phi(v)$ is an edge in $H$. Since our graphs are simple, a graph map induces a mapping from the edges of $G$ to the edges of $H$, and we will consider $\phi$ as a function on the edges as well. Let $st(v)$ denote the subgraph induced by the edges incident with $v$. A *covering* is a graph map that induces an isomorphism from $st(v)$ to $st(\phi(v))$ for each $v \in V(G)$. We call $G$ a *cover* of $H$. Observe that if the graphs are endowed with the usual topology, then a covering is a topological covering map. Figure 1 shows a covering of $K_5$ by a planar graph $G$. Here, as elsewhere, we can define the map $\phi$ by labeling the vertices of $G$ with their image $\phi(v)$.

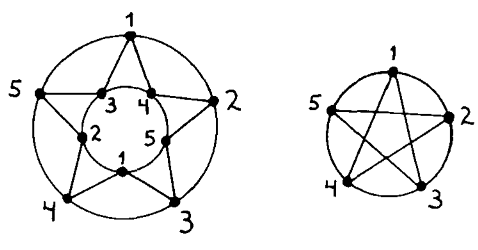


FIGURE 1

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Several observations about coverings are immediate. For example, coverings preserve degrees, that is, $\deg_G(v) = \deg_H(\phi(v))$. Also, if $e = uv$ is an edge of $H$, then $\phi^{-1}(e)$ gives a bijection between $|\phi^{-1}(u)|$ and $|\phi^{-1}(v)|$. Since $H$ is connected, it follows that $|\phi^{-1}(v)|$ is constant, say $n$. We call this constant the *fold number* of the covering, and say that $\phi$ is an *n-fold* covering. The main result of this paper is

**Theorem 1.** If $G$ is a $n$-fold planar cover of a nonplanar graph $H$, then $n$ is even.

An $n$-fold covering where $n$ is odd will be called an *odd-fold* covering. Thus Theorem 1 states that a nonplanar graph does not have an odd-fold cover.

Negami [5] conjectures that a graph has a planar cover if and only if the graph embeds in the real projective plane. He has proven this in the special case of regular planar covers [5]. Using techniques similar to those in our Lemma 3, it suffices to show that the 35 minor minimal graphs that do not embed in the projective plane [1] have no planar covers. Using the techniques of our Lemma 4, this set of 35 reduces to a set of 11. Of this set of eleven, 9 have been proven to have no planar cover. Thus, to prove Negami's conjecture, it suffices to prove that $K_{4,4} - K_2$ and $K_7 - 3K_2$ have no planar cover (these are the two remaining cases). Theorem 1 offers some evidence of the verity of Negami's conjecture.

The proof of Theorem 1 is broken into two parts. In Section 1 we show that Theorem 1 reduces to showing that $H = K_5$ has no odd-fold planar covering. In Section 2 we prove this special case. In Section 3 we give some concluding remarks. We proceed with the proofs.

## 1. THE REDUCTION TO $H = K_5$

In this section we reduce the proof of Theorem 1 to the case where $H = K_5$. We will first show that we may reduce to the case where $H$ is either $K_{3,3}$ or $K_5$. Recall that a graph $H$ is a *minor* of $G$ if we can form $H$ from $G$ by a sequence of edge deletions, edge contractions, and deletion of isolated vertices.

**Lemma 2.** If $G$ is a cover of $H$, then any minor of $H$ is covered by a (possibly disconnected) minor of $G$ with the same fold number.

***Proof.*** We shall show that the lemma is true for a single edge deletion, a single edge contraction, and the deletion of a single vertex. The lemma will follow by induction. Observe that by performing all edge deletions prior to performing edge contractions the graphs in the intermediate steps have no loops nor parallel edges.

We first observe that $G - \phi^{-1}(e)$ is a (possibly disconnected) cover of $H - e$. Since $G - \phi^{-1}(e)$ is a minor of $G$, the lemma is true for edge deletions. Similarily, let $H/e$ denote the graph obtained from $H$ by contracting the edge $e$ to a point, and let $G/\phi^{-1}(e)$ denote the graph obtained from $G$ by contracting (one at a time) the edges in $\phi^{-1}(e)$. Since $G/\phi^{-1}(e)$ is a covering of $H/e$, the lemma is true for edge contractions. The lemma is easily seen to be true for the deletion of an isolated vertex. Thus it is true for arbitrary minors. ∎

**Lemma 3.** If a nonplanar graph $H$ has an odd-fold planar cover, then either $K_{3,3}$ or $K_5$ has a (connected) odd-fold planar cover.

***Proof.*** By Kuratowski's Theorem, a nonplanar $H$ must contain a subgraph homeomorphic to either $K_{3,3}$ or to $K_5$. Equivalently, every nonplanar graph contains $K_{3,3}$ or $K_5$ as a minor. By Lemma 2, the planar cover $G$ has a (possibly disconnected) minor $G'$ that covers the $K_{3,3}$ or $K_5$ with the same fold number as $G$ covers $H$. This $G'$ is a minor of a planar graph, so it is also planar. If $G'$ is disconnected, then each component is a planar cover of a Kuratowski graph. Since the total fold number is odd, at least one of these components has an odd fold number, which satisfies the lemma. ∎

We will show that if $K_{3,3}$ has as $n$-fold planar cover, then so does $K_5$. Let $v$ be a cubic vertex of a graph $H$ that is not in a triangle. Let $H'$ be the graph formed from $H$ by deleting $v$ and its three incident edges, and adding three edges joining pairwise the three vertices adjacent with $v$ (see Figure 2). We call $H'$ a $Y\Delta$-*transformation* of $H$ at $v$.

Let $G$ be a graph covering $H$, and let $H'$ be a $Y\Delta$-transformation of $H$ at $v$. Let $G'$ be the graph formed from $G$ by performing a $Y\Delta$-transformation at each vertex in $\phi^{-1}(v)$. Note that $G'$ covers $H'$, and this covering has the same fold number as the original covering. Also note that if $G$ is planar, then so is $G'$. Thus the property of having an $n$-fold planar cover is closed under $Y\Delta$-transformations.

**Lemma 4.** If $K_{3,3}$ has an $n$-fold planar cover, then $K_5$ has an $n$-fold planar cover.

***Proof.*** Let $G$ be a $n$-fold planar cover of $H = K_{3,3}$, and let $uv$ be an edge of $H$. Form $H'$ from $H$ by replacing the edge $uv$ by a path of length 2. Similarly form $G'$ from $G$ by replacing each edge in $\phi^{-1}(uv)$ by a path of length 2. Note that $G'$ is a planar cover of $H'$. We now create $K_5$ from $H'$ by performing a $Y\Delta$-transformation at both $u$ and $v$. The graph resulting from performing the corresponding $Y\Delta$-transformations on $G'$ is an $n$-fold planar cover of $K_5$. ∎

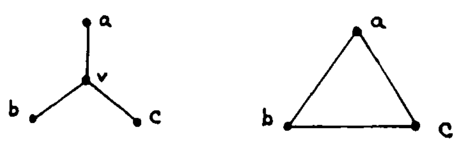


FIGURE 2

## 2. THE PROOF WHEN $H = K_5$

In this section we will show that if $K_5$ has an $n$-fold planar cover, then $n$ is even. Let $G$ be an $n$-fold planar cover of $K_5$, and label the vertices of $G$ with their image $\phi(v)$. Thus the four edges incident with a vertex labeled $v$ are also incident with vertices receiving the other four labels. In the (oriented) planar embedding of $G$, these edges appear in one of six possible cyclic permutations, called the *rotation* at this vertex. The six possible rotations fall into three sets when we pair a rotation $\rho$ with the rotation $\rho^{-1}$. We will show that in $\phi^{-1}(v) \subset V(G)$, vertices with rotation $\rho$ occur precisely as often as those with rotation $\rho^{-1}$. As a result, we can conclude

**Lemma 5.** If $G$ is an $n$-fold planar cover of $K_5$, then $n$ is even.

***Proof.*** As above, it suffices to show that a rotation $\rho$ occurs equally often with its inverse rotation $\rho^{-1}$. Label the vertices of $K_5$ by $1, \ldots, 5$. Without loss of generality, let $v = 1$, $\rho = (2345)$, and $\rho^{-1} = (2543)$. Let $T$ be the triangle $(124)$. Then $\phi^{-1}(T)$ is a 2-regular subgraph of $G$, and hence is the disjoint union of simple cycles. Let $S$ be one of these cycles. By the Jordan Curve Theorem, $S$ separates the plane into two components, an inside and an outside. Let $A$ be the set of vertices $v \in \phi^{-1}(1)$ where both of the edges labeled 13 and 15 incident with $v$ lie inside of $S$. Note that $A$ contains every vertex of $\phi^{-1}(1)$ that lies inside $S$, as well as possibly some of the vertices on $S$. Define $B$ as those vertices in $\phi^{-1}(1)$ incident with an edge 13 lying inside of $S$ and with an edge 15 lying outside of $S$. Similarly, define $C$ as those vertices in $\phi^{-1}(1)$ where 13 lies outside of $S$ and 15 lies inside of $S$. Note that the sets $B$ and $C$ are precisely the vertices in $S \cap \phi^{-1}(1)$ that receive rotations $\rho$ and $\rho^{-1}$. Thus to prove the claim, it suffices to prove that $|B| = |C|$.

The edges in $\phi^{-1}(35)$ give a bijection between the number of vertices in $\phi^{-1}(3)$ that lie inside of $S$ and the number of vertices in $\phi^{-1}(5)$ inside of $S$. The edges in $\phi^{-1}(13)$ that lie inside of $S$ give a bijection between the vertices inside of $S$ labeled 3 and the vertices in $A \cup B$. Similarly, the edges in $\phi^{-1}(15)$ inside of $S$ give a bijection between the vertices labeled 5 inside of $S$ and the vertices in $A \cup C$. It follows that $|A \cup B| = |A \cup C|$, and hence that $|B| = |C|$, as desired. ∎

## 3. CONCLUDING REMARKS

In Lemma 3 we showed that if a nonplanar graph $H$ has an odd-fold planar cover $G$, then so does either $K_{3,3}$ or $K_5$. In Lemma 4 we showed that, in fact, $K_5$ must have an odd-fold planar cover. Having shown in Lemma 5 that no such cover exists, the proof of Theorem 1 is complete.

The hypothesis that $H$ is nonplanar is necessary in Theorem 1. Let $\phi_n$ be the map from the plane to itself described in polar coordinates by $(\rho, \theta) \to (\rho, n\theta)$.

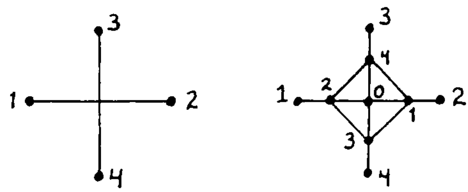


FIGURE 3

Provided that the origin is in the interior of a noninfinite face, $G_n = \phi_n^{-1}(H)$ is a connected planar $n$-fold cover of $H$ for all $n$. Furthermore, if $H$ embeds in the projective plane, then the 2-fold covering of the projective plane by the sphere yields a planar 2-fold covering of $H$. The preceding construction then gives an $n$-fold planar covering of $H$ for all even $n$.

Kleitman [4] has shown that any drawing of $K_5$ in the plane has an odd number of nonadjacent edge crossings (see [3] for an introduction to crossing numbers). The following proposition generalizes both Lemma 5 and Kleitman's result.

**Proposition 6.** In any drawing of any cover $G$ of $K_5$ in the plane, the number of crossings of $G$ plus the fold number of the covering is even.

***Proof.*** The proof proceeds by induction on the number of crossings. The start of the induction is provided by Theorem 1, where the number of crossings is zero. Let $G$ be a cover of $K_5$ drawn in the plane with an edge crossing as show in the left half of Figure 3. Replace the crossing edges with the subgraph shown in the right half of Figure 3. The $G'$ thus formed has one fewer crossing than does $G$, but the fold number is increased by one. By induction, the sum of these parameters for $G'$ is even and the proposition follows. ∎

The authors would find it interesting to further pursue the relationship between the genus of a graph and the minimum genus of its covering graphs. For example, consider a graph that embeds in the projective plane but that has large orientable genus (such graphs are known to exist by a result of Auslander et al. [2]). Using the 2-fold cover of the projective plane by the sphere, we obtain a planar 2-fold covering of a graph with arbitrarily large orientable genus. By slightly modifying the embeddings we can construct, for example, a toroidal covering of a nontoroidal graph. However, a construction of this type yields only even-fold coverings. It is unknown, for example, if there is an odd-fold toroidal covering of a nontoroidal graph.

## References

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