XronoMorph: Algorithmic Generation of Perfectly Balanced and Well-Formed Rhythms

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ABSTRACT

We present an application—XronoMorph—for the algorithmic generation of rhythms in the context of creative composition and performance, and of musical analysis and education. XronoMorph makes use of visual and geometrical conceptualizations of rhythms, and allows the user to smoothly morph between rhythms. Sonification of the user generated geometrical constructs is possible using a built-in sampler, VST and AU plugins, or standalone synthesizers via MIDI. The algorithms are based on two underlying mathematical principles: *perfect balance* and *well-formedness*, both of which can be derived from coefficients of the discrete Fourier transform of the rhythm. The mathematical background, musical implications, and their implementation in the software are discussed.

Author Keywords

music, rhythm, scales, balance, evenness, perfect balance, well-formedness, discrete Fourier transform

ACM Classification

H.5.5 [Information Interfaces and Presentation] Sound and Music Computing --- Systems

1. INTRODUCTION

The composition and performance of interesting rhythms is often hindered by the limitations of traditional music notation and conceptualization. Simultaneously, many new music students struggle in generating an intuitive understanding of poly-rhythmic structures. This paper presents XronoMorph, an application designed for the visual and geometrical exploration and construction of interesting rhythms based on two underlying mathematical principles: *perfect balance* (*PB*) and *well-formedness* (*WF*).

As shown in Figures 2–4, the temporal structure of periodic rhythms, meters, riffs and ostinatos can be conveniently represented as points on a circle: the clockwise angle of each point indicates when it is sounded and the circularity represents the rhythm's periodicity.

1.1 An Introduction to XronoMorph

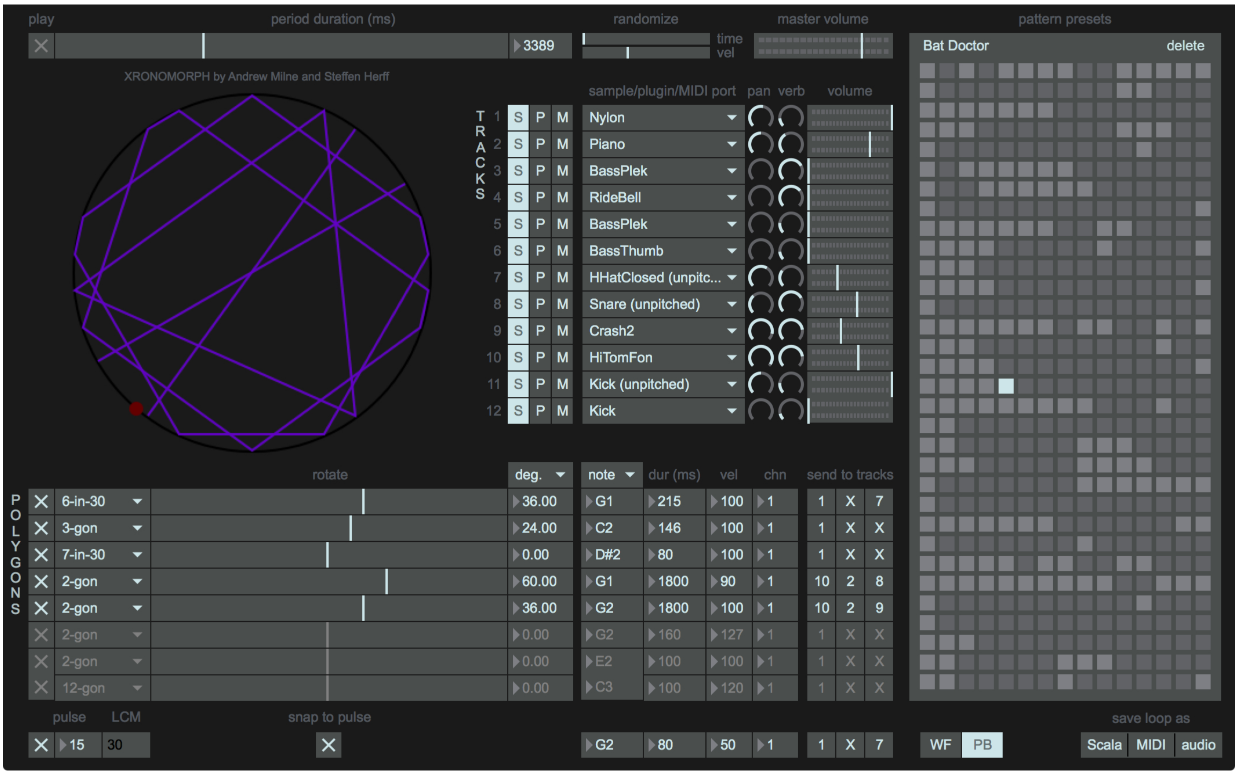
XronoMorph has a PB mode and a WF mode for the respective classes of rhythms (PB mode in Fig. 1a, WF mode in Fig. 1b). The rhythmic patterns are visualized by polygons inscribed in a circle. A small disk rotates clockwise around the circle and when it hits a polygon vertex a MIDI event is triggered. The speed at which the disk rotates (the length of the period) is controlled by the long horizontal slider at the top. Each rhythm is visualized with its own configuration of underlying polygons. Each such polygon can be assigned a MIDI pitch, velocity, duration, and channel, and directed to up to three, out of a total of twelve, *tracks*. Each of these twelve tracks can be thought of as an "instrumentalist" who plays any polygon being sent to it. Each track produces sound from a built-in sampler, from a plugin AU or VST synthesizer, or directs the MIDI to a port to drive a standalone software or hardware synthesizer. In this way an "ensemble" of twelve "instrumentalists" can be formed, and each polygon can be played by up to three of these "instrumentalists". This means that the orchestration/sonification of a given rhythm can be easily changed during performance and composition.

A large number of user presets can be stored allowing rhythms to be easily switched between during live performance. Rhythms can also be saved as MIDI or audio loops for later processing in a sequencer or digital audio workstation; alternatively, they can be saved as Scala scale files, allowing XronoMorph to be used for designing perfectly balanced and well-formed microtonal scales.

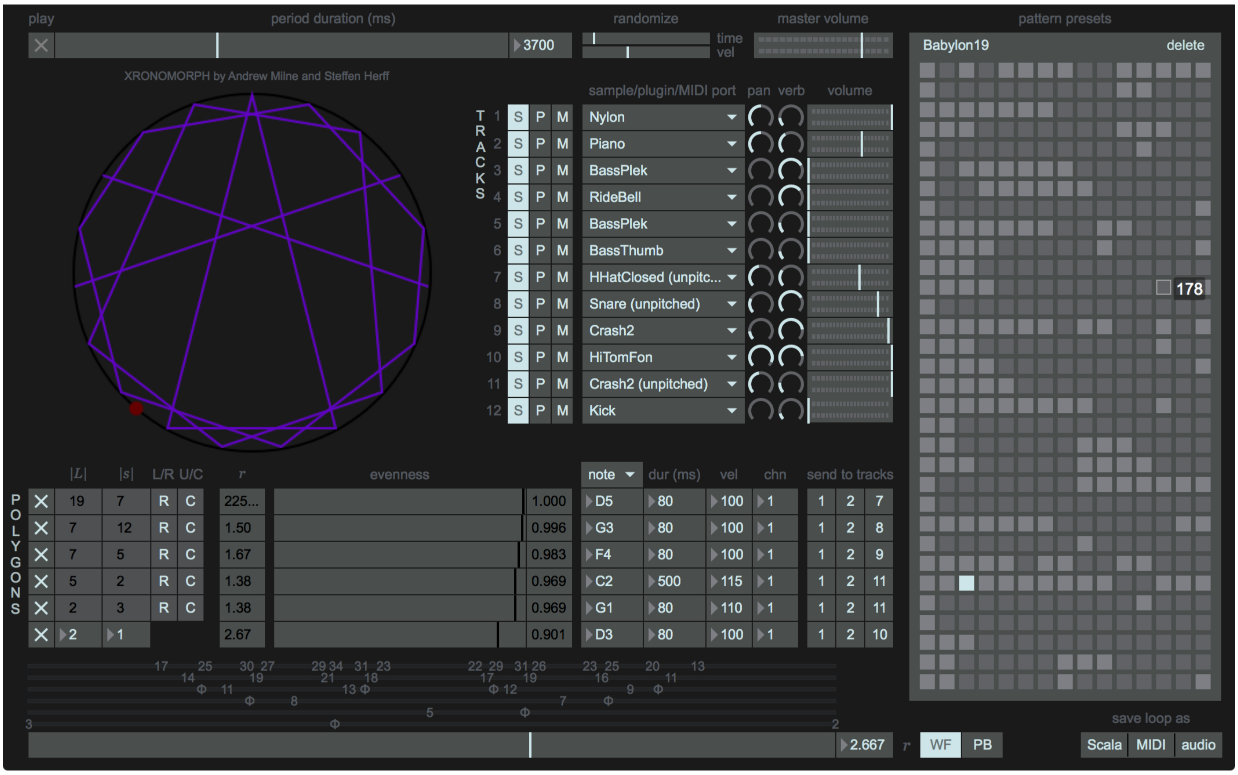
XronoMorph realizes three principal novelties. Firstly, perfect balance is a recently developed concept [12], which has not previously been instantiated in a musical application. Secondly, although well-formedness (equivalently moments of symmetry) is a well-established theoretical concept applied to scales and rhythms [20, 5], it has not been realized in a rhythm application (existing pitch-based applications include Hex [15] and associated synthesizers [14, 18]); furthermore, our parameterization of WF and the sounding of a full hierarchy of WF rhythms is novel [13]. Thirdly, because the PB and WF rhythms have continuously variable parameters, differing rhythms can be smoothly morphed between. We are aware of only one other rhythm app—Rhythmorpher—designed for rhythmic morphing, and this uses a very different set of rhythmic parameters [23].

The mathematical principles of evenness and balance,

NIME'16, July 11-15, 2016, Griffith University, Brisbane, Australia.



(a) XronoMorph in PB mode: A sum of perfectly balanced rhythms is depicted by polygons inscribed in a circle. The controls below the circle allow the type of polygon to be chosen and its rotation to be smoothly adjusted.



(b) XronoMorph in WF mode: A hierarchy of well-formed rhythms is depicted by polygons inscribed in a circle. The large horizontal slider at the bottom (the r-slider) is used to smoothly control beat size ratio.

Figure 1: The user interface of XronoMorph in PB mode (a) and WF mode (b). The rhythm is represented by polygons inscribed in a circle. A "playhead", depicted as a small disk, rotates around the circle and, whenever it "hits" a polygon vertex, a MIDI note is sent out with a pitch, duration, and channel specific to that polygon. The controls below-right the circle allow each polygon's MIDI parameters to be specified. At the top is a slider to control the length of the period of rhythmic repetition (the tempo). To the right of the polygons are tracks, which can play built-in samples, host plugins to sonify the polygons, or send MIDI to standalone synthesizers. To the right is a large bank of slots where users can store their rhythms as presets.

which underlie well-formedness and perfect balance, as well as their implementation in XronoMorph, are now detailed.

1.2 Mathematical Background

A natural mathematical representation of points on a unit circle (e.g., rhythmic events) is as a vector of unit-magnitude complex numbers arranged in circular order. Building on research by Lewin [10], Quinn [16], and Amiot [2], in recent work [12, 13] we have shown how the magnitudes of the first two coefficients of the discrete Fourier transform of this vector identify two musically relevant properties of the resulting rhythm. Unity minus the magnitude of the zeroth coefficient quantifies the rhythm's *balance*, whilst the magnitude of the first coefficient quantifies the rhythm's *evenness*. Balance measures the distance of the rhythm's centre of gravity (mean position) from the centre of the circle; evenness measures the similarity of the rhythm to an isochronous rhythm with the same number of events (ignoring their relative phases).

More formally, the vector $\boldsymbol{x} \in [0,1)^K$ has K real-numbered time values (for the K rhythmic events) normalized to lie between between 0 and 1 (the period has a size of 1), and ordered by size so $x_0 < x_1 < \cdots < x_{K-1}$. For example, for the ♩♩♪♩♩♩♪ "diatonic" rhythm used in Sub-Saharan African music [17], $\boldsymbol{x} = \left(\frac{0}{12}, \frac{2}{12}, \frac{4}{12}, \frac{5}{12}, \frac{7}{12}, \frac{9}{12}, \frac{11}{12}\right)$. The elements of this vector are then mapped to the unit circle in the complex plane with $\boldsymbol{z}[k] = \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{x}[k]} \in \mathbb{C}$, so $\boldsymbol{z} \in \mathbb{C}^K$. Each complex element $\boldsymbol{z}[k]$ of $\boldsymbol{z}$ has unit magnitude, and its angle represents its time location as a proportion of the period (whose angle is 2π).

The tth coefficient of the discrete Fourier transform of the scale vector is given by

$$\mathcal{F}\boldsymbol{z}[t] = \frac{1}{K} \sum_{k=0}^{K-1} \boldsymbol{z}[k]\, \mathrm{e}^{-2\pi \mathrm{i} tk/K} . \tag{1}$$

As outlined above, the zeroth and first coefficients respectively quantify balance and evenness.

2. BALANCE AND PERFECT BALANCE: THE ZEROTH COEFFICIENT

Unity minus the magnitude of the zeroth coefficient of the DFT of $\boldsymbol{x}$ gives the *balance* of the rhythm:

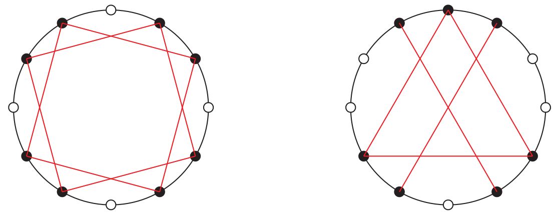
$$\mathit{balance} = 1 - |\mathcal{F}\boldsymbol{z}[0]| \in [0,1]\,, \text{ where}$$

$$\mathcal{F}\boldsymbol{z}[0] = \frac{1}{K} \sum_{k=0}^{K-1} \boldsymbol{z}[k]\,. \tag{2}$$

The maximum possible value for balance is 1, and this is termed *perfect balance* (*PB*). The balance of a rhythm can be thought of as the distance of the rhythm's centre of gravity (mean position) from the centre of the circle. If the balance is 1, the centre of gravity is precisely at the centre of the circle. This means that if each rhythmic event were a weight placed onto the rim of a vertical bicycle wheel, the wheel will have no preferred rotation.

Rhythms with equally-sized steps (isochronous rhythms, which are also perfectly even) are PB. Isochronous rhythms can be visualized by a regular K-gon placed within a circle (e.g. one of the squares in Figure 2a). However, there is also a complicated manifold of irregular (non-isochronous) perfectly balanced rhythms. Obtaining perfect balance under additional constraints provides a useful way to narrow down the manifold of possibilities to a smaller selection of musically interesting rhythms.

An important consequence of the definition of perfect balance is that the sum of any two or more PB rhythms is also



(a) PB shuffle rhythm comprising two squares (4-gons). The greatest common divisor of 4 and 4 is 4, hence the rhythm has rotational symmetry.

(b) PB rhythm comprising two digons (2-gons) and an equilateral triangle (3-gon). The greatest common divisor of 2, 2, and 3 is 1, hence this rhythm does not have rotational symmetry.

Figure 2: Two perfectly balanced rhythms in a 12-fold isochronous "grid". The first rhythm is PB over the whole period, but not over its fundamental period of repetition, which is one quarter of the circle. The second rhythm is one of just two PB rhythms in 12 that are PB over their fundamental period of repetition.

PB (as shown in Figs. 2 and 3); as we show later, this enables complex multilayered rhythms to be constructed from the summation of simpler PB rhythms.

2.1 Perfectly Balanced Sums

In order to constrain perfect balance, we start with a requirement for the rhythm to be a subset of N isochronous pulses (i.e., all its events align with an N-fold *grid*); though this constraint is later relaxed when more than one PB rhythm is combined.

For any N that is prime, there is only one perfectly balanced pattern, which is simply a regular N-gon (prime N are, therefore, not of great interest with perfect balance).

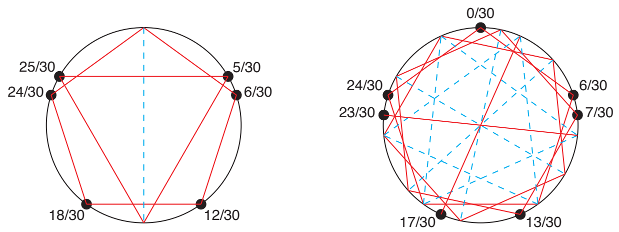
However, when N is not prime, perfectly balanced rhythms can be formed from the sum (union) of regular K-gons where $K|N$ (which means K divides N).

For example, when $N = 12$, we can combine 2-gons (digons), 3-gons (equilateral triangles), 4-gons (squares), and 6-gons (regular hexagons), each such regular polygon being placed in one of its N/K distinct rotations. Any sum of such polygons will produce a rhythm that is, in total, balanced. Of particular interest are those rhythms where the set of differently sized K are coprime (no common divisors greater than 1)—such rhythms do not have rotational symmetry and are perfectly balanced over their fundamental period of repetition. This is illustrated in Figure 2.

For any N that is a product of no more than two distinct primes, all possible perfectly balanced rhythms can be formed by summing regular K-gons where K is prime and $K|N$; thus the regular K-gons are *elemental*.

Those N that are the product of three or more distinct primes, however, are particularly interesting. Between 1 and 100, the only such values are 30, 42, 60, 66, 70, 78, 84, and 90. They contain PB patterns that cannot be created from a simple sum of regular K-gons; these PB patterns can only be created from *integer combinations* of regular K-gons (i.e., *subtracting* as well as adding polygons, thereby allowing vertices to be cancelled out [12]). (Indeed, all PB patterns in an N-fold grid can be produced from an integer combination of regular polygons [12, Thm. 5].) Figure 3 shows two such patterns in $N = 30$.

These two patterns exemplify a useful property, which is that there is no PB polygon that can be subtracted without producing a sonically "nonsensical" (hence illegal) negative



(a) triangle + pentagon − digon make a 6-element pattern in a 30-fold period.

(b) 2 digons + 3 pentagons − 3 digons − 2 triangles in a 30-fold period [3].

Figure 3: Perfectly balanced integer combinations of intersecting regular polygons in a 30-fold isochronous grid. When the vertex of one positive-weighted polygon (solid line) coincides with the vertex of one negative-weighted polygon (dashed line) they cancel out to zero.

weight. Thus these patterns are also elemental, albeit irregular. So, for any N, a set of elemental rhythms exists, such that all possible PB rhythms can be constructed by only summing elemental rhythms (subtraction no longer being necessary). If N has more than two prime factors, then some of its elemental rhythms are irregular: in $N = 30$, there are 6 irregular elemental PB patterns; in $N = 42$, there are 18 such patterns; in $N = 66$, there are more than 100. Clearly, for larger values of N with three or more prime factors, the number of such patterns explodes.

2.2 Perfect Balance in XronoMorph

In order to accommodate a musically sufficient number of possibilities, XronoMorph allows the following PB rhythms to be chosen and summed: all regular K-gons up to 12, all regular prime-K-gons up to 29, and all six irregular elemental polygons in 30. This allows a wide variety of PB rhythms to be produced (in future versions, we plan to allow a wider variety of polygons to be specified or generated by the user and stored as presets). Each such polygon can be independently rotated—either snapping to a specified N-grid, or smoothly (thereby allowing PB rhythms that are not grid-based).

The principal user-parameters for defining PB rhythms are the choice of polygons (up to 8 may be simultaneously sounded) and the independent rotation of each of these polygons. The circle in Figure 1a shows a rhythm that consists of five underlying PB geometrical shapes, each of which has been independently rotated.

3. EVENNESS AND WELL-FORMEDNESS: THE FIRST COEFFICIENT

As first shown by Amiot and Noll [2], the magnitude of the first coefficient of the DFT of $\boldsymbol{x}$ gives the *evenness* of the rhythm:

$$\textit{evenness} = |\mathcal{F}\boldsymbol{z}[1]| \in [0,1]\,, \text{ where}$$

$$\mathcal{F}\boldsymbol{z}[1] = \frac{1}{K}\sum_{k=0}^{K-1} \boldsymbol{z}[k]\,\mathrm{e}^{-2\pi \mathrm{i} k/K}\,. \tag{3}$$

The maximum possible value for evenness is 1, and this is termed *perfect evenness*. The evenness of a K-event rhythm can be thought of as a quantification of its similarity to a K-equal division of the period that has been rotated so as to maximize this similarity. Following from this, the only rhythms that are perfectly even are those with equally-sized steps (isochronous rhythms, or regular K-gons). This is quite different to perfect balance, where there is a continuum of possibilities. However, as we show later, when evenness is maximized under constraints that imply perfect evenness is unobtainable, musically interesting results occur—notably, when we constrain the rhythm to contain no more than two interonset intervals (*IOI*s), the resulting rhythms are *well-formed* [5].

3.1 Well-Formed Hierarchies

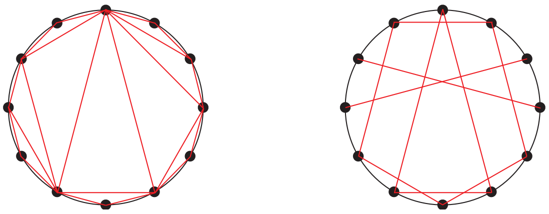
There are two commonly discussed types of periodic rhythm (or, analogously, scale) that result from maximizing evenness under musically sensible constraints. The first are *Euclidean* rhythms [19] (which can be generated by existing apps such as SequenceApp, Rhythm Necklace, Euclidean sequencer, Gibber, and many others), which result when evenness is maximized under the constraint of K events in an N-fold isochronous grid (interesting when N and K are coprime). The second are *well-formed* (*WF*) rhythms or scales [5], (also known as *moments of symmetry* [20]), which result when evenness is maximized under the constraint of no more than two sizes of IOIs. This means a WF rhythm can be described by a *word* such as ℓs, $s\ell ss\ell$, or $\ell\ell\ell s$, etc., where ℓ denotes a large interonset interval, s denotes a small interonset interval; for any given number of large and small IOIs, their maximally even arrangement always forms a well-formed word [6]).

XronoMorph uses well-formed rhythms for the following three reasons: a) they are a superset of Euclidean; b) they can produce rhythms that do not fit into an isochronous grid (although grid-based rhythms are of obvious utility in music, subtle deviations from the grid are vitally important, as is the interesting possibility of deeply non-isochronous rhythms—discussed below—that are maximally distant from any possible grid); c) they invite a principled approach for producing a hierarchy of interlocking WF rhythms.

Let the length of ℓ divided by the length of s be denoted r (a real number between 1 and infinity). Any given WF pattern is a subset of a higher-level WF pattern that is derived by the use of two different *morphisms* [21, 4] (or, equivalently in this context, *parallel rewrites* [11]): when $r < 2$, the morphism is $\ell \mapsto \ell s$ and $s \mapsto \ell$; when $r \geq 2$, the morphism is $\ell \mapsto \ell s$ and $s \mapsto s$ [13]. A consequence of this is that when the r-value for the lowest level is a rational number, a higher level (and all levels higher than that level) will be isochronous (when r is irrational, no higher level is ever precisely isochronic).

This can be perhaps most simply explained by reference to musical scales in 12-tone equal temperament. The WF tetractys D, G, A ($\ell s\ell$) is a subset of the WF pentatonic scale D, F, G, A, C ($\ell ss\ell s$), which is a subset of the WF diatonic scale D, E, F, G, A, B, C ($\ell s\ell\ell\ell s\ell$), which is a subset of the evenly tempered ("isochronous") chromatic scale D, D♯, E, F, F♯, G, G♯, A, A♯, B, C, C♯ (where $\ell = s$). If these patterns are interpreted rhythmically, and each such level is considered as a separate rhythmic stream (perhaps each played with a distinct timbre), then combining them makes a complex hierarchy of rhythms. This rhythmic hierarchy is illustrated in Fig. 4a.

This method of generating successive levels results in every rhythmic event being duplicated in all higher levels. For example, all three beats in the lowest level are additionally played by the remaining three higher levels. Naturally, this gives a strong accent to low-level beats, and amplifies the inherently hierarchical nature of WF rhythmic structures. However, XronoMorph allows an interesting alternative strategy, which is to treat each successive level as the complement of all lower levels, so it plays only when no lower



(a) A WF hierarchy with universal levels.

(b) A WF hierarchy with complementary levels.

Figure 4: Four levels in the tetractys-pentatonic-diatonic-chromatic well-formed hierarchy, shown as universal and complementary forms (as defined in the main text).

level is also playing (this is done by toggling the "U/C"—universe/complement—buttons for each level in the eponymous column). For example, consider a lower level which, if expressed as a scale rather than as a rhythm, corresponds to the white-note diatonic scale, while the next higher level corresponds to a twelve-pitch chromatic scale. When "C" is selected for the chromatic level, instead of playing all twelve events in the latter rhythm, only those events not occurring in the lower-level pattern are played. Using the scalic analogy, this means using only the black-note pentatonic scale, which is the complement of the white-note diatonic in a chromatic universe.

Interestingly, these *complementary* well-formed rhythms are themselves well-formed [1, Prop. 3.2], but they are displaced with respect to each other, so they never coincide. This non-redundant rhythmic structure is somewhat reminiscent of the multiple interlocking parts used by Latin percussion or gamelan percussion ensembles—although each individual part is relatively simple, in combination, they produce a complex and interwoven totality. This complementary hierarchy is illustrated in Figure 4b.

The number of levels that need to be ascended before isochrony is reached is a function of the ratio of the large and small IOIs of the lowest level; indeed, as mentioned above, if this ratio is irrational, isochrony will never be reached (though it may be closely approximated). Interestingly, there are a number of ratios based on the golden section that ensure isochrony is never closely approximated [22, 13].

3.2 Well-Formedness in XronoMorph

The principal user-parameters for defining WF rhythms are: a) the number of large IOIs of the lowest-level rhythm, b) the number of small IOIs of the lowest level rhythm, c) the ratio of the large IOI and the small IOI. This ratio is controlled by the large horizontal r-slider at the bottom of the interface, and it traverses the range 1 to ∞ (using the mapping $ratio = 1/(1-t)$, where $t \in [0, 1)$ is the left-right position of the slider). With these values chosen, a hierarchy of six WF rhythms is constructed, each of which can be switched on or off, and between complementary and universal mode (as described earlier).

As the r-slider is moved, the visualization and sonification of the rhythmic hierarchy continuously updates. Above the r-slider are six levels of numbers. When the r-slider lines up with one of these numbers on a given level (or when one of these numbers is clicked on), the corresponding level (and all higher levels) of the resulting rhythm is isochronous with that number of pulses. As the r-slider is smoothly moved away from these numbers, each previously isochronous level smoothly shifts to having two differing IOIs. Certain r-slider locations are indicated with a ϕ symbol. These ratios are related to the golden section and, at such positions, no rhythmic level approaches isochrony—they are maximally distant from any isochronous grid (of whatever granularity). For this reason, we term these rhythms *deeply non-isochronous*. Perhaps counter-intuitively, we have found these rhythms to be rather groovy. Figure 1b shows a WF hierarchy with six levels; this is related to the previously mentioned tetractys-pentatonic-diatonic-chromatic hierarchy, but the r-slider has a value of 5/3, which results in the highest level being a 19-fold (rather than 12-fold) isochronous pulse. All levels are in complementary mode.

4. CASE STUDIES

4.1 Education

Part of a course in audio engineering at the University of Wisconsin (ECE401) studies the perception of sound. This is focused at two levels: on timbre (where both temporal and spectral influences are important) and on rhythm. One module considers a taxonomy of rhythm: from isochronous pulses to polyrhythms, and then "upwards" through the metrical hierarchy. XronoMorph provides an excellent experimental platform for the demonstration and investigation of the various ways of characterizing rhythmic patterns. For example, one homework set considers rhythms that are (1) isochronous, (2) polyrhythmic, (3) well-formed, (4) perfectly balanced, (5) Euclidean, and (6) rotationally symmetric. Students are asked to create examples of each of these, and then to provide examples of (n) that are not (m), for instance, well-formed rhythms that are not Euclidean, or PB rhythms that are not rotationally symmetric. The software makes the task feasible, and allows instant feedback on the perceptible meaning of the various definitions.

The final project in this class is a relatively free assignment where students choose their own subject (within the audio realm) and prepare a term paper. In the fall semester 2015, one student (Matthew Cortner) conducted a pilot study with the goal of determining if the properties of "balance" and "evenness" were perceptually salient. Is it possible to tell, just by listening, if a rhythm is perfectly balanced or even (or neither)? Because of the difficulty of describing to naive listeners what is meant by terms such as balance and evenness, an experiment was designed to test whether listeners could better distinguish perturbations of isochronous rhythms and non-isochronous PB rhythms than they could perturbations of unbalanced and uneven rhythms. Positive results would show that the specified property (balance or evenenness) is perceptually salient, at least in the sense that it matters in discrimination experiments. The results of the pilot study [7] are encouraging, though the small sample size precludes any statistically significant results.

4.2 Composition and Performance

XronoMorph facilitates the production of a wide variety of complex rhythms, many of which would be hard to compose or perform manually. The ability to smoothly transition between rhythms as well as abruptly switching between complex but related rhythms also opens up novel compositional and performative possibilities.

The sonification can be done with unpitched sounds, in which case purely rhythmic patterns can be created. Some of the WF rhythms are reflective of rhythms found in non-Western music; for example, *aksak* additive rhythms like $2+3+3+2+3$ are often well-formed [9], as are Sub-Saharan rhythms such as the previously mentioned "diatonic" rhythm [17]. The PB rhythms include polyrhythms (e.g., 3 against 2, or 3 against 4) that are also common in

Sub-Saharan music; they also include polyrhythms where the individual streams are phase-shifted so they never coincide (e.g., Fig. 3b, where the 3-fold rhythm and two 2-fold rhythms are respectively displaced). Beyond these displaced polyrhythms, we also find the fascinating rhythmic structures formed by the irregular elemental PB patterns, which sonify combinations of positively- and negatively-weighted (cancelling) polyrhythms.

When the sonification is made with pitched sounds, we may find that melodies (hockets) emerge from perceptual streaming of proximal pitches between levels. When the user changes the rotation or pitch of each polygon, the emergent melody also changes. An interesting feature of such melodies is that they typically arise without compositional forethought, but since they arise from such a highly organized structure, they frequently exhibit æsthetic promise.

Another possibility is to use the raw MIDI output to seed other algorithmic generation systems. For example, austraLYSIS (`http://www.australysis.com`) have performed using WF rhythms to drive Serial Collaborator [8] to produce rhythmically informed serial transformations of previously written tone rows.

5. CONCLUSION

We have introduced XronoMorph, an application for the algorithmic generation of perfectly balanced and well-formed rhythms. The software makes use of visualizations and sonifications of a geometrical conceptualization of rhythms to allow a novel approach for their construction. It is built around two underlying mathematical principles whose mathematical background and implementation in XronoMorph have been detailed.

The multilevel rhythmic structures generated by XronoMorph have levels that are PB, or WF, both individually and in combination. This leads to interwoven structures evoking a sense of deep organization and self similarity that is reminiscent of fractals.

Using the algorithmic approach described here, intelligent compositional input is still required—not all well-formed and perfectly balanced rhythms, or transitions between them, will sound appropriate. Furthermore, effective choices for pitches and durations are still required. But we have found this tool to be both inspiring and surprising in its musical output. We hope that the visual and geometrical conceptualization of rhythms demonstrated in XronoMorph will help in the composition and performance of new and interesting rhythms, and facilitate an intuitive understanding of complex rhythms found in real-world music.

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