Mathematics for Celestial Navigation

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Abstract

The equations of spherical trigonometry are derived via three dimensional rotation matrices. These include the spherical law of sines, the spherical law of cosines and the second spherical law of cosines. Versions of these with appropriate symbols and aliases are also provided for those typically used in the practice of celestial navigation. In these derivations, surface angles, e.g., azimuth and longitude difference, are unrestricted, and not limited to 180 degrees.

Additional rotation matrices and derivations are considered which yield further equations of spherical trigonometry. Also addressed are derivations of "Ogura's Method " and "Ageton's Method", which methods are used to create short-method tables for celestial navigation.

It is this author's opinion that in any book or paper concerned with three-dimensional geometry, *visualization* is paramount; consequently, an abundance of figures, carefully drawn, is provided for the reader to better visualize the positions, orientations and angles of the various lines related to the three-dimensional object. 44 pages, 4MB. **RicLAO.**

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1 Celestial Navigation

Consider a model of the earth with a Cartesian coordinate system and an embedded spherical coordinate system. The origin of coordinates is at the center of the earth and the x-axis points through the meridian of Greenwich (England). This spherical coordinate system is referred to as the *celestial equator system of coordinates*, also know as the *equinoctial system*. Initially, all angles are measured in standard mathematical format; for example, the (longitude) angles have positive values measured toward the east from the x-axis.

Initially in this paper we will measure all angles in this standard mathematical format, that is, in the sense that a right-hand screw would turn were it to advance from the south pole to the north pole. In the *Nautical Almanac* most angles are tabulated westerly, in keeping with the direction of apparent travel of the sun over a point on the surface of the earth.

We will be using two sets of symbols for angles (aliases of one to the other), one of which is frequently used in the practice of celestial navigation. In this paper, for economy of space in figures, the symbol P will often be employed instead of GP to represent the geographical position of an observed celestial body. Likewise, the symbol M will be used to represent the position of the observer, e.g., the assumed position AP or dead reckon position DR. The reader should note that in works of other authors the symbol M oftentimes represents the GP instead of the position of the observer.

Spherical Coordinate Angle Symbols and Aliases

$\gamma =$ $\phi_B =$ co-altitude of the celestial body P.
$\Delta =$ $\phi_A =$ co-declination of the celestial body P.
$\Lambda =$ $\theta_A =$ east longitude of the celestial body P.
$A =$ $\theta_B =$ co-azimuth of the celestial body P relative to observer M.
$\phi =$ (no alias) co-latitude of the observer M.
$\lambda =$ $\theta =$ east longitude of the observer M.
$l =$ $\Delta\theta$ (or $\Delta\lambda$) $=$ $\Lambda - \lambda$, difference in east longitudes.

The celestial equator system of coordinates.

Angles $\theta_A = \Lambda$, $\theta = \lambda$ and $\Delta\theta$ are measured in customary *mathematical format*; that is, they are positive when measured in an *easterly* direction and expressed as east longitudes. For westerly longitudes, λ is negative.

The symbol λ will also be used to generically represent an easterly longitude of any other point specified on the celestial sphere in the text and understood in the context of that text.

Longitude can also be measured in customary *navigational format*; that is, positive when measured in a *westerly* direction.

I call the rotational motion of the earth *boreal motion*, which usage I adopted from Skilling[1]. As Skilling notes, the word *boreal* is derived from the rotation of the earth and signifies a northerly direction compared to the rotation of the earth. An analogous "right-hand screw", corresponding to this motion, would be driven along the polar axis from the South pole up through the North pole.

[1] Skilling, Hugh Hildreth, Fundamentals of Electric Waves, 2^{nd} edition, 1948, page 87, reprinted by Robert E. Krieger Publishing Company, Inc., 1974. ISBN 0-88275-180-8

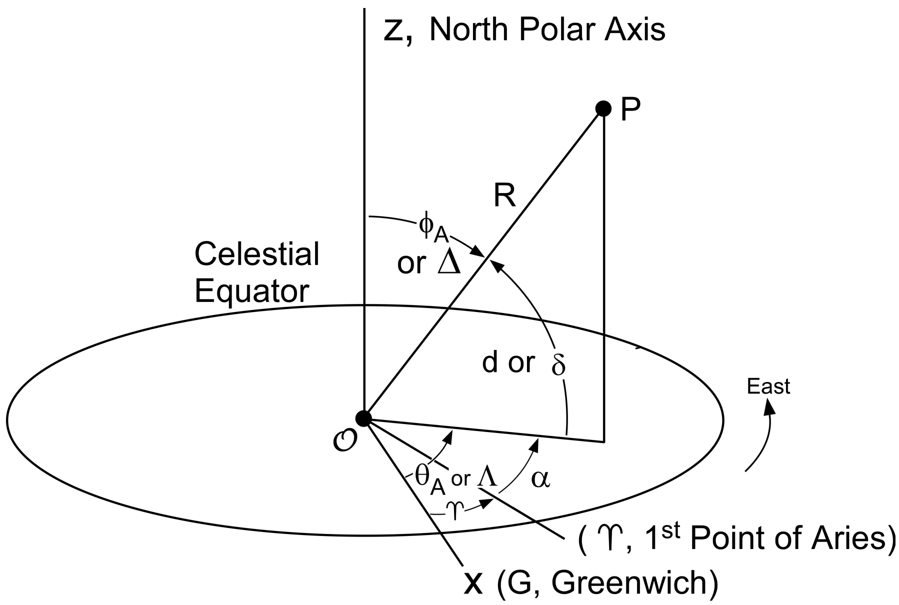


Figure 1: the Celestial Equator System of Coordinates.

where $\alpha =$ Right Ascension (RA) of P
$\phi_A = \Delta =$ Co-declination of P
$\theta_A = \Lambda = \Upsilon + \alpha$

and the symbol Υ (the "ram's horns") is known as *the first point of Aries*.

Remember that the "right hand screw rotates to the EAST". Υ is the longitude of this point, the first point of Aries or the vernal (spring) equinox, measured easterly in the equatorial plane from the Greenwich meridian.

Now consider the horizon system of coordinates.

$R_\star$ is the distance from the center of the earth to the observed celestial body P. Suppose that R_{obs} is the distance from the observer M on the surface of the earth to the same celestial body. Since these distances are extremely large compared to the radius of the earth, for computation purposes we may regard these two distances as equal to one another. Furthermore, we may regard spherical coordinate angles of the celestial body as equal to one another whether measured from the center of the earth or from the observer's position on the surface. Let R be the radius of the earth, considered constant in this paper. The *geographical position* GP of a celestial body is the the point on the earth's surface directly below the celestial body, that is, the point of intersection with the earth's surface of a line through the body and the center of the earth.

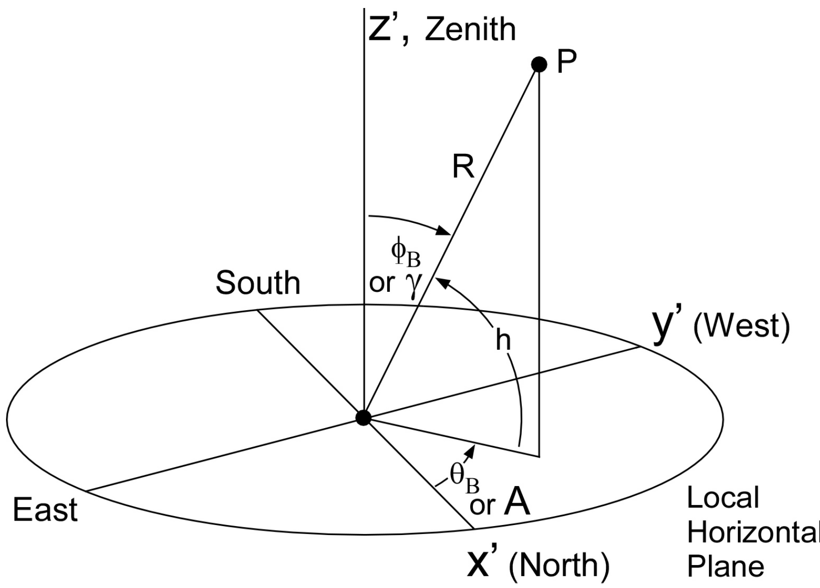


Figure 2: The Horizon System of Coordinates.

In the practice of celestial navigation, angles are usually measured in degrees, minutes and seconds of arc rather than in radians. We shall adhere to that convention. Furthermore, since the apparent motion of the sun in the sky relative to an observer on the surface of the earth is westward, navigational angles are usually measured westward.

$$\text{The Greenwich Hour Angle} \quad GHA(P) = 360^\circ - \Lambda \quad (1)$$

For example, $GHA\odot$ is the longitude of the sun measured westerly. The symbol λ_W will also be used to generically represent a westerly longitude of any other point specified on the celestial sphere in the text and understood in the context of that text.

Whether we are expressing all longitudes in the easterly direction (standard mathematical angle format) or some in the westerly direction (navigational format), the longitude of the observer M is the same in both systems.

$$\begin{aligned} \lambda &= \lambda_E \text{ and } \lambda_W = -\lambda; \quad \lambda_W = -\lambda_E. \\ \lambda &> 0 \text{ , Easterly;} \quad \lambda < 0 \text{ , Westerly} \end{aligned}$$

l (or $\Delta\lambda$ or $\Delta\theta$) is *defined* as $l \stackrel{D}{=} \Lambda - \lambda$ with λ and Λ in *customary mathematical format* for measuring angles, that is, measured positively in an easterly direction. Angles in celestial navigation are traditionally measured in *navigational format*, that is, positively in a westerly direction.

Most angles (with one exception) tabulated or computed in celestial navigation are positive. If a computed angle is negative, it is changed to a positive angle by adding to it 360°. Moreover if an angle is greater than 360°, we subtract 360° from it.

$$\text{Let } \; LHA(P) = \; 360° - \Delta\theta, \; \text{ the local hour angle of P}$$

$$GHA = 360° - \Lambda \quad \Rightarrow \quad \Lambda = 360° - GHA$$

Then

$$LHA = 360° - (\Lambda - \lambda)$$

$$LHA = \lambda + (360° - \Lambda) = \lambda + GHA$$

l is the angle by which the celestial body P is east of the observer M.
LHA is the angle by which celestial body P is west of observer M.

We may then write

$$\boxed{LHA(P) = GHA(P) + \lambda} \qquad (2)$$

For example, see Figure 3 below.

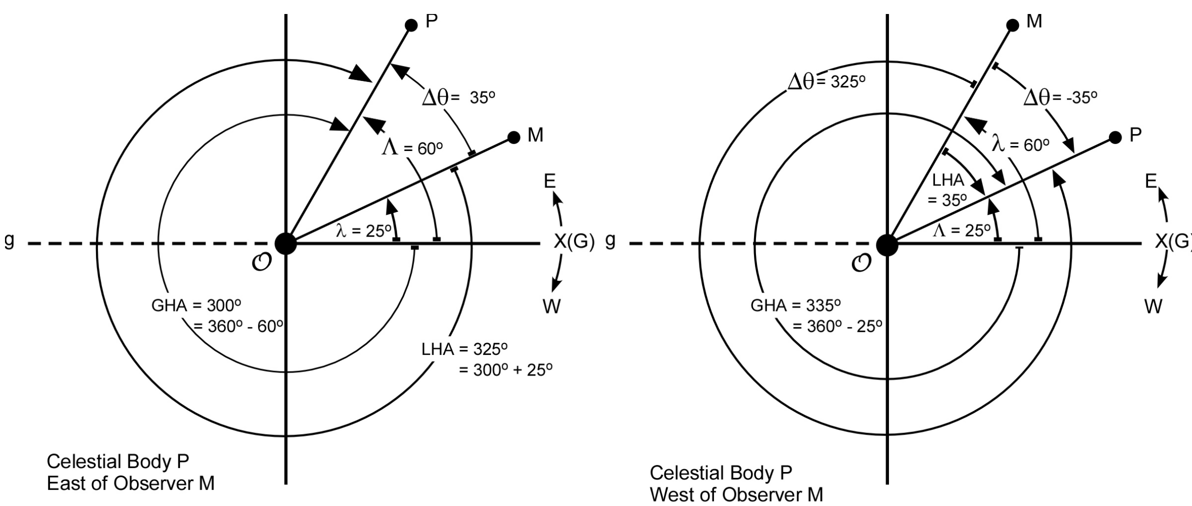


Figure 3: The Equality of $360° - \Delta\theta$ and LHA.

If the three points on the globe, the North Pole, M and P are connected by great circles, there are two possible navigational (spherical) triangles. In celestial navigation, we are interested in the smaller of these, the spherical triangle which has the smaller angle between the meridian of the celestial body and the meridian of the observer. However, the equations in this paper derived via rotation matrices apply to any spherical triangle.

The longitude difference LHA is frequently supplemented by the measure $t \, \varepsilon \, [0, 180°)$, the *meridian angle*, the smaller of the two angles between the meridian of the observer M and the meridian of the geographical position of the celestial body observed [10]. It is measured east or west, $t = t_E$ or t_W, depending upon its value.

If $LHA \leq 180°$, then $t_W = LHA$, celestial body P west of observer's meridian.

If $LHA > 180°$, then $t_E = 360° - LHA$, celestial body P east of observer's meridian.

Both of these meridian angles are positive.

In this paper all declinations north of the equator are positive; those south of the equator are negative. Until relatively recently, before modern calculators and computers were available, arguments of trigonometric functions were tabulated for angles in the first trigonometric quadrant, that is, 0° to 90°. If any sign changes of the trigonometric functions of angles used in the navigational calculations were necessary for angles residing in any quadrant other than the first, rules were used to assign these signs.

1.1 Derivation of The Navigation Equations

In the derivation of the equations of spherical trigonometry used in celestial navigation, there are 3 rotations of coordinates to be performed.

I typically use the symbols (x, y, z) to refer to coordinates of a vector $\mathbf{R}$ in a coordinate system S and (x', y', z') to refer to coordinates of the same vector $\mathbf{R}$ in the rotated coordinate system S' (rotated relative to system S). Frequently, system S is referred to as the "laboratory system" with (x, y, z) as the "space axes" and system S' as the "body axis system" with (x', y', z') as the "body axes". Here, the laboratory system is the celestial equator (equinoctial) system and the "body axis system" is the horizon system.

I know θ_A and ϕ_A. I want to determine ϕ_B and θ_B, or conversely.

We proceed as follows:

1.1.1 Sequence of Three Rotations to be Performed

For the purpose of this section, we will undertake a *temporary* reassignment of symbols for the Cartesian coordinates involved.

Step 1. The Cartesian coordinates in the equinoctial system of celestial body P are (x_0, y_0, z_0). *The first rotation* is around the z_0 axis by angle $\theta = \lambda_E(M)$, the east longitude of the observer M. The new coordinates of P are (x_1, y_1, z_1). (At the end of these three coordinate rotations, we will relabel (x_0, y_0, z_0) as (x, y, z)).

Step 2. *The second rotation* is around the y_1 axis by angle ϕ, the colatitude of the observer M. The new coordinates of P are (x_2, y_2, z_2).

Step 3. *The third rotation* is around the z_2 axis by angle $\psi = 180°$, so that the new x-axis points in the northerly direction. The new coordinates of P are (x_3, y_3, z_3).

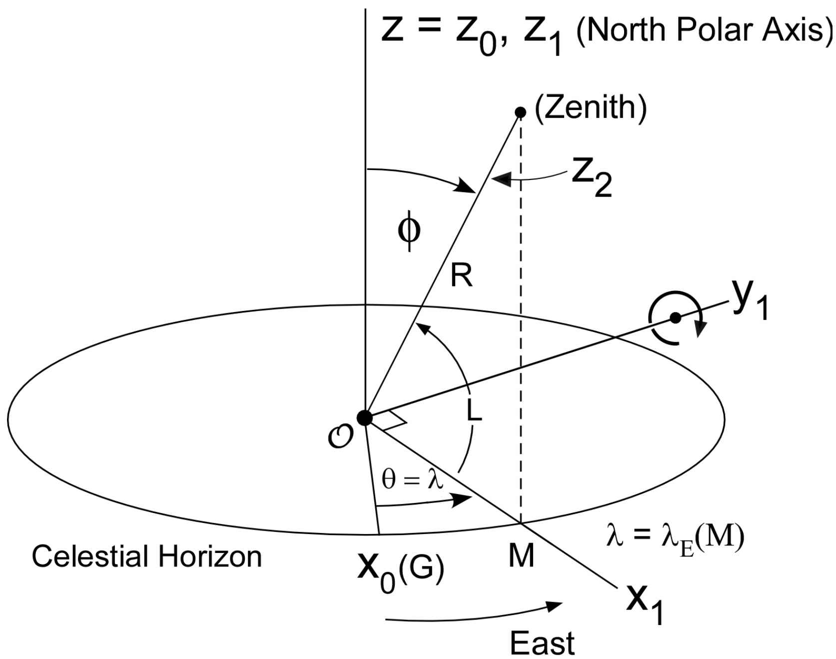


Figure 4: After Two Rotations.

Now, reassign $(x_0, y_0, z_0) \leftarrow (x, y, z)$ and $(x_3, y_3, z_3) \leftarrow (x', y', z')$.
Proceeding in this way as we have done before, we write

Suppose that $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = R \begin{pmatrix} \sin\phi_A \cos\theta_A \\ \sin\phi_A \sin\theta_A \\ \cos\phi_A \end{pmatrix}$ are the components of a vector $\mathbf{R}$ in the celestial equator (equinoctial system), and that this coordinate system is copied and then rotated via $G(\theta, \phi, \psi) = Z(\psi) \cdot Y(\phi) \cdot Z(\theta)$, which we call the *gyro rotation matrix.*

$$G(\theta,\phi,\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi\cos\theta\cos\psi - \sin\theta\sin\psi & \cos\theta\sin\psi + \cos\phi\cos\psi\sin\theta & -\sin\phi\cos\psi \\ -\sin\theta\cos\psi - \cos\phi\cos\theta\sin\psi & \cos\theta\cos\psi - \cos\phi\sin\theta\sin\psi & \sin\phi\sin\psi \\ \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \end{pmatrix} \quad (3)$$

After the coordinate system has been rotated via $Z(\theta)$ and $Y(\phi)$, the new (carried) x-axis will be pointing away from the original z-axis, that is, it will be pointing in a south direction along the meridian to which it is tangent. But we require that the x-axis point in a northerly direction along the meridian, because North is the direction from which co-azimuth is measured. For this to occur, the coordinate system must be rotated by π around the latest z-axis, that is, we must have $\psi = 180°$. (This was described above).

$$G(\theta,\phi,\pi) = Z(\pi)\cdot Y(\phi)\cdot Z(\theta) = \begin{pmatrix} -\cos\theta\cos\phi & -\cos\phi\sin\theta & \sin\phi \\ \sin\theta & -\cos\theta & 0 \\ \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \end{pmatrix} \tag{4}$$

Figure 5 portrays the relevant lines and angles with which we are concerned. Keep in mind that the axes x' and y' lie in a different plane than axes x and y.

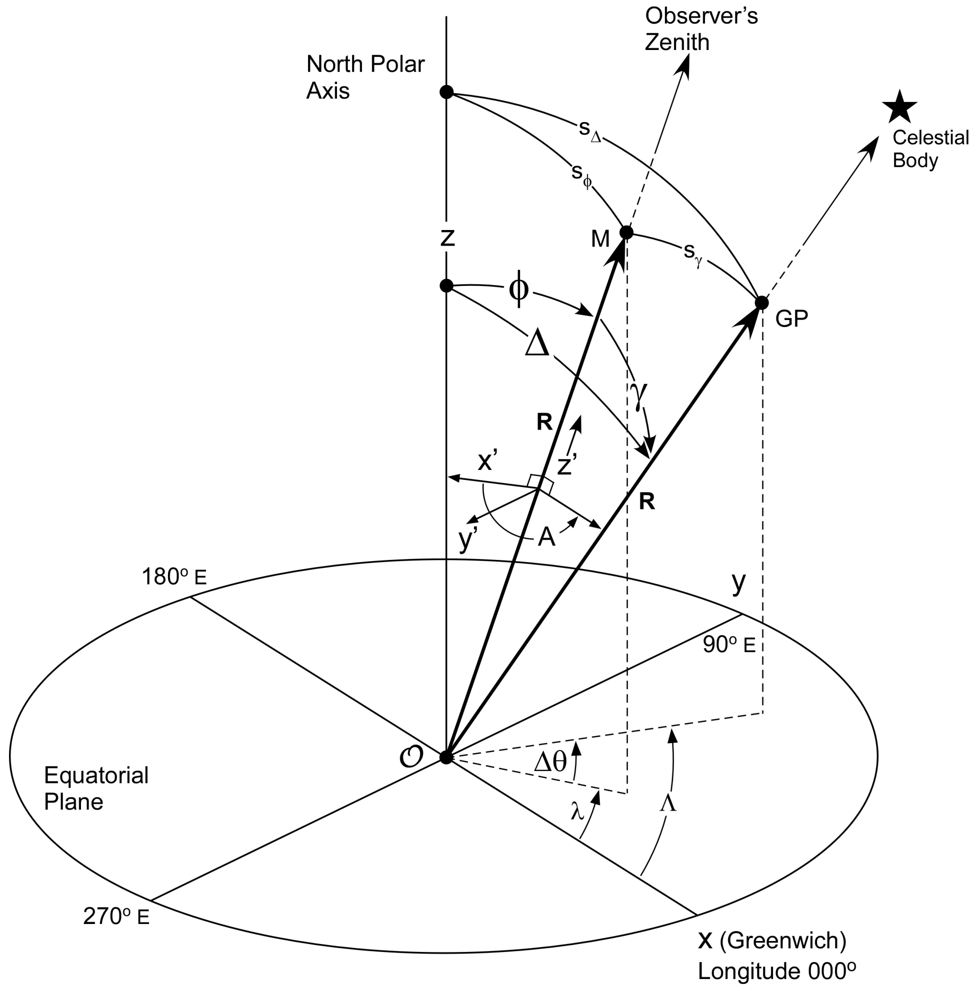


Figure 5: Position Vectors of the GP and the Observer's Position M.

The Euler rotation matrix rather than the Gyro rotation matrix can be used for these derivations with ultimately the same results.

Diagrams or figures constructed for the study of spherical trigonometry portray relevant lines, angles and great circles. These enable us to visualize the mutual geometrical relationships of these lines, angles and great circles, and to subsequently declare these relationships algebraically. Without such figures, it would be difficult to accomplish this task. Moreover, these figures, which represent three-dimensional entities, are produced on a two-dimensional sheet of paper as perspective drawings. We do not have three-dimensional (e.g., holographic) drawings, and visualization of perspective on a two-dimensional surface can be difficult if care is not taken in their creation. Furthermore, the figures can become cluttered if we attach all of the relevant lines and symbols, detracting from their visualization. For example, the angle of intersection of two great circles is measured by the angle between their tangents at the point of intersection. Conventionally however, we usually express this angle as between the circular arcs themselves as illustrated below.

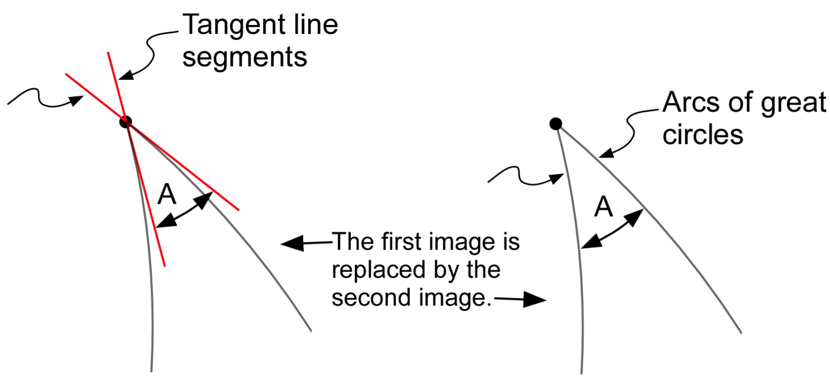


Figure 6: Arcs and Tangents.

In Figure 7A and 7B below we observe that the vertices of the spherical triangle are each connected to the origin $\mathcal{O}$ by equal radii R. The vertices are NP, M and P.

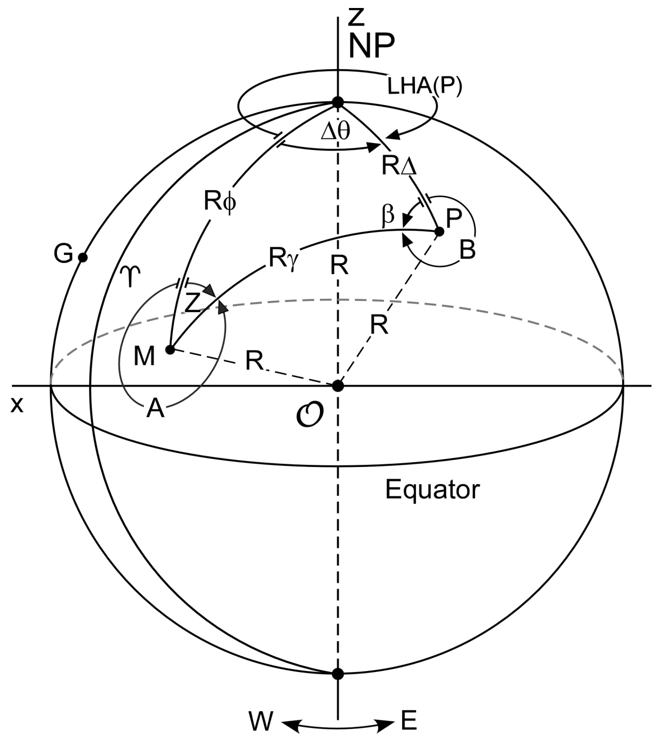


Figure 7A: Navigational Triangle, P East of M.

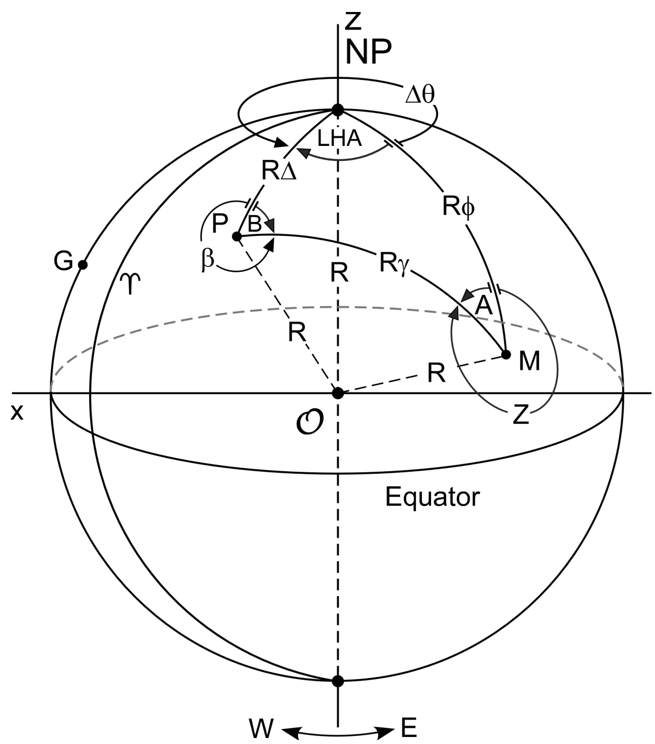


Figure 7B: Navigational Triangle, M East of P.

The intersection of the three great circles spanned by the central angles γ, Δ, ϕ

and the surface of the earth form a trihedron. If we include the chords and/or arcs between M, P and NP, we have a tetrahedron.

(e.g., the figure above created from line segments $\overline{OM}$, $\overline{OP}$, $\overline{O\ NP}$, $\overline{MP}$, $\overline{NP\ M}$, $\overline{NP\ P}$ and the corresponding great circle arcs). These and the spherical triangle M-P-NP possess threefold symmetry.

The angles $(\gamma,\, \Delta\theta)$ of Figure 9A or $(\gamma,\, LHA)$ of Figure 9B each subtend the circular arc of length $R\gamma$.

The angles $(\Delta,\, Z)$ of Figure 9A or $(\Delta,\, A)$ of Figure 9B each subtend the circular arc of length $R\Delta$.

The angles $(\phi,\, \beta)$ of Figure 9A or $(\Delta,\, B)$ of Figure 9B each subtend the circular arc of length $R\phi$.

Equations derived from the analysis of *the tetrahedron alone* are the same for the three angle pairs except for their interior angle arguments. As will be shown shortly, because of the symmetry inherent in the tetrahedron, we may permute the symbols in equations 4, 5, 6. These equations are known as the *spherical trigonometric sine and cosine equations.* Derivations of these appear in the Appendix.

However, when the tetrahedron is embedded in the Cartesian coordinate system S overlaid by spherical coordinates (using the results of the coordinate rotations):

1. The same cosine equations continue to be threefold symmetrical, except now their arguments also include exterior angles. These angles are not the same as the interior angles, but are arithmetically related to them.
2. Three of the sine equations are symmetrical.
3. Additional equations are generated via the coordinate rotation process. These provide information to uniquely justify the trigonometric quadrants and are not usually derived via the "classical" method appearing in the Appendix.

If the radii and chords of the tetrahedron were characterized by (overlaid with) direction cosines rather than by spherical coordinates, there would be complete threefold symmetry, because direction cosine angles are all measured in the same manner. However, the two spherical coordinate angles (colatitude and longitude) are not measured in a similar manner to one-another. Longitude for both M and P are measured in the same plane, whereas the co-declination and co-latitude are each measured in different planes.

1.2 A Perspective on the Measurement Angles

In Figure 8, the surface of the earth in the neighborhood of the observer's position M is represented by the surface of the rotor of a mechanical gyroscope. This is a supplementary figure introduced here merely to provide, in this author's opinion, a better visualization of the relevant angles. In the neighborhood of M, the surface of the earth may, for computational purposes, be regarded as being flat.

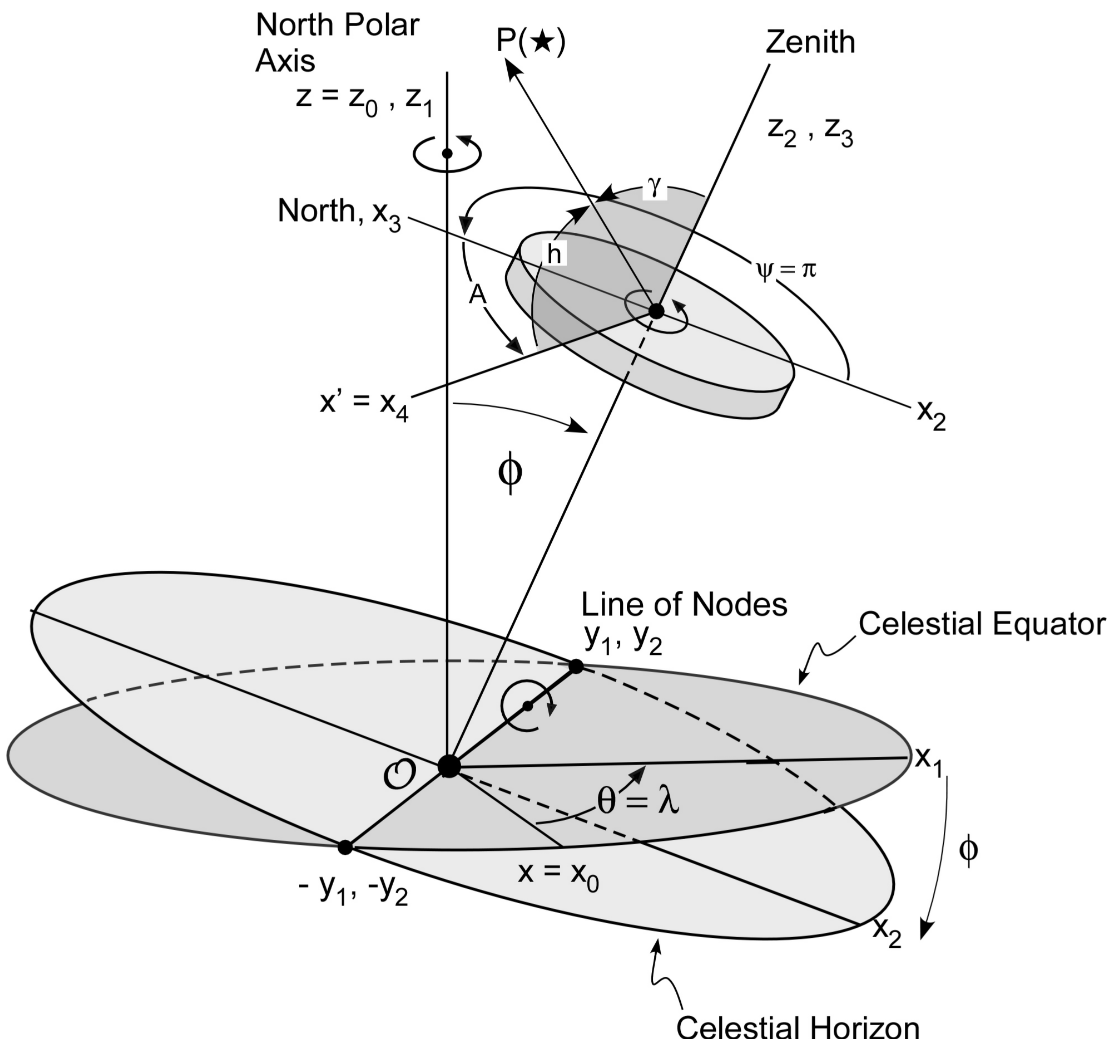


Figure 8: Celestial Equator and Horizon Systems of Coordinates.

Referring to Figure 5, suppose that we replace symbols,

$$\theta \leftarrow \lambda, \;\; \phi = \phi \text{ (unchanged)}, \; \phi_A \leftarrow \Delta, \;\; \phi_B \leftarrow \gamma, \;\; \theta_A \leftarrow \Lambda, \;\; \theta_B \leftarrow A$$

The components of vector $\mathbf{R}$ in the original Cartesian coordinate system (the celestial equator system) are

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = R \begin{pmatrix} \sin\phi_A \cos\theta_A \\ \sin\phi_A \sin\theta_A \\ \cos\phi_A \end{pmatrix} = R \begin{pmatrix} \sin\Delta\cos\Lambda \\ \sin\Delta\sin\Lambda \\ \cos\Delta \end{pmatrix} \qquad (5)$$

The components of vector $\mathbf{R}$ in the new rotated Cartesian coordinate system (the horizon system) are

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R \begin{pmatrix} \sin\phi_B \cos\theta_B \\ \sin\phi_B \sin\theta_B \\ \cos\phi_B \end{pmatrix} = R \begin{pmatrix} \sin\gamma\cos A \\ \sin\gamma \sin A \\ \cos\gamma \end{pmatrix} \qquad (6)$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = G(\theta,\phi,\pi) \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad (7)$$

where $G(\theta,\phi,\pi) = G(\lambda,\phi,\pi)$. That is,

$$\begin{pmatrix} \sin\gamma\cos A \\ \sin\gamma\sin A \\ \cos\gamma \end{pmatrix} = \begin{pmatrix} -\cos\lambda\cos\phi & -\cos\phi\sin\lambda & \sin\phi \\ \sin\lambda & -\cos\lambda & 0 \\ \cos\lambda\sin\phi & \sin\lambda\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \sin\Delta\cos\Lambda \\ \sin\Delta\sin\Lambda \\ \cos\Delta \end{pmatrix} \qquad (8)$$

$$= \begin{pmatrix} \cos\Delta\sin\phi - \cos\lambda\cos\Lambda\cos\phi\sin\Delta - \cos\phi\sin\lambda\sin\Delta\sin\Lambda \\ \cos\Lambda\sin\lambda\sin\Delta - \cos\lambda\sin\Delta\sin\Lambda \\ \cos\Delta\cos\phi + \cos\lambda\cos\Lambda\sin\Delta\sin\phi + \sin\lambda\sin\Delta\sin\Lambda\sin\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\Delta\sin\phi - \cos\phi\sin\Delta\,(\cos\lambda\cos\Lambda + \sin\lambda\sin\Lambda) \\ \sin\Delta\,(\sin\lambda\cos\Lambda - \cos\lambda\sin\Lambda) \\ \cos\Delta\cos\phi + \sin\phi\sin\Delta\,(\cos\lambda\cos\Lambda + \sin\lambda\sin\Lambda) \end{pmatrix}$$

That is, $$\begin{pmatrix} \sin\gamma\cos A_B \\ \sin\gamma\sin A \\ \cos\gamma \end{pmatrix} = \begin{pmatrix} \cos\Delta\sin\phi - \cos\phi\sin\Delta\cos(\Lambda-\lambda) \\ -\sin\Delta\sin(\Lambda-\lambda) \\ \cos\Delta\cos\phi + \sin\phi\sin\Delta\cos(\Lambda-\lambda) \end{pmatrix}$$

$$\begin{pmatrix} \sin\gamma\cos A \\ \sin\gamma\sin A \\ \cos\gamma \end{pmatrix} = \begin{pmatrix} \cos\Delta\sin\phi - \cos\phi\sin\Delta\cos l \\ -\sin\Delta\sin l \\ \cos\Delta\cos\phi + \sin\phi\sin\Delta\cos l \end{pmatrix}, \quad \text{where } l = \Delta\lambda = \Lambda - \lambda$$

Displayed below are the equations of spherical trigonometry using different combinations of (alias) symbols, (e.g., $l = \Delta\theta = \Delta\lambda$).

Angles measured in standard mathematical format, that is, easterly:

$$\sin\gamma\cos\lambda_B = \cos\phi_A\sin\phi - \cos\phi\sin\phi_A\cos\Delta\theta \qquad (9.1)$$

$$\sin\gamma\sin\lambda_B = -\sin\phi_A\sin\Delta\theta \qquad (9.2)$$

$$\cos\gamma = \cos\phi_A\cos\phi + \sin\phi_A\sin\phi\cos\Delta\theta \qquad (9.3)$$

where $\Delta\theta = \theta_A - \theta = \Lambda - \lambda$ (alias l or $\Delta\lambda$)

Displayed below are the equations of spherical trigonometry using different combinations of (alias) symbols:

$$\sin\gamma\cos A = \cos\Delta\sin\phi - \cos\phi\sin\Delta\cos l \qquad (10.1)$$

$$\sin\gamma\sin A = -\sin\Delta\sin l \qquad (10.2)$$

$$\cos\gamma = \cos\Delta\cos\phi + \sin\Delta\sin\phi\cos l \qquad (10.3)$$

$$\text{where} \quad l = \Delta\theta = \Delta\lambda = \Lambda - \lambda$$

If the third equation 10.3 is solved for $\cos l$ and substituted into equation 10.1, we obtain

$$\cos A = \frac{\cos\Delta - \cos\phi\cos\gamma}{\sin\phi\sin\gamma} \qquad (11)$$

$$\sin A = -\frac{\sin\Delta\sin l}{\sin\gamma} \qquad (12.1)$$

$$\cos A = \frac{\cos\Delta - \cos\phi\cos\gamma}{\sin\phi\sin\gamma} \qquad (12.2)$$

$$\cos\gamma = \cos\Delta\cos\phi + \sin\Delta\sin\phi\cos l \qquad (12.3)$$

$$\text{where} \quad l = \Delta\lambda = \Lambda - \lambda$$

Angles measured in navigational format with $LHA = 360° - l$, that is, westerly:

$$\cos h\cos Z = \sin d\cos L - \cos d\sin L\cos LHA \qquad (13.1)$$

$$\cos h\sin Z = \cos d\sin LHA \qquad (13.2)$$

$$\sin h = \sin d\sin L + \cos d\cos L\cos LHA \qquad (13.3)$$

$$\text{where} \quad LHA = 360° - l$$

If the third equation 13.3 is solved for $\cos LHA$ and substituted into equation 13.1, we obtain

$$\cos Z = \frac{\sin d - \sin L\sin h}{\cos L\cos h} \qquad (14)$$

h_c is computed altitude, d (or alias δ) is declination of celestial body P, and L is latitude of observer M. Let subscript "a" indicate "assumed", and subscript "c" indicate "computed".

Angles measured in navigational format, that is, westerly:

$$\sin Z = -\frac{\cos d \, \sin LHA}{\cos h} \tag{15.1}$$

$$\cos Z = \frac{\sin d - \sin L \sin h}{\cos L \cos h} \tag{15.2}$$

$$\sin h = \sin d \sin L + \cos d \cos L \cos LHA \tag{15.3}$$

$$\text{where} \quad LHA = GHA + \lambda$$

In the practice of celestial navigation, we usually use equations 15.1, 15.2 and 15.3. And, of course, equation (15.3) must be evaluated first for subsequent substitution in to (15.1) and (15.2). L is the assumed latitude of observer M, λ is the assumed longitude of observer M, with westerly observer longitudes replaced by their negatives. h is computed altitude of celestial body P. A subscript "c" might typically be appended to these symbols to represent "computed".

1.3 More Spherical Trigonometry Equations

Figures 9A, 9B, 10A, 10B below provide refinements to Figures 7A, 7B presented earlier. In these figures we notice a threefold symmetry.

The *internal* angles γ, Δ, ϕ subtend circular arcs. The *internal* angles $\Delta\theta, Z, \beta$ of Figure 9A or LHA, A, B of Figure 9B are angles on the surface of the celestial sphere between tangent lines of intersecting circular arcs.

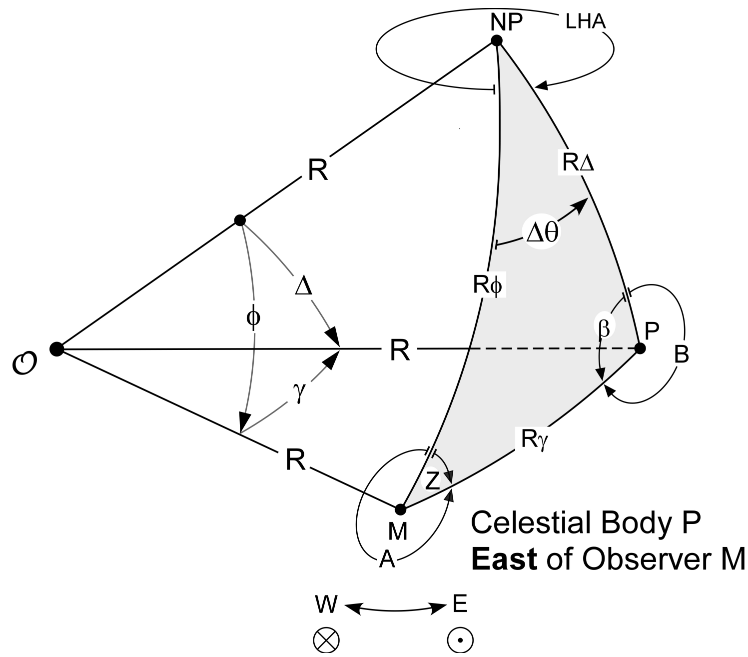


Figure 9A:

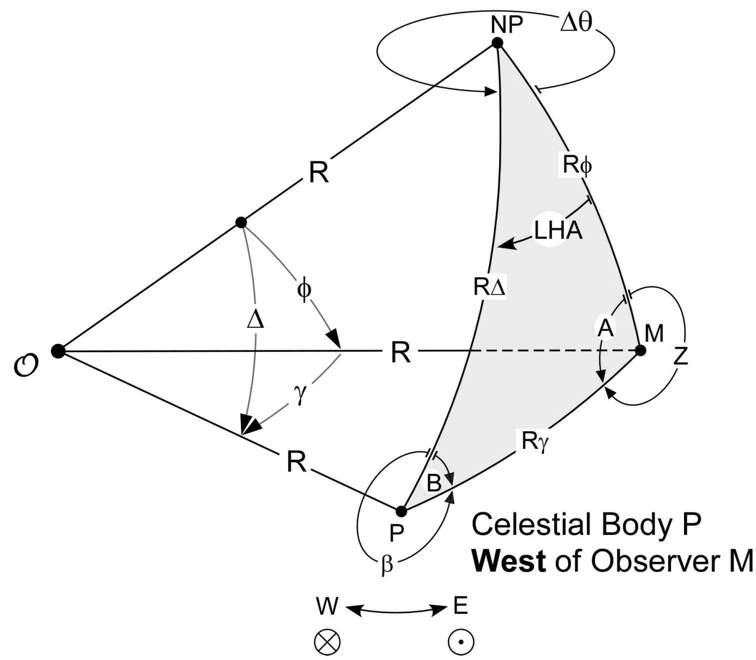


Figure 9B:

Celestial Body Coordinates Revisited

From previous derivations, we have:

- Coordinates of celestial body P (of GP) relative to Greenwich G in system S,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = R \begin{pmatrix} \sin\Delta\cos\Lambda \\ \sin\Delta\sin\Lambda \\ \cos\Delta \end{pmatrix} \quad \text{(5, repeated);}$$

- Coordinates of celestial body P (of GP) relative to observer M in system S',

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R \begin{pmatrix} \sin\gamma\cos A \\ \sin\gamma\sin A \\ \cos\gamma \end{pmatrix} \quad \text{(6, repeated);}$$

$$\underbrace{\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}}_{\substack{\uparrow \\ Position\ of\ P \\ in\ S'\ system \\ (What\ M\ sees)}} = \underbrace{G(\lambda, \phi, \pi)}_{\substack{\uparrow \\ Coordinates \\ of\ M}} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\substack{\uparrow \\ Position\ of\ P \\ in\ S\ system \\ (What\ G\ sees)}}$$

$$\text{that is,} \begin{pmatrix} \sin\gamma\cos A \\ \sin\gamma\sin A \\ \cos\gamma \end{pmatrix} = G(\lambda, \phi, \pi) \begin{pmatrix} \sin\Delta\cos\Lambda \\ \sin\Delta\sin\Lambda \\ \cos\Delta \end{pmatrix}$$

where

$$G(\lambda, \phi, \pi) = \begin{pmatrix} -\cos\phi\cos\lambda & -\cos\phi\sin\lambda & \sin\phi \\ \sin\lambda & -\cos\lambda & 0 \\ \sin\phi\cos\lambda & \sin\phi\sin\lambda & \cos\phi \end{pmatrix}$$

$$\begin{pmatrix} \sin\gamma\cos A \\ \sin\gamma\sin A \\ \cos\gamma \end{pmatrix} = \begin{pmatrix} -\cos\lambda\cos\phi & -\cos\phi\sin\lambda & \sin\phi \\ \sin\lambda & -\cos\lambda & 0 \\ \cos\lambda\sin\phi & \sin\lambda\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \sin\Delta\cos\Lambda \\ \sin\Delta\sin\Lambda \\ \cos\Delta \end{pmatrix}$$

We may now consider the inverse relationship,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = G^{-1}(\lambda, \phi, \pi) \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \equiv G^{T}(\lambda, \phi, \pi) \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$$\begin{pmatrix} \sin\Delta\cos\Lambda \\ \sin\Delta\sin\Lambda \\ \cos\Delta \end{pmatrix} = \begin{pmatrix} -\cos\lambda\cos\phi & \sin\lambda & \cos\lambda\sin\phi \\ -\cos\phi\sin\lambda & -\cos\lambda & \sin\lambda\sin\phi \\ \sin\phi & 0 & \cos\phi \end{pmatrix} \begin{pmatrix} \sin\gamma\cos A \\ \sin\gamma\sin A \\ \cos\gamma \end{pmatrix}$$

$$= \begin{pmatrix} \sin\lambda\sin\gamma\sin A + \cos\lambda\,(\sin\phi\cos\gamma - \cos\phi\sin\gamma\cos A) \\ -\cos\lambda\sin\gamma\sin A + \sin\lambda\,(\sin\phi\cos\gamma - \cos\phi\sin\gamma\cos A) \\ \cos\phi\cos\gamma + \sin\phi\sin\gamma\cos A \end{pmatrix} \quad \Rightarrow$$

$$\boxed{\begin{aligned} &\sin\Delta\cos\Lambda = \sin\lambda\sin\gamma\sin A + \cos\lambda\,(\sin\phi\cos\gamma - \cos\phi\sin\gamma\cos A) && (16.1) \\ &\sin\Delta\sin\Lambda = -\cos\lambda\sin\gamma\sin A + \sin\lambda\,(\sin\phi\cos\gamma - \cos\phi\sin\gamma\cos A) && (16.2) \\ &\cos\Delta = \cos\phi\cos\gamma + \sin\phi\sin\gamma\cos A && (16.3) \end{aligned}}$$

$$(Eq.16.1) \times \cos\lambda + (Eq.16.2) \times \sin\lambda \quad \Rightarrow$$

$$\begin{aligned}\sin\Delta\cos l &= \sin\phi\cos\gamma - \cos\phi\sin\gamma\cos A \qquad (17)\\ \text{where} \quad l &= \Lambda - \lambda\end{aligned}$$

Observer Coordinates Revisited

- Coordinates of observer M in system S, $\begin{pmatrix} u \\ v \\ w \end{pmatrix} = R \begin{pmatrix} \sin\phi\cos\lambda \\ \sin\phi\sin\lambda \\ \cos\phi \end{pmatrix}$ (18)

- Coordinates of observer M in system S', $\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = R \begin{pmatrix} \sin\gamma\cos\beta \\ \sin\gamma\sin\beta \\ \cos\gamma \end{pmatrix}$ (19)

$$\underset{\substack{\uparrow \\ Position\ of\ M \\ in\ S'\ system \\ (What\ P\ sees)}}{\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix}} = \underset{\substack{\uparrow \\ Coordinates \\ of\ P}}{G(\Lambda, \Delta, \pi)} \underset{\substack{\uparrow \\ Position\ of\ M \\ in\ S\ system \\ (What\ G\ sees)}}{\begin{pmatrix} u \\ v \\ w \end{pmatrix}} \qquad (20)$$

$$G(\Lambda, \Delta, \pi) = \begin{pmatrix} -\cos\Lambda\cos\Delta & -\cos\Delta\sin\Lambda & \sin\Delta \\ \sin\Lambda & -\cos\Lambda & 0 \\ \cos\Lambda\sin\Delta & \sin\Lambda\sin\Delta & \cos\Delta \end{pmatrix} \qquad (21)$$

$$\begin{pmatrix} \sin\gamma\cos\beta \\ \sin\gamma\sin\beta \\ \cos\gamma \end{pmatrix} = \begin{pmatrix} -\cos\Lambda\cos\Delta & -\cos\Delta\sin\Lambda & \sin\Delta \\ \sin\Lambda & -\cos\Lambda & 0 \\ \cos\Lambda\sin\Delta & \sin\Lambda\sin\Delta & \cos\Delta \end{pmatrix} \begin{pmatrix} \sin\phi\cos\lambda \\ \sin\phi\sin\lambda \\ \cos\phi \end{pmatrix} \qquad (22)$$

$$= \begin{pmatrix} \sin\Delta\cos\phi - \cos\Delta\sin\phi\cos(\Lambda-\lambda) \\ \sin(\Lambda-\lambda)\sin\phi \\ \cos\Delta\cos\phi + \cos(\Lambda-\lambda)\sin\Delta\sin\phi \end{pmatrix}, \text{ that is,}$$

$$\begin{pmatrix} \sin\gamma\cos\beta \\ \sin\gamma\sin\beta \\ \cos\gamma \end{pmatrix} = \begin{pmatrix} \sin\Delta\cos\phi - \cos\Delta\sin\phi\cos l \\ \sin l\sin\phi \\ \cos\Delta\cos\phi + \sin\Delta\sin\phi\cos l \end{pmatrix}$$

$$\boxed{\begin{aligned} &\sin\gamma\cos\beta = \sin\Delta\cos\phi - \cos\Delta\sin\phi\cos l \quad (23.1)\\ &\sin\gamma\sin\beta = \sin\phi\sin l \quad (23.2)\\ &\cos\gamma = \cos\Delta\cos\phi + \sin\Delta\sin\phi\cos l \quad (23.3)\end{aligned}}$$

With the exception of a different variable (requiring a minus sign) in equation (10.2), equations (23.2) and (10.2) are of the same form. With the substitution of $A = 360° - Z$ into equation 10.2 we may restore the symmetry.

$$\sin\gamma \sin Z = \sin\Delta \sin l \tag{24}$$

Equation (23.3) is identical to (10.3).

From Equations 10.2 and 23.2, we observe that

$$\boxed{\frac{\sin l}{\sin\gamma} = \frac{\sin\beta}{\sin\phi} = \frac{\sin Z}{\sin\Delta} \qquad (25)}$$

The three equations 25 are known as the *Spherical Law of Sines*. In Figure 10A, we see that these are interior angles of the navigational tetrahedron.

Again, we may consider the inverse relationship, $\begin{pmatrix} u \\ v \\ w \end{pmatrix} = G^T(\Lambda, \Delta, \pi) \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix}$.

$$G^{-1}(\Lambda, \Delta, \pi) = G^T(\Lambda, \Delta, \pi) = \begin{pmatrix} -\cos\Delta\cos\Lambda & \sin\Lambda & \cos\Lambda\sin\Delta \\ -\cos\Delta\sin\Lambda & -\cos\Lambda & \sin\Delta\sin\Lambda \\ \sin\Delta & 0 & \cos\Delta \end{pmatrix}$$

$$\begin{pmatrix} \sin\phi\cos\lambda \\ \sin\phi\sin\lambda \\ \cos\phi \end{pmatrix} = \begin{pmatrix} -\cos\Delta\cos\Lambda & \sin\Lambda & \cos\Lambda\sin\Delta \\ -\cos\Delta\sin\Lambda & -\cos\Lambda & \sin\Delta\sin\Lambda \\ \sin\Delta & 0 & \cos\Delta \end{pmatrix} \begin{pmatrix} \sin\gamma\cos\beta \\ \sin\gamma\sin\beta \\ \cos\gamma \end{pmatrix}$$

$$\begin{pmatrix} \sin\phi\cos\lambda \\ \sin\phi\sin\lambda \\ \cos\phi \end{pmatrix} = \begin{pmatrix} \sin\Lambda\sin\beta\sin\gamma + \cos\Lambda\left(\sin\Delta\cos\gamma - \cos\Delta\sin\gamma\cos\beta\right) \\ \sin\Lambda\sin\Delta\cos\gamma - \sin\gamma\left(\cos\Lambda\sin\beta + \cos\Delta\sin\Lambda\cos\beta\right) \\ \cos\Delta\cos\gamma + \sin\Delta\sin\gamma\cos\beta \end{pmatrix}$$

That is,

$$\boxed{\begin{aligned} &\sin\phi\cos\lambda = \sin\Lambda\sin\gamma\sin\beta + \cos\Lambda\left(\sin\Delta\cos\gamma - \cos\Delta\sin\gamma\cos\beta\right) && (26.1) \\ &\sin\phi\sin\lambda = \sin\Lambda\sin\Delta\cos\gamma - \sin\gamma\left(\cos\Lambda\sin\beta + \cos\Delta\sin\Lambda\cos\beta\right) && (26.2) \\ &\cos\phi = \cos\gamma\cos\Delta + \sin\gamma\sin\Delta\cos\beta && (26.3) \end{aligned}}$$

Equations 10.3, 16.3 and 26.3 are known as the *Spherical Law of Cosines* and are, for convenience, restated below.

$$\boxed{\begin{aligned} &\cos\gamma = \cos\Delta\cos\phi + \sin\Delta\sin\phi\cos l && \text{(10.3, repeated)} \\ &\cos\Delta = \cos\phi\cos\gamma + \sin\phi\sin\gamma\cos A && \text{(16.3, repeated)} \\ &\cos\phi = \cos\gamma\cos\Delta + \sin\gamma\sin\Delta\cos\beta && \text{(26.3, repeated)} \end{aligned}}$$

Note: $\cos Z \equiv \cos A$, so that (16.3) may used with that substitution if required. Elsewhere, $\sin Z = -\sin A$ may be used.

A traditional derivation of the spherical law of sines and of cosines is provided in the Appendix.

1.4 A Second Spherical Law of Cosines

We have already derived equations (10.3), (16.3) and (26.3), the three instances of the spherical law of cosines. These equations are repeated below for convenience.

$$\begin{aligned}
\cos\gamma &= \cos\Delta\cos\phi + \sin\Delta\sin\phi\cos l \quad \text{(10.3, repeated)}\\
\cos\Delta &= \cos\phi\cos\gamma + \sin\phi\sin\gamma\cos A \quad \text{(16.3, repeated)}\\
\cos\phi &= \cos\gamma\cos\Delta + \sin\gamma\sin\Delta\cos\beta \quad \text{(26.3, repeated)}
\end{aligned}$$

These equations express the cosines of the interior angles γ, Δ and ϕ as functions of the other interior angles and the one surface angle l, Z, or β, repectively, corresponding to the interior angle on the left hand side of these equations. There exist three complementary or converse equations of rather similar but not identical form, with interior angles and surface angles interchanged. These equations are provided below, followed by their derivations from equations (10.3), (16.3) and (26.3). These converse equations express what we refer to as *the second spherical law of cosines.*

$$\boxed{\begin{aligned}
\cos l &= -\cos Z\cos\beta + \sin Z\sin\beta\cos\gamma \quad (27.1)\\
\cos Z &= -\cos\beta\cos l + \sin\beta\sin l\cos\Delta \quad (27.2)\\
\cos\beta &= -\cos l\cos Z + \sin l\sin Z\cos\phi \quad (27.3)
\end{aligned}}$$

1.4.1 Some Auxiliary Equations

We first derive a set of equations from (10.3), (16.3) and (26.3), which equations will subsequently be used to derive (27.1), (27.2) and (27.3).

Derivation 1.

Substitute (3) into (1) to eliminate $\cos\phi$:

$$\cos\gamma = \cos\Delta\left(\cos\gamma\cos\Delta + \sin\gamma\sin\Delta\cos\beta\right) + \sin\Delta\sin\phi\cos l$$

$$\cos\gamma = \cos\gamma\cos^2\Delta + \sin\gamma\sin\Delta\cos\Delta\cos\beta + \sin\Delta\sin\phi\cos l$$

$$\cos\gamma\ \left(1-\cos^2\Delta\right) = \sin\gamma\sin\Delta\cos\Delta\cos\beta + \sin\Delta\sin\phi\cos l$$

$$\cos\gamma\sin^2\Delta = \sin\gamma\sin\Delta\cos\Delta\cos\beta + \sin\Delta\sin\phi\cos l$$

$$\cos\gamma\sin\Delta = \sin\gamma\cos\Delta\cos\beta + \sin\phi\cos l$$

$$\cos\gamma\frac{\sin\Delta}{\sin\gamma} = \cos\Delta\cos\beta + \frac{\sin\phi}{\sin\gamma}\cos l$$

But

$$\frac{\sin l}{\sin\gamma} = \frac{\sin Z}{\sin\Delta} = \frac{\sin\beta}{\sin\phi} \quad (4) \quad \Rightarrow \quad \frac{\sin\Delta}{\sin\gamma} = \frac{\sin Z}{\sin l} \quad \text{and} \quad \frac{\sin\phi}{\sin\gamma} = \frac{\sin\beta}{\sin l}$$

Making these substitutions,

$$\cos\gamma\frac{\sin Z}{\sin l} = \cos\Delta\cos\beta + \frac{\sin\beta}{\sin l}\cos l$$

$$\cos\gamma\sin Z = \cos\Delta\sin l\cos\beta + \cos l\sin\beta \quad (28.1)$$

<u>Derivation 2.</u>
Substitute (3) into (2) to eliminate $\cos\phi$:

$$\cos\Delta = (\cos\gamma\cos\Delta + \sin\gamma\sin\Delta\cos\beta)\cos\gamma + \sin\phi\sin\gamma\cos Z$$

$$\cos\Delta = \cos^2\gamma\cos\Delta + \sin\gamma\cos\gamma\sin\Delta\cos\beta + \sin\phi\sin\gamma\cos Z$$

$$\cos\Delta\ \left(1-\cos^2\gamma\right) = \sin\gamma\cos\gamma\sin\Delta\cos\beta + \sin\phi\sin\gamma\cos Z$$

$$\cos\Delta\ \sin^2\gamma = \sin\gamma\cos\gamma\sin\Delta\cos\beta + \sin\phi\sin\gamma\cos Z$$

$$\cos\Delta\ \sin\gamma = \cos\gamma\sin\Delta\cos\beta + \sin\phi\cos Z$$

$$\cos\Delta\ \frac{\sin\gamma}{\sin\Delta} = \cos\gamma\cos\beta + \frac{\sin\phi}{\sin\Delta}\cos Z$$

But

$$\frac{\sin l}{\sin\gamma} = \frac{\sin Z}{\sin\Delta} = \frac{\sin\beta}{\sin\phi} \quad (4) \quad \Rightarrow \quad \frac{\sin\gamma}{\sin\Delta} = \frac{\sin l}{\sin Z} \quad \text{and} \quad \frac{\sin\phi}{\sin\Delta} = \frac{\sin\beta}{\sin Z}$$

Making these substitutions,

$$\cos\Delta\ \frac{\sin l}{\sin Z} = \cos\gamma\cos\beta + \frac{\sin\beta}{\sin Z}\cos Z$$

$$\cos\Delta\sin l = \cos\gamma\sin Z\cos\beta + \sin\beta\cos Z \qquad \text{2nd eq.} \quad (28.2)$$

<u>Derivation 3.</u>
Substitute (2) into (3) to eliminate $\cos\Delta$:

$$\cos\phi = \cos\gamma\ (\cos\phi\cos\gamma + \sin\phi\sin\gamma\cos Z) + \sin\gamma\sin\Delta\cos\beta$$

$$\cos\phi = \cos\phi\cos^2\gamma + \sin\phi\sin\gamma\cos\gamma\cos Z + \sin\gamma\sin\Delta\cos\beta$$

$$\cos\phi\ \left(1-\cos^2\gamma\right) = \sin\phi\sin\gamma\cos\gamma\cos Z + \sin\gamma\sin\Delta\cos\beta$$

$$\cos\phi\ \sin^2\gamma = \sin\phi\sin\gamma\cos\gamma\cos Z + \sin\gamma\sin\Delta\cos\beta$$
$$\cos\phi\ \sin\gamma = \sin\phi\cos\gamma\cos Z + \sin\Delta\cos\beta$$
$$\cos\phi\,\frac{\sin\gamma}{\sin\phi} = \cos\gamma\cos Z + \frac{\sin\Delta}{\sin\phi}\cos\beta$$

But

$$\frac{\sin l}{\sin\gamma} = \frac{\sin Z}{\sin\Delta} = \frac{\sin\beta}{\sin\phi}\quad(4)\quad\Rightarrow\quad\frac{\sin\gamma}{\sin\phi} = \frac{\sin l}{\sin\beta}\quad\text{and}\quad\frac{\sin\Delta}{\sin\phi} = \frac{\sin Z}{\sin\beta}$$

Making these substitutions,

$$\cos\phi\,\frac{\sin l}{\sin\beta} = \cos\gamma\cos Z + \frac{\sin Z}{\sin\beta}\cos\beta$$

1

$$\cos\phi\ \sin l = \cos\gamma\cos Z\sin\beta + \sin Z\cos\beta\quad(28.3)$$

Derivation 4.

Substitute (2) into (1) to eliminate $\cos\Delta$:

$$\cos\gamma = (\cos\phi\cos\gamma + \sin\phi\sin\gamma\cos Z)\cos\phi + \sin\Delta\sin\phi\cos l$$
$$\cos\gamma = \cos^2\phi\cos\gamma + \sin\phi\cos\phi\sin\gamma\cos Z + \sin\Delta\sin\phi\cos l$$
$$\cos\gamma\ \left(1-\cos^2\phi\right) = \sin\phi\cos\phi\sin\gamma\cos Z + \sin\Delta\sin\phi\cos l$$
$$\cos\gamma\ \sin^2\phi = \sin\phi\cos\phi\sin\gamma\cos Z + \sin\Delta\sin\phi\cos l$$
$$\cos\gamma\ \sin\phi = \cos\phi\sin\gamma\cos Z + \sin\Delta\cos l$$
$$\cos\gamma\,\frac{\sin\phi}{\sin\gamma} = \cos\phi\cos Z + \frac{\sin\Delta}{\sin\gamma}\cos l$$

But

$$\frac{\sin l}{\sin\gamma} = \frac{\sin Z}{\sin\Delta} = \frac{\sin\beta}{\sin\phi}\quad(4)\quad\Rightarrow\quad\frac{\sin\phi}{\sin\gamma} = \frac{\sin\beta}{\sin l}\quad\text{and}\quad\frac{\sin\Delta}{\sin\gamma} = \frac{\sin Z}{\sin l}$$

Making these substitutions,

$$\cos\gamma\,\frac{\sin\beta}{\sin l} = \cos\phi\cos Z + \frac{\sin Z}{\sin l}\cos l$$

$$\cos\gamma\ \sin\beta = \cos\phi\sin l\cos Z + \sin Z\cos l\quad(28.4)$$

Derivation 5.
Substitute (1) into (2) to eliminate $\cos\gamma$:

$$\cos\Delta = \cos\phi\,(\cos\Delta\cos\phi + \sin\Delta\sin\phi\cos l) + \sin\phi\sin\gamma\cos Z$$

$$\cos\Delta = \cos\Delta\cos^2\phi + \sin\Delta\sin\phi\cos\phi\cos l + \sin\phi\sin\gamma\cos Z$$
$$\cos\Delta\ \left(1-\cos^2\phi\right) = \sin\Delta\sin\phi\cos\phi\cos l + \sin\phi\sin\gamma\cos Z$$
$$\cos\Delta\ \sin^2\phi = \sin\Delta\sin\phi\cos\phi\cos l + \sin\phi\sin\gamma\cos Z$$
$$\cos\Delta\ \sin\phi = \sin\Delta\cos\phi\cos l + \sin\gamma\cos Z$$
$$\cos\Delta\,\frac{\sin\phi}{\sin\gamma} = \frac{\sin\Delta}{\sin\gamma}\cos\phi\cos l + \cos Z$$

But

$$\frac{\sin l}{\sin\gamma} = \frac{\sin Z}{\sin\Delta} = \frac{\sin\beta}{\sin\phi}\quad(4)\quad\Rightarrow\quad\frac{\sin\phi}{\sin\gamma} = \frac{\sin\beta}{\sin l}\quad\text{and}\quad\frac{\sin\Delta}{\sin\gamma} = \frac{\sin Z}{\sin l}$$

Making these substitutions,

$$\cos\Delta\,\frac{\sin\beta}{\sin l} = \frac{\sin Z}{\sin l}\cos\phi\cos l + \cos Z$$

$$\cos\Delta\ \sin\beta = \sin Z\cos l\cos\phi + \sin l\cos Z\quad(28.5)$$

Derivation 6.
Substitute (1) into (3) to eliminate $\cos\gamma$:

$$\cos\phi = (\cos\Delta\cos\phi + \sin\Delta\sin\phi\cos l)\cos\Delta + \sin\gamma\sin\Delta\cos\beta$$
$$\cos\phi = \cos^2\Delta\cos\phi + \sin\Delta\cos\Delta\sin\phi\cos l + \sin\gamma\sin\Delta\cos\beta$$

$$\cos\phi\ \left(1-\cos^2\Delta\right) = \sin\Delta\cos\Delta\sin\phi\cos l + \sin\gamma\sin\Delta\cos\beta$$
$$\cos\phi\ \sin^2\Delta = \sin\Delta\cos\Delta\sin\phi\cos l + \sin\gamma\sin\Delta\cos\beta$$
$$\cos\phi\ \sin\Delta = \cos\Delta\sin\phi\cos l + \sin\gamma\cos\beta$$
$$\cos\phi\,\frac{\sin\Delta}{\sin\gamma} = \cos\Delta\frac{\sin\phi}{\sin\gamma}\cos l + \cos\beta$$

But

$$\frac{\sin l}{\sin\gamma} = \frac{\sin Z}{\sin\Delta} = \frac{\sin\beta}{\sin\phi}\quad(4)\quad\Rightarrow\quad\frac{\sin\Delta}{\sin\gamma} = \frac{\sin Z}{\sin l}\quad\text{and}\quad\frac{\sin\phi}{\sin\gamma} = \frac{\sin\beta}{\sin l}$$

Making these substitutions,

$$\cos\phi\,\frac{\sin Z}{\sin l} = \cos\Delta\frac{\sin\beta}{\sin l}\cos l + \cos\beta$$

$$\cos\phi\,\sin Z = \cos\Delta\sin\beta\cos l + \sin l\cos\beta \quad (28.6)$$

Summary of Auxiliary Equations:

$$\cos\gamma\sin Z = \cos\Delta\sin l\cos\beta + \cos l\sin\beta \quad (28.1)$$

$$\cos\Delta\sin l = \cos\gamma\sin Z\cos\beta + \sin\beta\cos Z \quad (28.2)$$

$$\cos\phi\,\sin l = \cos\gamma\cos Z\sin\beta + \sin Z\cos\beta \quad (28.3)$$

$$\cos\gamma\,\sin\beta = \cos\phi\sin l\cos Z + \sin Z\cos l \quad (28.4)$$

$$\cos\Delta\,\sin\beta = \sin Z\cos l\cos\phi + \sin l\cos Z \quad (28.5)$$

$$\cos\phi\,\sin Z = \cos\Delta\sin\beta\cos l + \sin l\cos\beta \quad (28.6)$$

1.4.2 Combining the Auxiliary Equations

First Instance of the Second Cosine Law:

$$\cos\gamma\sin Z = \cos\Delta\sin l\cos\beta + \sin\beta\cos l \quad (28.1)$$

$$\cos\Delta\sin l = \cos\gamma\sin Z\cos\beta + \sin\beta\cos Z \quad (28.2)$$

Substitute (28.2) into (28.1) to eliminate $\cos\Delta\sin l$:

$$\cos\gamma\sin Z = (\cos\gamma\sin Z\cos\beta + \sin\beta\cos Z)\cos\beta + \cos l\sin\beta$$

$$\cos\gamma\sin Z = \cos\gamma\sin Z\cos^2\beta + \sin\beta\cos\beta\cos Z + \cos l\sin\beta$$

$$\cos\gamma\sin Z\left(1-\cos^2\beta\right) = \sin\beta\cos\beta\cos Z + \cos l\sin\beta$$

$$\cos\gamma\sin Z\sin^2\beta = \sin\beta\cos\beta\cos Z + \cos l\sin\beta$$

$$\cos\gamma\sin Z\sin\beta = \cos\beta\cos Z + \cos l$$

$$\therefore \quad \boxed{\cos l = -\cos Z\cos\beta + \sin Z\sin\beta\cos\gamma} \quad (27.1,\ \text{repeated})$$

Second Instance of the Second Cosine Law:

$$\cos\Delta\,\sin\beta = \sin Z\cos l\cos\phi + \sin l\cos Z \quad (28.5)$$

$$\cos\phi\ \sin Z = \cos\Delta\sin\beta\cos l + \sin l\cos\beta \qquad (28.6)$$

Substitute (28.6) into (28.5) to eliminate $\cos\phi\ \sin Z$:

$$\cos\Delta\ \sin\beta = (\cos\Delta\sin\beta\cos l + \sin l\cos\beta)\cos l + \sin l\cos Z$$

$$\cos\Delta\ \sin\beta = \cos\Delta\sin\beta\cos^2 l + \sin l\cos l\cos\beta + \sin l\cos Z$$
$$\cos\Delta\ \sin\beta\left(1-\cos^2 l\right) = \sin l\cos l\cos\beta + \sin l\cos Z$$
$$\cos\Delta\ \sin\beta\sin^2 l = \sin l\cos l\cos\beta + \sin l\cos Z$$

$$\cos\Delta\ \sin\beta\sin l = \cos l\cos\beta + \cos Z$$
$$\therefore \quad \boxed{\cos Z = -\cos l\cos\beta + \sin\beta\sin l\cos\Delta} \qquad (27.2,\ \ \text{repeated})$$

Third Instance of the Second Cosine Law:

$$\cos\Delta\ \sin\beta = \sin Z\cos l\cos\phi + \sin l\cos Z \quad (28.5)$$

$$\cos\phi\ \sin Z = \cos\Delta\sin\beta\cos l + \sin l\cos\beta \qquad (28.6)$$

Substitute (28.5) into (28.6) to eliminate $\cos\Delta\sin\beta$:

$$\cos\phi\ \sin Z = (\sin Z\cos l\cos\phi + \sin l\cos Z)\cos l + \sin l\cos\beta$$
$$\cos\phi\ \sin Z = (\sin Z\cos l\cos\phi + \sin l\cos Z)\cos l + \sin l\cos\beta$$
$$\cos\phi\ \sin Z = \sin Z\cos^2 l\cos\phi + \sin l\cos l\cos Z + \sin l\cos\beta$$
$$\cos\phi\ \sin Z\ \left(1-\cos^2 l\right) = \sin l\cos l\cos Z + \sin l\cos\beta$$
$$\cos\phi\ \sin Z\sin^2 l = \sin l\cos l\cos Z + \sin l\cos\beta$$
$$\cos\phi\ \sin Z\sin l = \cos l\cos Z + \cos\beta$$
$$\therefore \quad \boxed{\cos\beta = -\cos l\cos Z + \sin l\sin Z\cos\phi} \qquad (27.3,\ \ \text{repeated})$$

2 Right Triangles and Short-Method Tables

For three points on the surface of the earth, NP, M and P, there are two possible polar spherical triangles, the smaller (minor) local triangle and the larger (major) remote triangle on the other side of the earth. The coaltitude γ spans the smaller spherical triangle, and it is this spherical triangle with which we are concerned. The major spherical triangle, spanned by $360° - \gamma$, is of no interest to us here.

The spherical triangles of Figures 7A and 7B can be cut with great circles into two adjacent right spherical triangles via an auxiliary great circle. An advantage of doing so is that the equations describing them are simplified. The sine of 90° degrees is one and the cosine of 90° is zero, rendering one or more terms in the equations constant, 0 or 1.

Several relatively short mathematical tables have been compiled using these equations to "solve the navigational triangle", that is, to compute the altitude and azimuth of a celestial body based upon its LHA and the observer's latitude L.

Two different methods are employed to accomplish this based upon an auxiliary great circle configured in one of two ways. In the past, a number of investigators have derived and published their short-method trigonometric derivations and tables, but all are somewhat similar to one-another.

Ogura's Method: The new great circle passes through the observer's position M and intersects the *declination* great circle at a right angle. Reference: Figures 10A and 10B. This method is employed in the short-method navigational tables of Sintiki Ogura (1884-1937), the *Line of Position Book* [11] of P.V.H Weems, H.O. 208 of Dreisonstock, among others.

Ageton's Method: The new great circle passes through the geographical position P (or GP) of the celestial body and intersects the *colatitude* great circle at a right angle. Reference: Figures 11A and 11B.

This method is employed in H.O 211 by Arthur Ageton (1931) and a modified H.O 211 by Allan Bayless (1980).

"H.O." is an abbreviation for "Hydrographic Office", the name of the American government agency that formerly dealt with navigational charts.

In Figures 10A, 10B, 11A and 11B, note the arcs of great circles between M and P' (Ogura's Method) and between P and M' (Ageton's Method), ($\overset{\frown}{MP'}$ and $\overset{\frown}{PM'}$ respectively). As described above, these great circles cut the navigational triangles into two adjacent right-angled spherical triangles. The two companion right-angled spherical triangles each have more simple algebraic and trigonometric forms than the original triangles, and are used in compiling the "short-method" mathematical tables for celestial navigation.

The original polar spherical triangle and the upper right-angled polar spherical triangle can be analyzed with the original equations (15.1, 15.2, 15.3). Both of these spherical triangles are directly described via spherical coordinates. The upper right-angled polar spherical triangle with vertices NP, M, P' is sometimes referred to as the *Time triangle* [6], and the lower right-angled non-polar spherical triangle with vertices P', M, P is sometimes referred to as the *Altitude triangle* [6]. The altitude triangle cannot be *directly* described via spherical coordinates in the same manner as the other two triangles, because it does not have a vertex at NP.

For economy of space in the figures, the single letter symbols M and P are used. M represents the observer's position, usually designated as AP (assumed position), and P represents the geographical position of the celestial body, usually designated as GP. The symbols l and $\Delta\theta$ are aliases of each other.

In Figures 10A and 10B, the expressions "P East of M" and "P West of M" simply indicate that these figures represent the extreme configurations of the spherical triangle(s) for small and large l , i.e., for large and small GHA. The same trigonometric equations are used to "solve" these spherical triangles.

The domain of the coaltitude γ, as a standard spherical coordinate, is $[0, 180°]$. However, in marine celestial navigation, since $\gamma = 90° - h$, γ is further restricted to $[0, 90°]$. Altitudes h below the horizon are not typically measured by the marine sextant. Hence, if we know $\sin\gamma$, we can *uniquely* determine γ, an angle in the first trigonometric quadrant.

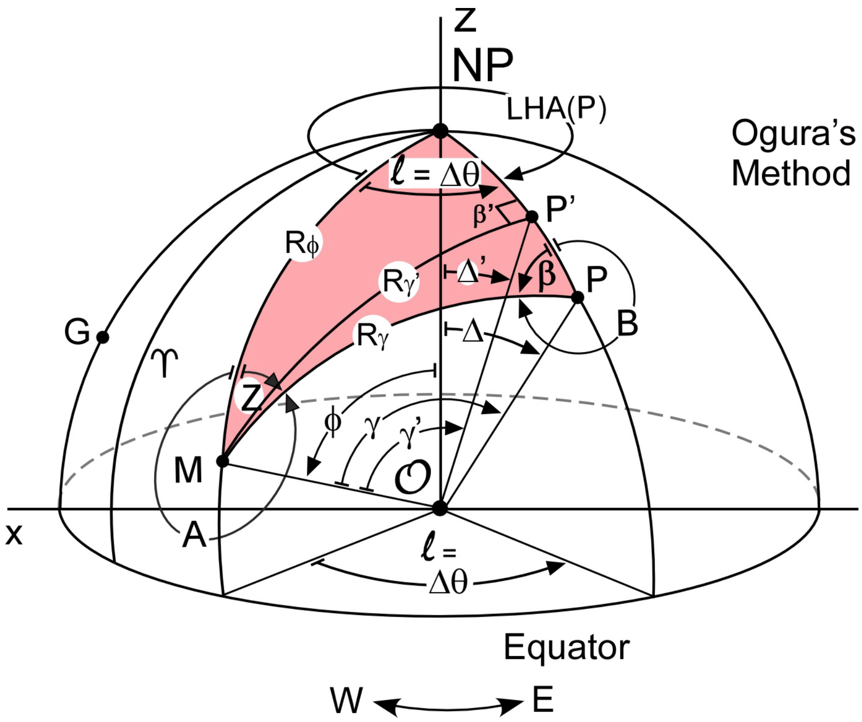


Figure 10A: Spherical Triangle, P East of M.

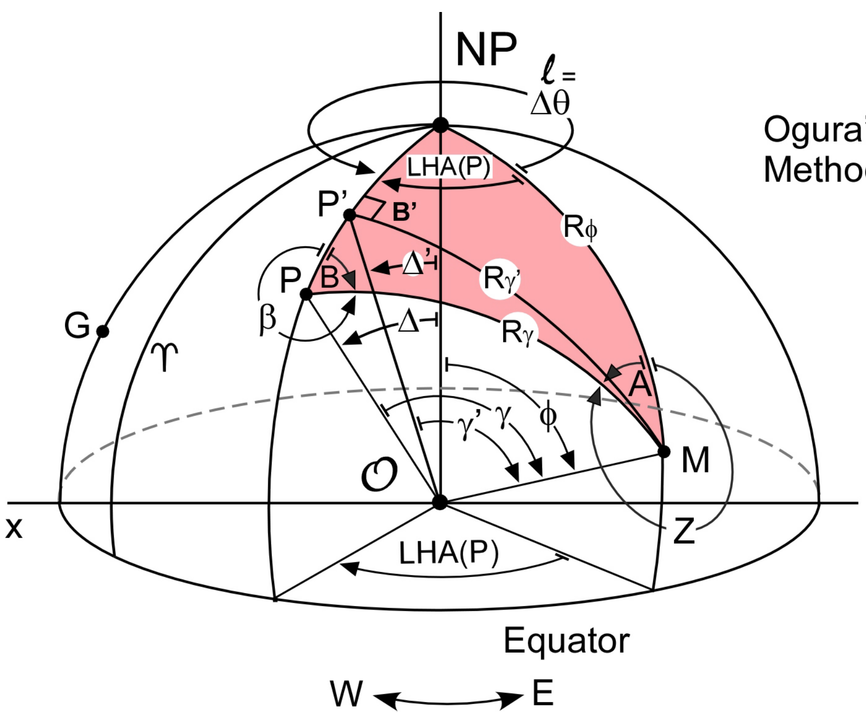


Figure 10B. Spherical Triangle 1, P West of M.

2.1 Splitting the Codeclination, Ogura's Method

For the upper spherical triangle, we use the spherical sine formulas.

$$\frac{\sin l}{\sin \gamma} = \frac{\sin \beta}{\sin \phi} = \frac{\sin Z}{\sin \Delta} \quad \text{(25, repeated)}$$

Domains of the angles:

$$\begin{aligned} D(\gamma) &= [0, 180°] \quad \Rightarrow \quad (\gamma) : \sin \gamma \geq 0 \\ D(\phi) &= [0, 180°] \quad \Rightarrow \quad (\phi) : \sin \phi \geq 0 \end{aligned}$$

However, $D(\beta) = D(l) = [0, 360°]$, so $\sin \beta$ and $\sin l$ may be positive or negative.

But for the ratios

$$\frac{\sin \beta}{\sin l} = \frac{\sin \phi}{\sin \gamma} \geq 0 \quad \text{and} \quad \frac{\sin \beta'}{\sin l} = \frac{\sin \phi}{\sin \gamma'} \geq 0$$

$$\therefore \; sign(\sin \beta) = sign(\sin l).$$

$$\beta' = 90° \;\Rightarrow\; \sin \beta' = 1 \;\text{ and }\; \beta' = 270° \;\Rightarrow\; \sin \beta' = -1.$$

$$\boxed{\sin \gamma' = \sin \phi \, \frac{\sin l}{\sin \beta'} = \sin \phi \; |\sin l| \geq 0} \quad (29)$$

$$\sin \gamma \sin \beta = \sin \phi \sin l \quad \text{and} \quad \sin \gamma' \sin \beta' = \sin \phi \sin l$$

$$\sin \gamma' \sin \beta' = \sin \gamma \sin \beta$$

$$\frac{\sin \gamma'}{\sin \gamma} = \frac{\sin \beta}{\sin \beta'}$$

Observe in Figure 10A that as l decreases with M moving to the right, β increases and P' moves down the codeclination great circle and Δ' increases. When $l = 0$ (i.e., $360°$), $\beta = 180°$, M and P lie on the same meridian. As M moves further *to the right (east)* of P and l decreases from $360°$, β decreases, the point P' moves up the codeclination great circle, Δ' decreases until finally $\beta' = 270°$.

As M moves from its original position to the left (westward) from P, l increases and Δ' decreases. When $l = 180°$, P' moves up to NP. The angle β' is always equal to $90°$ or $270°$.

$sign\ \beta' = sign\ \beta$. Therefore, $\dfrac{\sin \beta}{\sin \beta'} \geq 0$ and $sign\,(\sin \gamma') = sign\,(\sin \gamma)$.

Derivation.

$$\sin \gamma \cos \beta = \sin \Delta \cos \phi - \cos \Delta \sin \phi \cos l \quad \text{(23.1, repeated)}$$

$$\sin\gamma'\cos\beta' = \sin\Delta'\cos\phi - \cos\Delta'\sin\phi\cos l$$

$\beta' = 90^\circ$ or $\beta' = 270^\circ \quad \Rightarrow \; \cos\beta' = 0.$

$$\therefore \quad \sin\Delta'\cos\phi = \cos\Delta'\sin\phi\cos l \qquad (30)$$
$$\Rightarrow \quad \tan\Delta' = \tan\phi\cos l$$

Δ' is the codeclination of point P' and its domain is $[0, 180^\circ]$. If $\tan\Delta' \geq 0$, then Δ' lies in the 1^{st} trigonometric quadrant; if $\tan\Delta' < 0$, then Δ' lies in the 2^{nd} trigonometric quadrant. Thus, $\Delta' = \arctan(\tan\Delta')$ *uniquely* determines the value of the angle Δ'.

Derivation.

$$\cos\phi = \cos\gamma\cos\Delta + \sin\gamma\sin\Delta\cos\beta \quad \text{(26.3, repeated)}$$

$$\cos\phi = \cos\gamma'\cos\Delta' + \sin\gamma'\sin\Delta'\cos\beta'$$

$$\therefore \quad \cos\phi = \cos\Delta'\cos\gamma'$$

$$\boxed{\cos\Delta' = \frac{\cos\phi}{\cos\gamma'}} \qquad (31)$$

Substituting $\cos\phi = \cos\Delta'\cos\gamma'$ into equation (30), we have...

$$\sin\Delta'\cos\gamma' = \sin\phi\cos l \qquad (32)$$

Derivation.

$$\cos\gamma = \cos\Delta\cos\phi + \sin\Delta\sin\phi\cos l \quad \text{(10.3, repeated)}$$

From (31), $\cos\phi = \cos\Delta'\cos\gamma'$ and from (32), $\sin\Delta'\cos\gamma' = \sin\phi\cos l$. Making these replacements,

$$\cos\gamma = \cos\Delta\,(\cos\Delta'\cos\gamma') + \sin\Delta\,(\sin\Delta'\cos\gamma')$$

$$\cos\gamma = \cos\gamma'\,(\cos\Delta\cos\Delta' + \sin\Delta\sin\Delta')$$

$$\therefore \quad \boxed{\cos\gamma = \cos\gamma'\cos(\Delta - \Delta')} \qquad (33)$$

Since the cosine function $\cos(\Delta - \Delta') = \dfrac{\cos\gamma}{\cos\gamma'}$ and $\cos(\Delta - \Delta') < 1$, so $\dfrac{\cos\gamma}{\cos\gamma'} < 1$ and $\cos\gamma < \cos\gamma' \quad \Rightarrow \quad \gamma' < \gamma$.

Derivation.

Recall

$$\sin\phi\cos\lambda = \sin\Lambda\sin\gamma\sin\beta + \cos\Lambda\left(\sin\Delta\cos\gamma - \cos\Delta\sin\gamma\cos\beta\right) \quad \text{(26.1, repeated)}$$
$$\sin\phi\sin\lambda = \sin\Lambda\sin\Delta\cos\gamma - \sin\gamma\left(\cos\Lambda\sin\beta + \cos\Delta\sin\Lambda\cos\beta\right) \quad \text{(26.2, repeated)}$$
$$\cos\phi = \cos\gamma\cos\Delta + \sin\gamma\sin\Delta\cos\beta \quad \text{(26.3, repeated)}$$

$\sin\beta' = \pm 1$, depending upon β' with $\cos\beta' = 0$. Therefore,

$$\begin{aligned}
&\sin\phi\cos\lambda = \sin\Lambda'\sin\gamma'\sin\beta' + \cos\Lambda'\sin\Delta'\cos\gamma' \\
&\sin\phi\sin\lambda = \sin\Lambda'\sin\Delta'\cos\gamma' - \sin\gamma'\cos\Lambda'\sin\beta' \\
&\cos\phi = \cos\gamma'\cos\Delta'
\end{aligned}$$

Re-arrange terms in second equation:

$$\begin{aligned}
&\sin\phi\cos\lambda = \sin\Lambda'\sin\gamma'\sin\beta' + \cos\Lambda'\sin\Delta'\cos\gamma' \\
&\sin\phi\sin\lambda = -\cos\Lambda'\sin\gamma'\sin\beta' + \sin\Lambda'\sin\Delta'\cos\gamma' \\
&\cos\phi = \cos\gamma'\cos\Delta'
\end{aligned}$$

Multiplying the 1^{st} equation by $\sin\lambda$ and the 2^{nd} equation by $\cos\lambda$,

$$\begin{aligned}
&\sin\phi\cos\lambda\sin\lambda = \sin\Lambda'\sin\gamma'\sin\beta'\sin\lambda + \cos\Lambda'\sin\Delta'\cos\gamma'\sin\lambda \\
&\sin\phi\sin\lambda\cos\lambda = -\cos\Lambda'\sin\gamma'\sin\beta'\cos\lambda + \sin\Lambda'\sin\Delta'\cos\gamma'\cos\lambda \\
&\cos\phi = \cos\gamma'\cos\Delta'
\end{aligned}$$

Subtract the 2nd equation from the 1st equation:

$$\begin{aligned}
0 \;=\;& \sin\Lambda'\sin\gamma'\sin\beta'\sin\lambda + \cos\Lambda'\sin\gamma'\sin\beta'\cos\lambda + \cos\Lambda'\sin\Delta'\cos\gamma'\sin\lambda \\
& - \sin\Lambda'\sin\Delta'\cos\gamma'\cos\lambda
\end{aligned}$$

$$0 = \sin\gamma'\sin\beta'\left(\cos\Lambda'\cos\lambda + \sin\Lambda'\sin\lambda\right) - \sin\Delta'\cos\gamma'\left(\sin\Lambda'\cos\lambda - \cos\Lambda'\sin\lambda\right)$$
$$\sin\gamma'\sin\beta'\cos\left(\Lambda' - \lambda\right) = \sin\Delta'\cos\gamma'\sin\left(\Lambda' - \lambda\right)$$
$$\tan\gamma'\sin\beta' = \sin\Delta'\tan\left(\Lambda' - \lambda\right)$$

But $\Lambda' = \Lambda$ and $l = \Lambda - \lambda$, so

$$\tan\gamma'\sin\beta' = \sin\Delta'\tan l \quad (34\ )$$

2.1.1 The Line of Position Book

In the Line of Position Book (LPB), Weems uses the methods of Sintiki Ogura to split the spherical navigational triangle into two right-angled spherical triangles [11]. The right angles occur on the declination great circle. Dreisonstock used a similar method for his tables, H.O. 208. The LPB splits the calculations into two parts, each part using fairly simple equations to solve the spherical triangle. Since the Line of Position Book tables are tabulated only for $\lambda - \Lambda \; \varepsilon \; (0, 180°)$, the meridian angle t (rather than LHA) and observer's latitude L (DR or AP) are input into the LPB tables. Upon completing the two part calculations, the final outputs are the computed altitude h_c and azimuth Z of the celestial body. However, in our derivations, we will use the more general LHA rather than meridian angle t.

Ogura's method and similar methods were developed before the advent of modern digital computers. In legacy practice of spherical trigonometry and celestial navigation, cosecants and secants were usually employed instead of sines and cosines, because these functions always possess values greater than or equal to one. Furthermore, the 10^5 factor was used to generate large integers. It is peculiar that in the LPB, Weems and Lee omit mention of the 10^5 factor in their explanation. Logarithms were then used to evaluate products or ratios of these positive integers; this practice, done with paper and pencil, made the computational process possible [11].

First Part Calculation. Entering with LHA and L, the output from the first part calculation consists of two numbers, $\mathcal{A}$ and $\mathcal{K}$, where

$$\mathcal{A} = 10^5 \log \sec \gamma' \; = 10^5 \log \sec (90° - h') = 10^5 \log \csc h' \quad (35.1)$$

$$\mathcal{K} = d' = 90° - \Delta' \quad (35.2)$$

in which we have written the symbols $\mathcal{A}$ and $\mathcal{K}$ in script (calligraphic) form to distinguish the first symbol from the symbol A which is already used to represent the coazimuth of P. γ' and Δ' are the coaltitude and codeclination that the celestial body would have if its altitude great circle and declination great circle intersected at $90°$ as shown in Figure 10A, that is, if it were located at point P' in that figure.

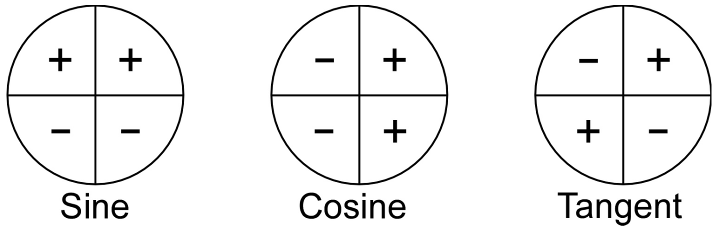
Hence, $(LHA, L) \Rightarrow (\mathcal{A}, \mathcal{K}) \Rightarrow (h', d')$, that is, $(l, \phi) \Rightarrow (\gamma', \Delta')$. The numbers $\mathcal{A}$ and $\mathcal{K}$ are obtained from Table A of the LPB.

Compute γ'.

The LPB uses (29) with $L = 90° - \phi$ to evaluate γ'.

$$\sin \gamma' = \sin \phi \; |\sin l| \quad \text{(29, repeated)}$$

In the First Part Calculation in the Line of Position Book, the angle γ' is calculated simply by taking the arcsine of $\sin\gamma'$. This is sufficient to uniquely evaluate the angle γ', because in marine navigation γ' is always an angle in the first trigonometric quadrant, that is, less than or equal to $90°$.



For aerial or space navigation, in which we might measure negative sextant angles (i.e., below the local horizontal plane), the coaltitudes would be angles in the second trigonometric quadrant. Keep in mind that the angles γ and γ' are normal spherical coordinates just like ϕ and Δ whose mathematical domains are $[0, 180°]$. In that case, equation 28 alone would be insufficient to uniquely determine γ'. We would require quadrant justification via, e.g., the cosine or the tangent function. This situation will be addressed shortly.

Compute d'.
Having just evaluated γ', equation 31 is now used to determine d'.

$$\begin{aligned} \cos\Delta' &= \frac{\cos\phi}{\cos\gamma'} \quad \text{(31, repeated)} \\ d' &= 90° - \Delta' \end{aligned}$$

End of first part calculations.

Second Part Calculation.
Definitions:

$$K \sim d \overset{D}{=} d' - d = (90° - \Delta') - (90° - \Delta) = \Delta - \Delta' \quad (35.3)$$

If the declinations d and d' are South declinations, they are negative numbers.

$$\mathcal{B} \overset{D}{=} 10^5 \log\sec(K \sim d) = 10^5 \log\sec(\Delta - \Delta') = 10^5 \log\frac{1}{\cos(\Delta - \Delta')} \quad (35.4)$$

Again, a script (calligraphic) font is used for this quantity to distinguish it from the angle B.

$K \sim d$ and $\mathcal{B}$ are obtained from Table B of the LPB. $\mathcal{A}$ and $\mathcal{B}$ are then added together.

$$\begin{aligned} \mathcal{A} + \mathcal{B} &= 10^5 \log\sec\gamma' + 10^5 \log\sec(\Delta - \Delta') \\ &= 10^5 \log[\sec\gamma' \sec(\Delta - \Delta')] \end{aligned}$$

But

$$\cos\gamma = \cos\gamma' \cos(\Delta - \Delta') \quad \text{(33, repeated)}$$

so

$$\sec\gamma = \sec\gamma' \sec(\Delta - \Delta')$$

and

$$\mathcal{A} + \mathcal{B} = 10^5 \log \sec\gamma$$

The sum $\mathcal{A}$ and $\mathcal{B}$ is then entered into Table B of the LPB to obtain the computed altitude h. The degrees of altitude appear on the bottom of the page and the minutes of altitude in the right margin.

.

Azimuth Calculation. In the Line of Position Book, the azimuth Z of P is obtained from "Rust's Diagram". This diagram is based upon (25) $\dfrac{\sin l}{\sin\gamma} = \dfrac{\sin Z}{\sin\Delta}$ $\Rightarrow \quad \sin Z = \dfrac{\sin\Delta \sin l}{\sin\gamma}$, except that Weems uses meridian angle t (i.e., HA) instead of l.

End of LPB calculations (Weems).

2.1.2 Numerical Example

Suppose that $L = 35°N \quad \Rightarrow \quad \phi = 55°, \;\; LHA = 48° \quad \Rightarrow \quad l = 312°$.

$d = 20°S = -20° \quad \Rightarrow \quad \Delta = 110°$.

$\sin\gamma' = \sin\phi \; |\sin l| = \sin 55° \; |\sin 312°| = (0.8195) \; |-0.74314| = 0.6088$

$\gamma' = \arcsin(\sin\gamma') = 37.499°$ or $142.501°$. In marine celestial navigation we are only interested in the value in the first trigonometric quadrant, so $\gamma' = 37.50°$. The LPB goes no further in trigonometric quadrant determination of angle γ'.

From (31), $\cos\Delta' = \dfrac{\cos\phi}{\cos\gamma'} = \dfrac{\cos 55°}{\cos 37.50°} = 0.7230$, and $\Delta' = arcos(0.7230) = 43.70°$.

$\Delta - \Delta' = 110° - 43.70° = 66.30°$.

From (33), $\cos\gamma = \cos\gamma' \cos(\Delta - \Delta') = \cos 37.50° \cos(66.30°) = 0.3189$, and $\gamma = arcos(0.3189) = 71.40° \quad \Rightarrow \quad h = 18.60°$.

Digression.

With computer computation for aerial or space navigation, wherein which h (and h') might assume negative values, we can proceed in a somewhat different manner than that employed in the LPB. Write

$$\tan\Delta' = \tan\phi \cos l \quad \text{(30, repeated)}$$

$\tan\Delta' = \tan 55° \cos 312° = 0.9556$. $\Delta' = \arctan(0.9556) \quad \Rightarrow \quad 43.70°$ with the positive sign $(+)$ of Δ' *uniquely* determining that this Δ' lies in the 1^{st} trigonometric quadrant (rather than in the 2^{nd}).

From (31), we write $\cos\gamma' = \dfrac{\cos\phi}{\cos\Delta'} = \dfrac{\cos 55°}{\cos 43.70°} = 0.793\,4$.

$\gamma' = arcos(0.793\,4) = 37.50°$ with the positive sign $(+)$ of γ' *uniquely* determining that γ' lies in the 1^{st} trigonometric quadrant (rather than in the 2^{nd}).

$\Delta - \Delta' = 110° - 43.70° = 66.30°$.
$\cos(\Delta - \Delta') = \cos(66.30°) = 0.4020$.
From (33), $\cos\gamma = \cos\gamma' \cos(\Delta - \Delta') = (0.7934)\,(0.4020) = 0.3190$.
$\gamma = arcos\ (0.3190) = 71.40° \ \Rightarrow \ h = 18.60°$.

Computations via the LPB Logarithmic Tables Round off to a whole number.

$$\mathcal{A} = 10^5 \log\sec\gamma' \ = 10^5 \log \frac{1}{\cos 37.50°} \ = 10,053$$

$$\mathcal{B} = 10^5 \log\sec(K \sim d) = 10^5 \log \frac{1}{\cos(\Delta - \Delta')} = 10^5 \log \frac{1}{0.40195} = 39,583$$

$$\mathcal{A} + \mathcal{B} = 10,053 + 39,583 = 49,636$$

$$(\mathcal{A} + \mathcal{B})\, 10^{-5} = 0.49636 = \log \frac{1}{\cos\gamma} \quad \Rightarrow \quad \cos\gamma = 10^{-0.49636} = 0.3189 > 0$$

that is, 1^{st} quadrant.

$$\gamma = arcos\ (0.3189) = 71.40°$$

or, since $\sin h = \sin(90° - \gamma) = \cos\gamma$, $\sin h = 0.3189 \quad \Rightarrow \quad h = 18.60°$ (confirmation).

Azimuth calculation:
Again, from (25),

$$\sin Z = \frac{\sin\Delta \sin l}{\sin\gamma} = \frac{\sin 110° \sin 312°}{\sin 71.40°} = -0.7369$$

$$Z = \arcsin(-0.7369) = -47.46° = 227.46° \ \text{ or } \ 312.54°$$

$\Rightarrow$ that is, this evaluation indicates that Z lies in the 3^{rd} or 4^{th} trigonometric quadrant.

The LPB predicts one of these values and provides directions for the navigator to eliminate the quadrant ambiguity. However, in this example, we will provide mathematical determination of the angle.

From (16.3),

$$\cos Z \equiv \cos A = \frac{\cos\Delta - \cos\phi\cos\gamma}{\sin\phi\sin\gamma} = \frac{\cos 110^\circ - \cos 55^\circ \overbrace{\cos 71.4042^\circ}^{0.3190}}{\sin 55^\circ \sin 71.4042^\circ}$$
$$= -0.6761 < 0$$

$$Z = arcos\ (-0.6761) = 132.54^\circ \quad \text{or} \quad 227.46^\circ$$

$\Rightarrow$ that is, this evaluation indicates that Z lies in the 2^{nd} or 3^{rd} trigonometric quadrant.

$$Z_n = \arctan 2\,(\cos Z, \sin Z) = \ 227.46^\circ$$

End of LPB Numerical Example.

2.2 Dividing the Colatitude, Ageton's Method

Figures 11A and 11B display the great circles

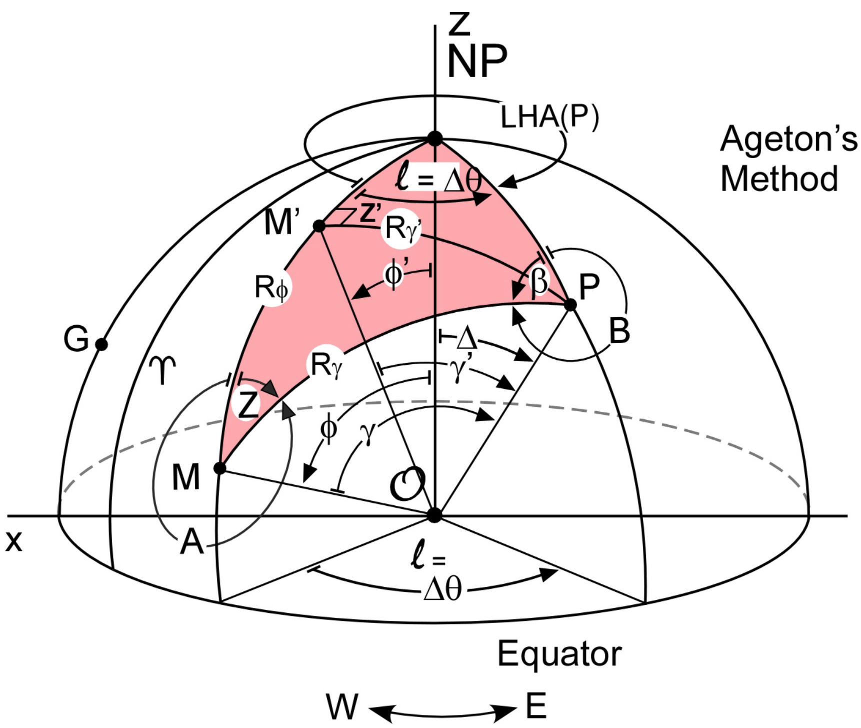


Figure 11A: Spherical Triangle, P East of M.

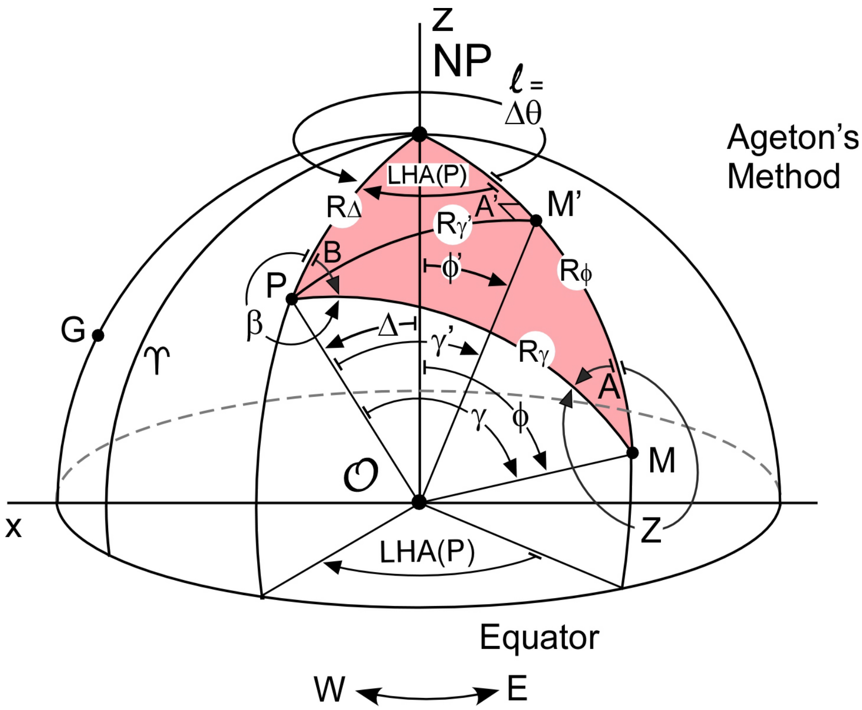


Figure 11B: Spherical Triangle 2, P West of M.

This is the geometry used by Ageton in his "short-method" tables. In order to avoid symbol congestion, primes (rather than, e.g., double primes) are attached to some of the angle symbols. **These must not be confused with the same primed symbols employed in the section of this paper dealing with Ogura's method.**

Again, for the upper spherical triangle in Figure 11A of Figure 11B, we make use of Equations 25 and write...

$$\sin\gamma \sin Z = \sin\Delta \sin l$$
$$\sin\gamma' \sin Z' = \sin\Delta \sin l$$

Domains of the angles:

$$\begin{aligned} D(\gamma) &= [0, 180°] \quad \Rightarrow \quad (\gamma) : \sin\gamma \geq 0 \\ D(\Delta) &= [0, 180°] \quad \Rightarrow \quad (\Delta) : \sin\Delta \geq 0 \end{aligned}$$

However, $D(Z) = D(l) = [0, 360°]$, so $\sin Z$ and $\sin l$ may be positive or negative.

But

$$\frac{\sin Z}{\sin l} = \frac{\sin\Delta}{\sin\gamma} \geq 0 \quad \text{and} \quad \frac{\sin Z'}{\sin l} = \frac{\sin\Delta}{\sin\gamma'} \geq 0$$
$$\therefore \; sign(\sin Z) = sign(\sin l).$$

$$\sin\gamma' = \sin\Delta\frac{\sin l}{\sin Z'}$$

$$\boxed{\sin\gamma' = \sin\Delta\,\frac{\sin l}{\sin Z'} = \sin\Delta\,\left|\sin l\right|} \quad (36)$$

$$\cos\Delta = \cos\phi\cos\gamma + \sin\phi\sin\gamma\cos A \quad (16.3,\text{ repeated})$$

$$\cos\Delta = \cos\phi'\cos\gamma' + \sin\phi'\sin\gamma'\cos A'$$

$$\cos\Delta = \cos\phi'\cos\gamma' + \sin\phi'\sin\gamma'\cos\left(360^\circ - Z'\right)$$

$$\cos\Delta = \cos\phi'\cos\gamma' + \sin\phi'\sin\gamma'\cos Z'$$

$Z' = 90^\circ \Rightarrow \quad \cos Z' = 0.$ Therefore, $\cos\Delta = \cos\phi'\cos\gamma'$.

$$\boxed{\cos\phi' = \frac{\cos\Delta}{\cos\gamma'}} \quad (37)$$

Derivation.

$$\sin\gamma\cos A = \cos\Delta\sin\phi - \cos\phi\sin\Delta\cos l \quad (10.1,\text{ repeated})$$

$$\sin\gamma'\cos A' = \cos\Delta\sin\phi' - \cos\phi'\sin\Delta\cos l$$

But $A' = 90^\circ$ or 270°; therefore, $\cos A' = 0$,

$$\cos\Delta\sin\phi' = \cos\phi'\sin\Delta\cos l$$

But from (37), $\cos\Delta = \cos\phi'\cos\gamma'$, so

$$\left(\cos\phi'\cos\gamma'\right)\sin\phi' = \cos\phi'\sin\Delta\cos l$$

$$\cos\gamma'\sin\phi' = \sin\Delta\cos l \quad (38)$$

$$\cos\gamma = \cos\Delta\cos\phi + \sin\Delta\sin\phi\cos l \quad (23.3,\text{ repeated})$$

$$\cos\gamma = \left(\cos\phi'\cos\gamma'\right)\cos\phi + \sin\phi\left(\cos\gamma'\sin\phi'\right)$$

$$\cos\gamma = \cos\gamma'\left(\cos\phi\cos\phi' + \sin\phi\sin\phi'\right)$$

$$\boxed{\cos\gamma = \cos\gamma'\cos\left(\phi - \phi'\right) \quad (39)}$$

End of calculations via Ageton's Method.

Recapitulation The sequence of calculations used in the LPB (Sintiki Ogura's Method):

1. Use (29) $\sin\gamma' = \sin\phi\,|\sin l|$ and $\gamma' = \arcsin(\sin\gamma')$, *assumed* to lie in the 1^{st} trigonometric quadrant.
2. Use (31), $\cos\Delta' = \dfrac{\cos\phi}{\cos\gamma'}$ and $\Delta' = arcos(\cos\Delta')$ to *uniquely* calculate the angle Δ'.
3. Use (33), $\cos\gamma = \cos\gamma'\cos(\Delta - \Delta')$ and $\gamma = arcos(\cos\gamma)$ to calculate the angle γ, *assumed* to lie in the 1^{st} trigonometric quadrant.

Another way of performing the calculations (This sequence not followed in the LPB):

1. Use (30), $\tan\Delta' = \tan\phi\cos l$ and $\Delta' = \arctan(\tan\Delta')$ to *uniquely* calculate the angle Δ'. $sign(\cos\Delta') = sign(\tan\Delta')$.
2. From (31), calculate $\cos\gamma' = \dfrac{\cos\phi}{\cos\Delta'}$.
3. Use (33), $\cos\gamma = \cos\gamma'\cos(\Delta - \Delta')$ and $\gamma = arcos(\cos\gamma)$ to *uniquely* calculate the angle γ.

The sequence of calculations used in Ageton's Method:

1. Use (36), $\sin\gamma' = \sin\Delta\,|\sin l|$ and $\gamma' = \arcsin(\sin\gamma')$, *assumed* to lie in the 1^{st} trigonometric quadrant.
2. Use (37), $\cos\phi' = \dfrac{\cos\Delta}{\cos\gamma'}$ and $\phi' = arcos(\cos\phi')$ to *uniquely* calculate the angle ϕ'.
3. Use (39), $\cos\gamma = \cos\gamma'\cos(\phi - \phi')$ and $\gamma = arcos(\cos\gamma)$ to calculate the angle γ, *assumed* to lie in the 1^{st} trigonometric quadrant.

2.3 Appendix

2.3.1 The Navigational Tetrahedron

Consider spherical triangles such as those of Figures 7A and 7B. Figure 12, below, is representative of a tetrahedron in a general spherical triangle. We will use this figure to derive equations suitable for its description.

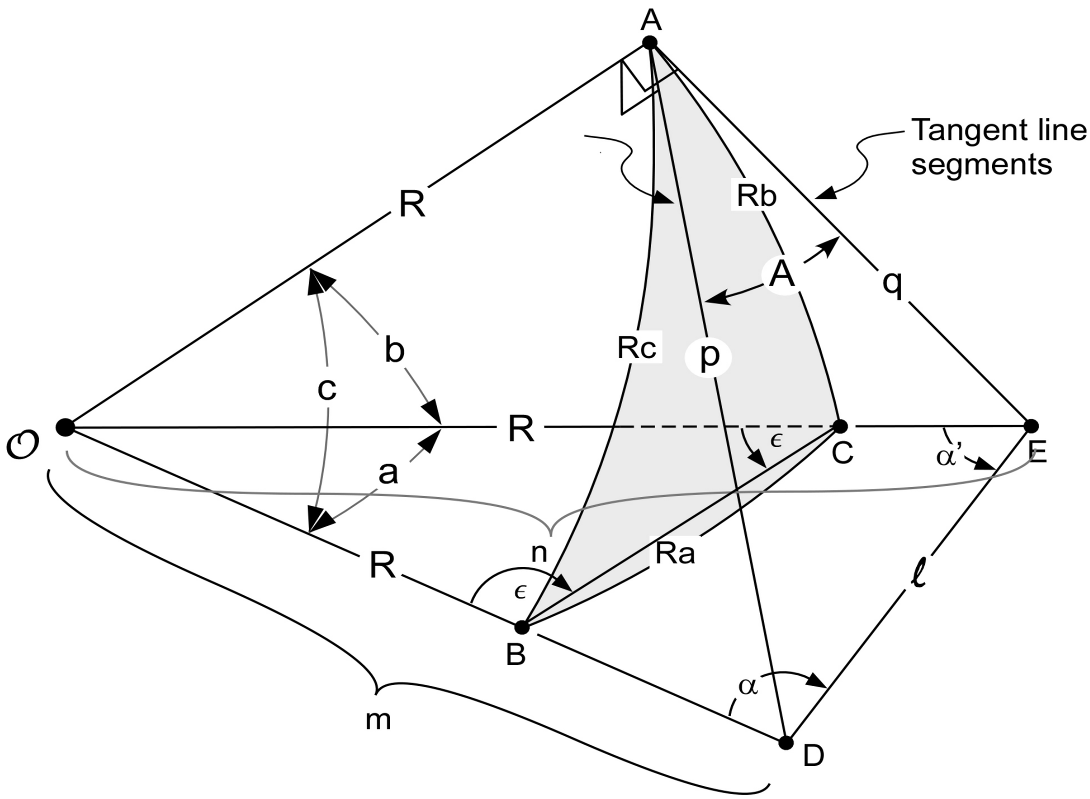


Figure 12: Details of the Navigational Tetrahedron.

The *internal angles* a, b, c (expressed in the lower case above) subtend circular arcs. The *surface angles* A, B, C are angles between tangent lines of intersecting circular arcs.

We derive equations on the basis of all angles less than 180° and greater than or equal to zero (no unique directions).

$$\begin{array}{lll} m = \overline{OE} & n = \overline{OD}, & l = \overline{DE} \\ p = \overline{AE}, & q = \overline{AD}, & R = \overline{OA} \\ R = \overline{OB}, & R = \overline{OC} & \end{array}$$

The Spherical Trigonometry Law of Cosines

$$\begin{aligned} l^2 &= m^2 + n^2 - 2mn\cos a \\ l^2 &= p^2 + q^2 - 2pq\cos A \\ m^2 &= R^2 + p^2 \\ n^2 &= R^2 + q^2 \end{aligned}$$

Subtract the second equation above from the first equation:

$$0 = m^2 - p^2 + n^2 - q^2 - 2mn\cos a + 2pq\cos A$$

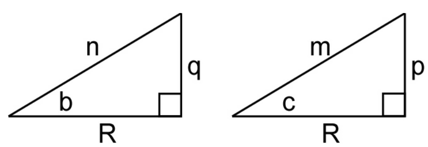
But $m^2 - p^2 = R^2$ and $n^2 - q^2 = R^2$, so

$$0 = R^2 - mn\cos a + pq\cos A$$

$$mn\cos a = R^2 + pq\cos A$$

$$\cos a = \frac{R}{n}\frac{R}{m} + \frac{p}{n}\frac{q}{m}\cos A$$

In Figure 15, notice that the angles $\angle OAD$ and $\angle OAE$ are right angles.



But $\quad \cos b = \dfrac{R}{n},\ \cos c = \dfrac{R}{m},\ \sin b = \dfrac{q}{n},\ \sin c = \dfrac{p}{m}.$

Then,

$$\boxed{\cos a = \cos b\cos c + \sin b\sin c\cos A} \quad (40.1)$$

Likewise for the other two tangent lines, chords and angles. By similar reckoning for the other vertices, lines and angles, we have the three "cosine equations" of spherical trigonometry: $a \leftarrow b,\ b \leftarrow c,\ c \leftarrow a$.

$$\boxed{\cos b = \cos c\cos a + \sin c\sin a\cos B} \quad (40.2)$$

$$\boxed{\cos c = \cos a\cos b + \sin a\sin b\cos C} \quad (40.3)$$

Temporary symbols $\epsilon, \alpha, \alpha'$ lying in the plane OED.

Problem: Determine whether $\overline{CB}//\overline{DE}$. $\overline{CB}//\overline{DE}$ if and only if angle $\alpha = \epsilon$.

In triangle OBC, the two sides $\overline{OB}$ and $\overline{OC}$ are both equal to R; Hence, the same angle ϵ on both sides of the triangle. In triangle ODE, we may not assume that

$m = n$; $m = \dfrac{R}{\cos c}$ and $n = \dfrac{R}{\cos b}$. But angle b is not necessarily equal to angle c, so m is not necessarily equal to n and thus, angle α' is not necessarily equal to angle α. Consequently, $\overline{CB}$ is not necessarily parallel to $\overline{DE}$, i.e., l.

<u>The Spherical Trigonometry Law of Sines</u>

From (40.1),

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$\sin^2 A = 1 - \cos^2 A = 1 - \frac{(\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$$

$$\begin{aligned} \sin^2 A &= \frac{\sin^2 b \sin^2 c - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c} \\ &= \frac{(1 - \cos^2 b)(1 - \cos^2 c) - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c} \end{aligned}$$

$$\begin{aligned} \sin^2 A &= \frac{\sin^2 b \sin^2 c - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c} \\ &= \frac{1 - \cos^2 b - \cos^2 c + \cos^2 b \cos^2 c - (\cos^2 a - 2 \cos a \cos b \cos c + \cos^2 b \cos^2 c)}{\sin^2 b \sin^2 c} \end{aligned}$$

$$\sin^2 A = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 b \sin^2 c}$$

$$\frac{\sin^2 A}{\sin^2 a} = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 a \sin^2 b \sin^2 c}$$

$$\frac{\sin A}{\sin a} = \frac{\sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}}{\sin a \sin b \sin c}$$

When we do likewise for $\dfrac{\sin B}{\sin b}$ and $\dfrac{\sin C}{\sin c}$, the right hand side of each of these equations is equal to

$$\frac{\sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}}{\sin a \sin b \sin c}.$$

From this we have what are known as the Law of Sines for Spherical Trigonometry:

$$\boxed{\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}} \qquad (41)$$

Equations 40.1, 40.2 and 40.3 express $\cos a$, $\cos b$ and $\cos c$ as trigonometric functions of angles a, b , c, A, B , C. Similarly, we would like to create some kind of converse relationship to these equations, expressing $\cos A$, $\cos B$ and $\cos C$ as trigonometric functions of angles a, b , c, A, B , C. We refer to this "converse relationship" as the *second spherical law of cosines.* Three instances of this law are presented below,

$$\boxed{\cos A = -\cos B \cos C + \sin B \sin C \cos a} \quad (42.1)$$

$$\boxed{\cos B = -\cos C \cos A + \sin C \sin A \cos b} \quad (42.2)$$

$$\boxed{\cos C = -\cos A \cos B + \sin A \sin B \cos c} \quad (42.3)$$

We have previously discussed this law at length in the main text:

Suppose that we introduce the following replacements of angle symbols:
$(a,\, A) \leftarrow (\gamma,\, l)$ for the circular arc $\widehat{\mathsf{PM}}$ of length $R\gamma$.
$(b,\, B) \leftarrow (\Delta,\, Z)$ for the circular arc $\widehat{\mathsf{P\,NP}}$ of length $R\Delta$.
$(c,\, C) \leftarrow (\phi,\, \beta)$ for the circular arc $\widehat{\mathsf{M\,NP}}$ of length $R\phi$.
Then, $\dfrac{\sin A}{\sin a} = \dfrac{\sin B}{\sin b} = \dfrac{\sin C}{\sin c}$ is replaced by $\dfrac{\sin\, l}{\sin\gamma} = \dfrac{\sin\beta}{\sin\phi} = \dfrac{\sin Z}{\sin\Delta}$ or $\dfrac{\sin\, LHA}{\sin\gamma} = \dfrac{\sin B}{\sin\phi} = \dfrac{\sin A}{\sin\Delta}$.

Hence, we see that $\sin\gamma \sin A = -\sin\Delta \sin l$, which we had earlier derived in the original rotation matrix derivation as equation 10.2.

Similarly, equations 40.1, 40.2, 40.3 are replaced by equations 10.3, 16.3 and 26.3 respectively (repeated below).

$$\begin{aligned}
\cos\gamma &= \cos\Delta \cos\phi + \sin\Delta \sin\phi \cos l \\
\cos\Delta &= \cos\phi \cos\gamma + \sin\phi \sin\gamma \cos A \\
\cos\phi &= \cos\gamma \cos\Delta + \sin\gamma \sin\Delta \cos\beta
\end{aligned}$$

Many authors derive the second spherical law of cosines by considering "polar triangles", which are discussed in some of the texts appearing in the bibliography. We will not address those concepts here, except to mention that the equations expressing the second law can be obtained by making the following replacements in the spherical law of cosines:

$$\begin{array}{lcl}
A \leftarrow 180^\circ - a & & a \leftarrow 180^\circ - A \\
B \leftarrow 180^\circ - b & \text{and} & b \leftarrow 180^\circ - B \\
C \leftarrow 180^\circ - c & & c \leftarrow 180^\circ - C
\end{array}$$

However, these replacements alone do not constitute mathematical proofs.

All of the derivations in this appendix are based upon the assumption that the domains of all surface angles, $D(A) = D(B) = D(C) = [0, 180^\circ]$. However, based upon arguments in the text derived via rotation matrices, we see that the domain of the surface angles is $[0, 360^\circ)$, provided that we adhere to the interpretation of those angles as given in the text.

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