Journal Homepage: -www.journalijar.com

INTERNATIONAL JOURNAL OF ADVANCED RESEARCH (IJAR)

ISSN NO. 2320-5407

Article DOI:10.21474/IJAR01/6696
DOI URL: http://dx.doi.org/10.21474/IJAR01/6696

RESEARCH ARTICLE

EXISTENCE AND UNIQUENESS OF SOLUTIONS TO SYSTEM OF LINEAR EQUATIONS AND INTEGRAL EQUATIONS USING BANACH FIXED-POINT THEOREM.

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Manuscript Info

Manuscript History

Received: 08 January 2018
Final Accepted: 10 February 2018
Published: March 2018

Keywords:-
Fixed point, Banach fixed-point theorem, System of linear equations, Fredholm integral equation.

Abstract

In this paper, using Banach fixed-point theorem, we study the existence and uniqueness of solution for a system of linear equations. Further, we prove the existence and uniqueness of the continuous solutions of linear and non-linear Fredholm integral equations over the Banach space $L^2[a,b]$. Our claim is also illustrated with the applications to respective examples for proving the existence results.

Introduction:-

Over the last few decades, fixed-point theory has become an important field of study in science. It provides a powerful tool for proving the existence of solutions of problems originating from various branches of mathematics. It has long been used in analysis to solve various kinds of differential and integral equations [1, 8]. Existence theorem for differential equation was first given by Cauchy [8]. Applications of fixed point results to integral equations have been studied in [7, 8]. In metric space, this theory begins with Banach fixed-point theorem (also known as Banach contraction principle) [2, 8]. Banach fixed-point theorem has many applications to linear and non-linear equations, to ordinary and semi-linear partial differential equations and to linear and non-linear integral equations [1, 4, 5, 8]. In this paper, we study the applications of Banach fixed-point theorem for proving existence results to solutions of system of linear equations and integral equations.

This paper is organized as follows. In section 2, we review some required background materials. In section 3, we investigate an existence and uniqueness result of the solution of a system of linear equations $Ax=b$, where A is a $n\times n$ co-efficient matrix and b is a constant matrix. In section 4, we study the existence and uniqueness of the solution of Fredholm integral equation

$$f(x)=\varphi(x)+\lambda\int_a^b K(x,\,t)\,f(t)dt \tag{1.1}$$

where the kernel $K(x,\,t)$ and $\varphi(x)$ are known functions and λ is a real parameter. Here, we seek the unknown function $f(x)$ as a solution of the integral equation (1.1) over the Banach space $L^2[a,\,b]$ of Lebesgue measurable functions.

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Preliminaries:-
To prove the existence results of solutions for system of linear equations and integral equations, we need the definitions and theorems of the following paragraph.

Definition 2.1:-
Let X be a non-empty set and $T: X \to X$ be a mapping. A fixed point of the mapping T is a point $x \in X$ such that $Tx = x.$ In other words, a fixed point of T is a solution of the functional equation $Tx = x$ for $x \in X.$

Definition 2.2:-
Let (X, d) be a metric space. A mapping $T: X \to X$ is said to be contraction mapping if there exists a constant $L \in [0, 1)$ such that

$$d(Tx, Ty) \leq Ld(x, y) \text{ for all } x, y \in X.$$

The main tool in the existence result of a solution is the Banach fixed-point theorem. It is based on the complete metric space.

Theorem 2.1:-
(*Banach fixed-point theorem*) [1, 2] Let (X, d) be complete metric space and suppose $T: X \to X$ is a contraction mapping. Then T has a unique fixed point $x_0 \in X.$ Furthermore, for any $x \in X$ we have

$$\lim_{n\to\infty} T^n x = x_0 \text{ with } d(T^n x, x_0) \leq \frac{L^n}{1-L} d(Tx, x).$$

Definition 2.3:-
Let $(X, \|.\|)$ be a normed linear space. Then a complete normed linear space is called a Banach space [1, 3]. Every Banach space $(X, \|.\|)$ also is a complete metric space (X, d) under $d(x, y) = \|x - y\|.$

A Banach space is chosen in such a way that the existence problem is converted into a fixed-point problem for an operator over this Banach space.

We define an operator $T: X \to X$ by

$$f(x) = \varphi(x) + \lambda \int_a^b K(x, t) f(t) dt .$$

Thus, a solution of integral equation (1.1) is a fixed point of the operator T over the Banach space X .

Existence and Uniqueness Results for a System of Linear Equations:-
In this section, we study the existence as well as the uniqueness of the solution of a system of linear equations. Under some conditions, the following theorem ensures the existence and uniqueness of a solution of the system of linear equations [6, 9].

Theorem 3.1:-
Consider a system of linear equations $Ax = b$ where

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \quad x = (x_1, \cdots, x_n)^T, \quad b = (b_1, \cdots, b_n)^T.$$

Then, there exists a unique solution if $|I - A| < L$ for $L \in (0, 1).$

Proof: Given system of equations $Ax = b$ can be re-written as

$$\left.\begin{array}{l} x_1 = (1-a_{11})x_1 - a_{12}x_2 - \cdots - a_{1n}x_n + b_1 \\ x_2 = -a_{21}x_1 + (1-a_{22})x_2 - \cdots - a_{2n}x_n + b_2 \\ \vdots \\ x_n = -a_{n1}x_1 - a_{n2}x_2 - \cdots + (1-a_{nn})x_n + b_n \end{array}\right\} \quad (3.1)$$

For $1 \le i,\ j \le n,$ set $\left(\alpha_{ij}\right) = I - A.$ So the above system of equations can be written as

$$x_i = \sum_{j=1}^{n} \alpha_{ij} x_j + b_i, \qquad \forall\ i = \{1,\ 2, \cdots,\ n\}.$$

Obviously, the system $Ax = b$ is equivalent to the problem $x - Ax + b = x.$

Now, define a mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $Tx = x - Ax + b.$

Thus, finding solutions of the system (3.1), is equivalent to finding fixed points of the map $T.$ Now, for $x = (x_1, x_2, \cdots, x_n)$ and $x^{'} = (x^{'}_1, x^{'}_2, \cdots, x^{'}_n) \in \mathbb{R}^n,$ we have

$$Tx - Tx^{'} = (x - Ax + b) - (x^{'} - Ax^{'} + b) = (x - x^{'}) - A(x - x^{'}) = (I - A)(x - x^{'}).$$

We claim that $Ax = b$ has a unique solution if

$$\sum_{j=1}^{n} \left|\alpha_{ij}\right| = \left|I - A\right| \le L < 1, \qquad \forall\ i = \{1,\ 2, \cdots,\ n\}.$$

We define a metric d on $\mathbb{R}^n$ by $d(x, x^{'}) = \sup_{1 \le i \le n} \left|x_i - x^{'}_i\right|.$ Then

$$d(Tx, Tx^{'}) = \sup_{1 \le i \le n} \left|Tx_i - Tx^{'}_i\right| = \sup_{1 \le i \le n} \left|\sum_{j=1}^{n} \alpha_{ij}(x_j - x^{'}_j)\right|$$

$$\le \sup_{1 \le i \le n} \sum_{j=1}^{n} \left|\alpha_{ij}\right| \sup_{1 \le i \le n} \left|x_j - x^{'}_j\right| = \left|I - A\right| d(x, x^{'}) \le L d(x, x^{'})$$

$$\text{i.e. } d(Tx,\ Tx^{'}) \le L d(x,\ x^{'}), \quad 0 < L < 1.$$

This shows that T is a contraction mapping. An application of Theorem 2.1 completes the proof.

Example 3.1:- Consider the system of linear equations $Ax = b$, where

$$A = \begin{pmatrix} 2 & 2 & 0 & -1 \\ -1 & 1 & 0 & 1 \\ 0 & 1 & 2/3 & 0 \\ -1 & 0 & 0 & 2/3 \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^T, \qquad b = \begin{pmatrix} 1 & 2 & 1 & 1 \end{pmatrix}^T,$$

it follows that $\left|I - A\right| = \begin{vmatrix} -1 & -2 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & \frac{1}{3} & 0 \\ 1 & 0 & 0 & \frac{1}{3} \end{vmatrix} = \frac{8}{9} \le L$ for $0 < L < 1.$

Then, there exists a unique solution $x = \begin{pmatrix} -3 & 2 & -\frac{3}{2} & -3 \end{pmatrix}^T$.

Remark 3.1:-
The Theorem 3.1 decreases the computation burden of determining the existence and uniqueness of the solutions to a system of linear equations.

Existence and Uniqueness Results for Integral Equations:-
In this section, we are interested in the study of the existence of continuous solutions of the Fredholm linear and non-linear integral equations over a Banach space. For this purpose, we have chosen the Banach space $L^2[a, b]$ of Lebesgue measurable functions.
Consider, the following Fredholm linear integral equation of second kind

$$f(x) = \varphi(x) + \lambda \int_a^b K(x, t)\, f(t)\, dt \tag{4.1}$$

where the kernel $K(x, t)$ is continuous on $L^2[a, b] \times L^2[a, b]$, λ is a real parameter and the function $\varphi(x)$ is continuous on $L^2[a, b]$. Now, define an operator $T : L^2[a, b] \to L^2[a, b]$ by

$$(Tf)(x) = \varphi(x) + \lambda \int_a^b K(x, t)\, f(t)\, dt .$$

Thus, a solution of Fredholm linear integral equation (4.1) is a fixed point of the operator T [5]. If in the Fredholm integral equation (4.1), we replace the upper integration limit b by the variable x, we obtain a Volterra integral equation. Under some conditions on the parameter λ, the following theorem ensures the existence and uniqueness of a solution of the Fredholm linear integral equation (4.1).

Theorem 4.1:-

Let $K(x, t)$ be a continuous measurable function on $L^2[a,b] \times L^2[a,b]$ with $\int_a^b \int_a^b |K(x,t)|^2\, dxdt < \infty$ and $f, \varphi \in L^2[a,b]$ then the Fredholm linear integral equation (4.1) has a unique solution if $|\lambda| \|K(x,t)\| < 1$.

Proof: By hypothesis, $K(x,t)$ is a continuous measurable function on $L^2[a,b] \times L^2[a,b]$ with $\int_a^b \int_a^b |K(x,t)|^2\, dxdt < \infty$ and $f, \varphi \in L^2[a,b]$. We need only to show that $\int_a^b K(x,t)\, f(t) dt$ is bounded measurable function, *i.e.* $\int_a^b K(x,t)\, f(t) dt \in L^2[a,b]$.

Now by Schwarz's inequality, we have

$$\left| \int_a^b K(x,t)\, f(t) dt \right| \le \int_a^b |K(x,t)\, f(t)|\, dt$$

$$\le \left(\int_a^b |K(x,t)|^2\, dt \right)^{1/2} \left(\int_a^b |f(t)|^2\, dt \right)^{1/2} .$$

Therefore,

$$\left|\int_a^b K(x,t)\,f(t)dt\right|^2 \le \left(\int_a^b |K(x,t)|^2\,dt\right)\left(\int_a^b |\,f(t)|^2\,dt\right)$$

$$\text{Or, }\int_a^b \left|\int_a^b K(x,t)\,f(t)dt\right|^2 dx \le \int_a^b \left(\int_a^b |K(x,t)|^2\,dt\right)\left(\int_a^b |\,f(t)|^2\,dt\right)dx$$

$$\le \left(\int_a^b\int_a^b |K(x,t)|^2\,dtdx\right)\left(\int_a^b |\,f(t)|^2\,dt\right).$$

Since, $\int_a^b\int_a^b |K(x,t)|^2\,dtdx < \infty$ and $\int_a^b |\,f(t)|^2\,dt < \infty$, we have $\int_a^b K(x,t)\,f(t)dt < \infty.$

Therefore,

$$\int_a^b K(x,t)\,f(t)dt \in L^2[a,b].$$

Define, an operator $T : L^2[a,b] \to L^2[a,b]$ by $Tf = f,$ where the metric d is the standard L^2 metric. For $f_1, f_2 \in L^2[a,b],$ we have

$$d(Tf_1, Tf_2) = \left(\int_a^b |\,f_1 - f_2|^2\,dx\right)^{1/2}$$

$$= |\lambda|\left(\int_a^b \left|\int_a^b K(x,t)\,f_1(t)dt - \int_a^b K(x,t)\,f_2(t)dt\right|^2 dx\right)^{1/2}$$

$$= |\lambda|\left(\int_a^b \left|\int_a^b K(x,t)\big(\,f_1(t) - f_2(t)\big)dt\right|^2 dx\right)^{1/2}$$

$$\le |\lambda|\left(\int_a^b \left|\int_a^b |K(x,t)||\,f_1(t) - f_2(t)|dt\right|^2 dx\right)^{1/2}$$

$$\le |\lambda|\left(\int_a^b \left(\int_a^b |K(x,y)|^2 dt\right)\left(\int_a^b |\,f_1(t) - f_2(t)|^2\right)dx\right)^{1/2}$$

$$= |\lambda|\left(\int_a^b\int_a^b |K(x,y)|^2 dtdx\right)^{1/2} d(\,f_1, f_2)$$

$$i.e.\;\; d(Tf_1, Tf_2) \le |\lambda|\,\|K(x,y)\|\,d(\,f_1, f_2).$$

Since $|\lambda| < \|K(x,y)\|^{-1},$ T is a contraction mapping. By Definition 2.1, there exists a unique solution f^* such that $Tf^* = f^*.$

Example 4.1:- Consider the integral equation,

$$f(x)=\varphi(x)+\lambda\int_0^1 e^{(x-t)/2} f(t)dt\,,$$ where φ is a given function.

Since $\int_0^1\int_0^1 \left(e^{(x-t)/2}\right)^2 dxdt=\frac{(e-1)^2}{e}$. It follows that, there exists a unique solution whenever $|\lambda|<\frac{e^{1/2}}{e-1}$.

Remark 4.1:-

Applying Theorem 4.1, we examine the existence and uniqueness of solution of the Fredholm linear integral equation.

Theorem 4.2:-

[3] Consider, the following hypothesis

(a) $\left\|\int_a^b K\left(x,t,\,f(t)\right)dt\right\|\le M\,\|f\|$,

(b) $|K(x,t,w_1)-K(x,t,w_2)|\le N(x,t)|w_1-w_2|$ for all $x,t,w_1,w_2\in[a,b]$,

(c) $\int_a^b\int_a^b |N(x,t)|^2\,dxdt=\mu^2<\infty$.

Then the non-linear Fredholm integral equation

$$f(x)=\varphi(x)+\lambda\int_a^b K\left(x,t,\,f(t)\right)dt \tag{4.2}$$

has a unique solution in $L^2[a,b]$ provided $|\lambda|\mu<1$, $K\left(x,t,\,f(t)\right)$ is continuous and $\varphi\in L^2[a,b]$.

Proof: Define an operator $T:L^2[a,b]\to L^2[a,b]$ by

$$(Tf)(x)=\varphi(x)+\lambda\int_a^b K(x,t,\,f(t))dt\,.$$

Then,

$$d(Tf_1,\,Tf_2)=\left(\int_a^b |f_1-f_2|^2\,dx\right)^{1/2}$$

$$=|\lambda|\left\|\int_a^b\left(K\left(x,t,\,f_1(t)\right)-K\left(x,t,\,f_2(t)\right)\right)dt\right\|$$

$$\le|\lambda|\left(\int_a^b\left(\int_a^b\left|K\left(x,t,\,f_1(t)\right)-K\left(x,t,\,f_2(t)\right)\right|dt\right)^2 dx\right)^{1/2}$$

$$\le|\lambda|\left(\int_a^b\left(\int_a^b N(x,t)\,|f_1(t)-f_2(t)|\,dt\right)^2 dx\right)^{1/2}$$

$$=|\lambda|\left(\int_a^b\int_a^b |N(x,t)|^2\,dxdt\right)^{1/2}\left(\int_a^b |f_1(t)-f_2(t)|^2\,dt\right)^{1/2}$$

$$i.e.\ \ d(Tf_1, Tf_2) \le |\lambda| \mu d(f_1, f_2).$$

Clearly, if $|\lambda| \mu < 1, \ T$ is a contraction mapping, so that it has a unique fixed point and that fixed point is the solution of the non-linear Fredholm integral equation (4.2).

Conclusion:-

Banach fixed point theorem has many applications in various branches of science. Here, we have studied some existence and uniqueness results to the solutions in solving system of linear equations and integral equations as applications of Banach fixed point theorem.

Conflicts of Interest:-

The authors declare that there is no conflicts of interest regarding the publication of this paper.

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