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SOME COMMON FIXED POINT THEOREMS IN B-METRIC SPACES

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Abstract:

In this paper we prove some common fixed point theorems for four mappings using some control functions in b-metric spaces.

1. Introduction and Preliminaries:

After Banach contraction theorem [4] many generalizations of this principle were introduced in metric spaces [10]. Also, in recent years some important generalizations of usual metric spaces have been defined. *b*-metric space [3, 5] is one of them. Also several interesting results about the existence and uniqueness of fixed point were proved in *b*-metric spaces [2, 3, 5, 6, 8, 9]. To complete the paper we collect the following definitions: The following definition is introduced in Bakhtin [3] and Czerwik [5].

Definition 1 [3, 5]:

Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d: X \times X \to [0, \infty)$ is a *b*-metric on X if, for all $x, y, z \in X$, the following conditions hold:

- ✓ $d(x, y) = 0$ if and only if $x = y$,
- ✓ $d(x, y) = d(y, x)$,
- ✓ $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a *b*-metric space (metric type space). It should be noted that, the class of *b*-metric spaces is effectively larger than that of metric spaces, every metric is a *b*-metric with $s = 1$, while the converse is not true.

We Collect the Following Example from [10]:

Let (X, d) be a metric space and $\rho(x, y) = (d(x,y))^p$, where $p > 1$ is a real number. Then ρ is a *b*-metric with $s = 2^{p-1}$.

However, if (X, d) is a metric space, then (X, ρ) is not necessarily a metric space.

For example, if $X = \mathbb{R}$ is the set of real numbers and $d(x, y) = |x - y|$ is usual Euclidean metric, then $\rho(x, y) = (x-y)^2$ is a *b*-metric on $\mathbb{R}$ with $s = 2$. But is not a metric on $\mathbb{R}$.

Definition 2 [6]:

Let $\{x_n\}$ be a sequence in a *b*-metric space (X, d).

- ✓ $\{x_n\}$ is called *b*-convergent if and only if there is $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$.
- ✓ $\{x_n\}$ is a *b*-Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

A *b*-metric space is said to be complete if and only if each *b*-Cauchy sequence in this space is *b*-convergent.

It is to be noted that in a b-metric space every convergent sequence has a unique limit and also every convergent sequence is Cauchy but it is remarkable to note that b-metric is not continuous while metric space is continuous [6].

Definition 3 [cf.10]:

Let (X, d) be a *b*-metric space. A subset $Y \subset X$ is called closed if and only if for each sequence $\{x_n\}$ in Y which *b*-converges to an element x, we have $x \in Y$.

Aamri and Moutawakil [1] introduced (*E.A*)-property. In this paper, we prove a common fixed point theorem for two pairs of mappings which satisfy the *b*-(*E.A*) property in *b*-metric spaces [8].

Definition 4 [7]:

Let (X, d) be a *b*-metric space and f and g be self mappings on X.

- ✓ f and g are said to compatible if whenever a sequence $\{x_n\}$ in X is such that $\{fx_n\}$ and $\{gx_n\}$ are *b*-convergent to some $t \in X$, then $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$.
- ✓ f and g are said to non compatible if there exists at least one sequence $\{x_n\}$ in X is such that $\{fx_n\}$ and $\{gx_n\}$ are *b*-convergent to some $t \in X$, but $\lim_{n\to\infty} d(fgx_n, gfx_n)$ is either nonzero or does not exist.
- ✓ f and g are said to satisfy the *b*-(*E.A*) property[8] if there exists a sequence $\{x_n\}$ such that
$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t, \text{ for some } t \in X.$$

Remark:

Noncompatibility implies *b*-(*E.A*)-property.

Ozturk Turkoglu [8] have shown with the example that there are mappings which satisfy (E-A) condition but they are non- compatible.

We collect the following definition from the Jungck [7]:

Definition 5[7]:

f and *g* be given self-mappings on a set *X*. The pair (f, g) is said to be weakly compatible if *f* and *g* commute at their coincidence points (i.e. $fgx = gfx$ whenever $fx = gx$).

2. Main Results:

Theorem 1:

Let(X, d) be a b-metric space with $s > 1$ and $f, g, S, T: X \to X$ be a mappings with $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$ such that

$$s^{\varepsilon} d(fx, gy) \leq \emptyset(d(Sx,Ty), d(fx,Sx), d(gy,Ty), d(fx,Ty), \frac{d(Sx,gy)}{s}) \text{ for all x,y in X}$$

$\varepsilon > 1$ is a constant and where $\emptyset : R_+^5 \to R_+$, R_+ is the set of all non negative real numbers and $\emptyset(\xi,\xi,\xi,\xi,\xi) \leq \xi$ for $\xi > 0$, $\emptyset$ is non decreasing in each coordinate and upper semi continuous.. Suppose that one of the pairs (f, S) and (g, T) satisfy the b-(E.A)-property and that one of the subspaces f(X), g(X), S(X) and T(X) is b-closed in X. Then the pairs (f, S) and (g, T) have a point of coincidence in X. Moreover, if the pairs (f, S) and (g, T) are weakly compatible, then f, g, S and T have a unique common fixed point

Proof:

We suppose that the pair (f, S) satisfies the *b*-(*E.A*)-property, there exists a sequence $\{x_n\}$ in *X* satisfying

$$\lim_n fx_n = \lim_n Sx_n = q$$

for some $q \in X$. As $f(X) \subseteq T(X)$ there exists a sequence $\{y_n\}$ in *X* such that $fx_n = Ty_n$. Hence $\lim_{n\to\infty} Ty_n = q$. We wish to show that $\lim_{n\to\infty} gy_n = q$.

If $T(X)$ is closed subspace of *X*, then there exists a $r \in X$, such that $Tr = q$. We shall show that $gr = q$. Indeed, we have

By (1),

$$s^{\varepsilon} d(fx_n, gy_n) \leq \emptyset(d(Sx_n,Ty_n), d(fx_n,Sx_n), d(gy_n,Ty_n), d(fx_n,Ty_n), \frac{d(Sxn,gyn)}{s})$$

$$= \emptyset(d(Sx_n,fx_n), d(fx_n,Sx_n), d(gy_n, fx_n), d(fx_n, fx_n), \frac{d(Sxn,gyn)}{s})\},\leq \emptyset(d(Sx_n,fx_n), d(fx_n,Sx_n), d(gy_n, fx_n), d(fx_n, fx_n), s[\frac{d(Sxn,fxn)}{s} + \frac{d(fxn,gyn)}{s}]) \quad (2)$$

In (2), on taking limit superior, we obtain

$\lim_n sup\, s^{\varepsilon} d(fx_n, gy_n) \leq \emptyset(0,0, \lim_n sup\, d(gy_n, fx_n),0, \lim_n sup\, d(gy_n, fx_n))$

because $d(Sx_n, fx_n) \to 0$ as $n \to \infty$. Since $s^{\varepsilon} > s > 1$, we have

$\lim_n sup\, d(gy_n, fx_n))=0$,

that is, $\lim_{n\to\infty} d(gy_n, fx_n))=0$. Further, we have

$$d(q, gy_n) \leq s [d(q, fx_n) + d(gy_n, fx_n)] \to 0 \text{ as } n \to \infty. \textit{Hence, } gy_n \to q.$$

$$d(q, gr) \leq s [d(q, fx_n) + d(gr, fx_n)] \ldots\ldots.. \quad (3)$$

Now,

$$s^{\varepsilon} d(fx_n, gr) \leq \emptyset(d(Sx_n,Tr), d(fx_n,Sx_n), d(gr,Tr), d(fx_n,Tr), \frac{d(Sxn,gr)}{s})$$

$$= \emptyset(d(Sx_n,q), d(fx_n,Sx_n), d(q,qr), d(fx_n,q), \frac{d(Sxn,q)+d(q,gr)}{s}) \quad (4)$$

Using (3) and (4) and letting $n \to \infty$,

$$s^{\varepsilon} d(q, gr) \leq \emptyset(0,0, d(q, gr), 0, d(q, gr)) \quad (5)$$

From (3) and (5), we have

$$d(q, gr) \leq s[0+\frac{\emptyset(0,0,d(q,gr),0,d(q,gr))}{s^{\varepsilon}}]$$

i.e., d(q, gr) =0. Hence, q= gr.

Hence, q=gr=Tr.

As $g(X) \subseteq S(X)$, there exists a point $z \in X$ such that $q = Sz$. We claim that $Sz = fz$.

By (1), we have

$$s^{\varepsilon} d(fz, gr) \leq \emptyset(d(Sz,Tr), d(fz,Sz), d(gr,Tr), d(fz,Tr), \frac{d(Sz,gr)}{s})$$

$$= \emptyset(0, d(fz,q), 0, d(fz,q), 0)$$

Implies d(fz, q) =0 i.e.,fz = q.

Thus fz = sz =q. Therefore, fz = Sz = Tr =gr =q

By the weak compatibility of the pairs (f, S) and (g, T), we obtain that $fq = Sq$ and $gq = Tq$.

We will show that *q* is a common fixed point of *f*, *g*, *S* and *T*. Using (1), we get

$$s^{\varepsilon} d(fq, q) = s^{\varepsilon} d(fq, gr) \leq \emptyset(d(Sq,Tr), d(fq,Sq), d(gr,Tr), d(fq,Tr), \frac{d(Sq,gr)}{s})$$

$$\leq \emptyset(d(fq,q), 0, 0, d(fq,q), \frac{s[d(Sq,fq)+d(fq,qr)]}{s})$$

$$= \emptyset(d(fq,q),0, 0, d(fq,q), 0) \ i.e.,$$

$$d(fq,q) \leq 0 \text{ implies } fq = q.$$

Hence, fq = Sq =q. Similarly it can be shown that gq = Tq =q.

We now prove the unicity of the common fixed point. If possible let p be another common fixed point of f,S.g.T. Using (1), we get,

$$s^{\varepsilon} d(fp, gq) \leq \emptyset(d(Sp,Tq), d(fp,Sp), d(gq,Tq), d(fp,Tq), \frac{d(Sp,gq)}{s})$$

$$\emptyset(d(p,q), 0 , 0, d(p,q), d(p,q))$$

Implies $s^{\varepsilon} d(fp, gq) \leq d(p,q)$ i.e., $d(p,q) \leq 0$ i.e., p=q.

This completes the proof of the theorem.

Corollary 1:

Let(X, d) be a b-metric space and f, T:X → X be mappings such that

$$s^{\varepsilon} d(fx, gy) \leq \emptyset(d(Tx,Ty), d(fx,Tx), d(fy,Ty), d(fx,Ty), \frac{d(Tx,fy)}{s}) \text{ for all x, y in X}$$

where $\varepsilon > 1$is a constant. Suppose that the pair (f, T) satisfies the b-(E.A)-property and T(X) is closed in X. Then the pair (f, T) has a unique point of coincidence in X. Moreover, if the pair (f, T) is weakly compatible, then f and T have a unique common fixed point.

Corollary 2:

Let (X, d) be a b-metric space and f, T: X → X be mappings such that

$$s^2 d(fx, fy)) \leq M_s(x, y) \text{ for all } x, y \in X,$$

where $\varepsilon > 1$is a constant and $s^2 d(fx, fy) \leq \emptyset(d(Tx,Ty), d(fx,Tx), d(fy,Ty), d(fx,Ty), \frac{d(Tx,fy)}{s})$ for all x,y in X

Suppose that the pair (f, T) satisfies the b-(E.A)-property and T(X) is closed in X. Then the pair (f, T) has a unique point of coincidence in X. Moreover, if the pair (f, T) is weakly compatible, then f and T have a unique common fixed point.

3. Conclusion:

In this paper, we have proved fixed point theorems for mappings satisfying *b*-(*E.A*)-property in *b*-metric spaces. Our results extended *b*-(*E.A*)-property results in the literature.

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