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Cross Check of NOvA Oscillation Probabilities

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Abstract

In this note we perform a cross check of the programs used by NOvA to calculate the 3-flavor oscillation probabilities with a independent program using a different method. The comparison is performed at 6 significant figures and the agreement, $|\Delta P|/P$ is better than 10^{-5}, as good as can be expected with 6 significant figures. In addition, a simple and accurate alternative method to calculate the oscillation probabilities is outlined and compared in the L/E range and matter density relevant for the NOvA experiment.

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I. SETUP

Neutrino parameters relevant for oscillations:

$$\begin{aligned} \Delta m^2_{32} = \pm\ 2.5 \times 10^{-3}\ \text{eV}^2, &\quad \Delta m^2_{21} = +\ 7.5 \times 10^{-5}\ \text{eV}^2 \\ \sin^2\theta_{12} = 0.31, &\quad \sin^2\theta_{23} = 0.43 \\ \sin^2\theta_{13} = 0.022, &\quad \delta = -69^\circ = -23\pi/60 \end{aligned} \tag{1}$$

Where $\Delta m^2_{32} > 0$ gives a normal ordering (NO) neutrino spectrum and $\Delta m^2_{32} < 0$ for inverted ordering (IO). Note we have avoided the special points: $\theta_{23} = \pi/4$ as well as $\delta = 0, \pm\pi/2, \pi$.

Experimental setup parameters, approximate that of the NOvA experiment:

$$\begin{aligned} L &= 800\ \text{km} \\ Y_e\rho &= 1.4\ \text{g.cm}^{-3} \\ 0.7\ \text{GeV} \leq\ E &\leq 3.7\ \text{GeV}. \end{aligned} \tag{2}$$

The comparison is performed every 0.1 GeV for both ν_μ and $\bar{\nu}_\mu$ disappearance as well as ν_e and $\bar{\nu}_e$ appearance for both NO and IO (8 channels and 31 energy points per channel).

II. OSCILLATION PROBABILITIES

NOvA has two independent methods for calculating the oscillation probabilities in constant matter: one based on [1] and the other based on [2]. These two methods have been compared to each other by the collaboration and they agree.

The two codes are located in the `OscLib` package in the NOvA software repository. The `PMNS.h,cxx` codes follow [1] which diagoalizes the 3×3 neutrino hamiltonian in closed form and provides exact solutions for its eigenvectors and eigenvalues. Starting from a unitary matrix in the flavor basis, the `PMNS` code accumulates the product of the transition amplitudes A_{ij} for steps of propagation distance L across matter with electron density N_e for flavor i to oscillate to flavor j. The final oscillation probabilities are then the magnitudes of these final transition amplitudes, $P_{ij} = A^*_{ij}A_{ij}$. The second codes, `PMNSOpt.h,cxx` are also based an exact calculation but completes the diagonalization of the 3×3 hamiltonian matrix in fewer internal calculations. As these routines are computationtioanlly more efficient, they are the ones used in NOvA physics analyses.

In this note we compare the NOvA calculations to a completely different method based on an exact analytical calculation based on paper [3] and programmed independently of the NOvA collaboration ie by the author SP.[1]

Beside the parameters in the previous section, we also had to agree on $\hbar c$ as well as the conversion factor going from $Y_e\rho$ to the matter potential $a = 2\sqrt{2}G_F N_e E_\nu$ to the relevant precision.

The comparison was make every 0.1 GeV from 0.7 GeV to 3.7 GeV, more than covering the energy window of NOvA . The results are given in Fig. 1 and Fig. 2. As the comparison

[1] This paper has some typos which have been corrected in the numerical program.

was only performed at 6 significant figures, having a fraction difference,

$$|\Delta P|/P \leq 10^{-5}$$

is as good as one can expect. Therefore the conclusion here that all three methods of calculating the oscillation probability agree beyond what is need for the NOvA experiment.

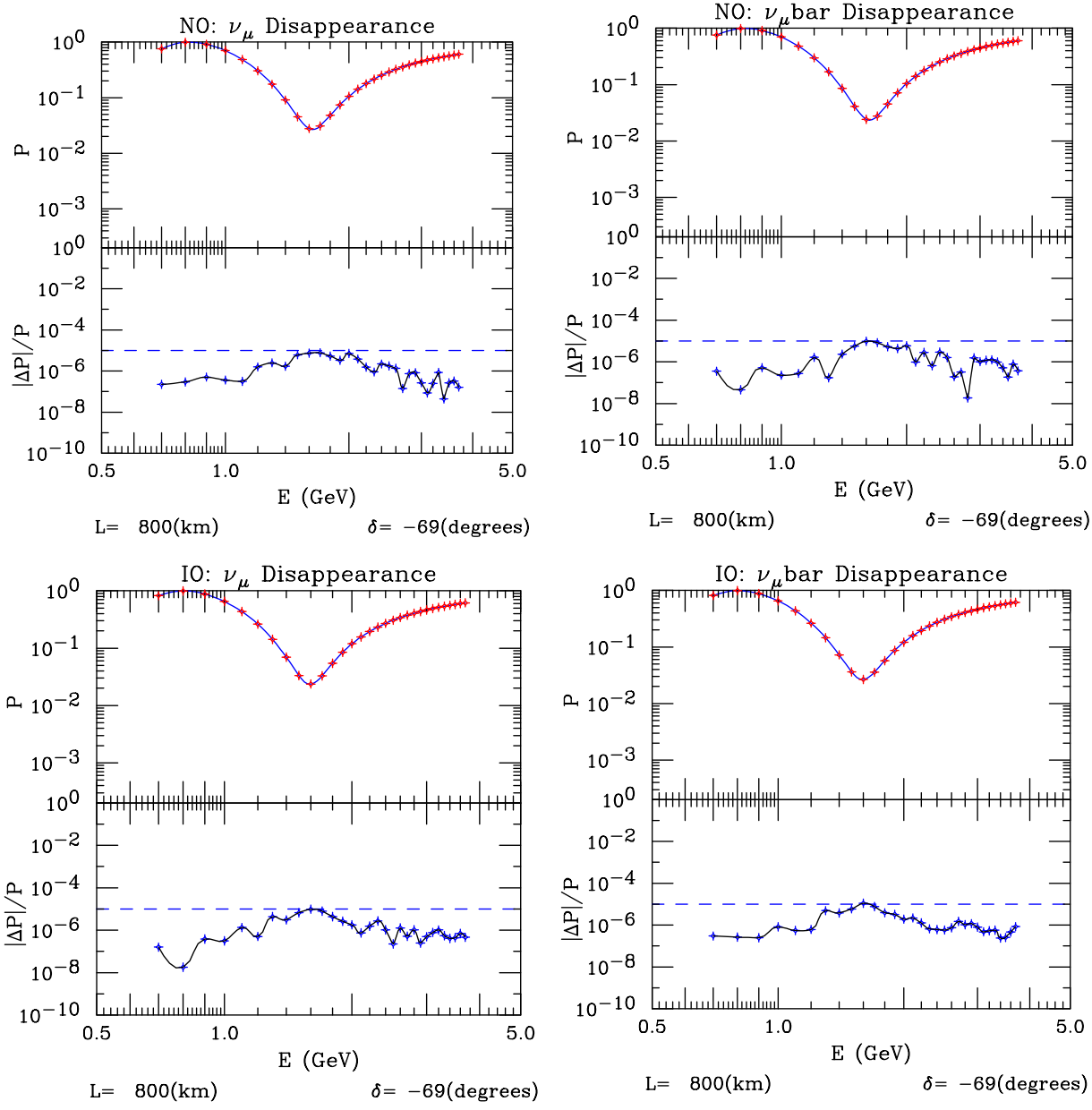

FIG. 1: Disappearance: $P(\nu_\mu \to \nu_\mu)$ and $P(\bar{\nu}_\mu \to \bar{\nu}_\mu)$ for NO (first row) and IO (second row).

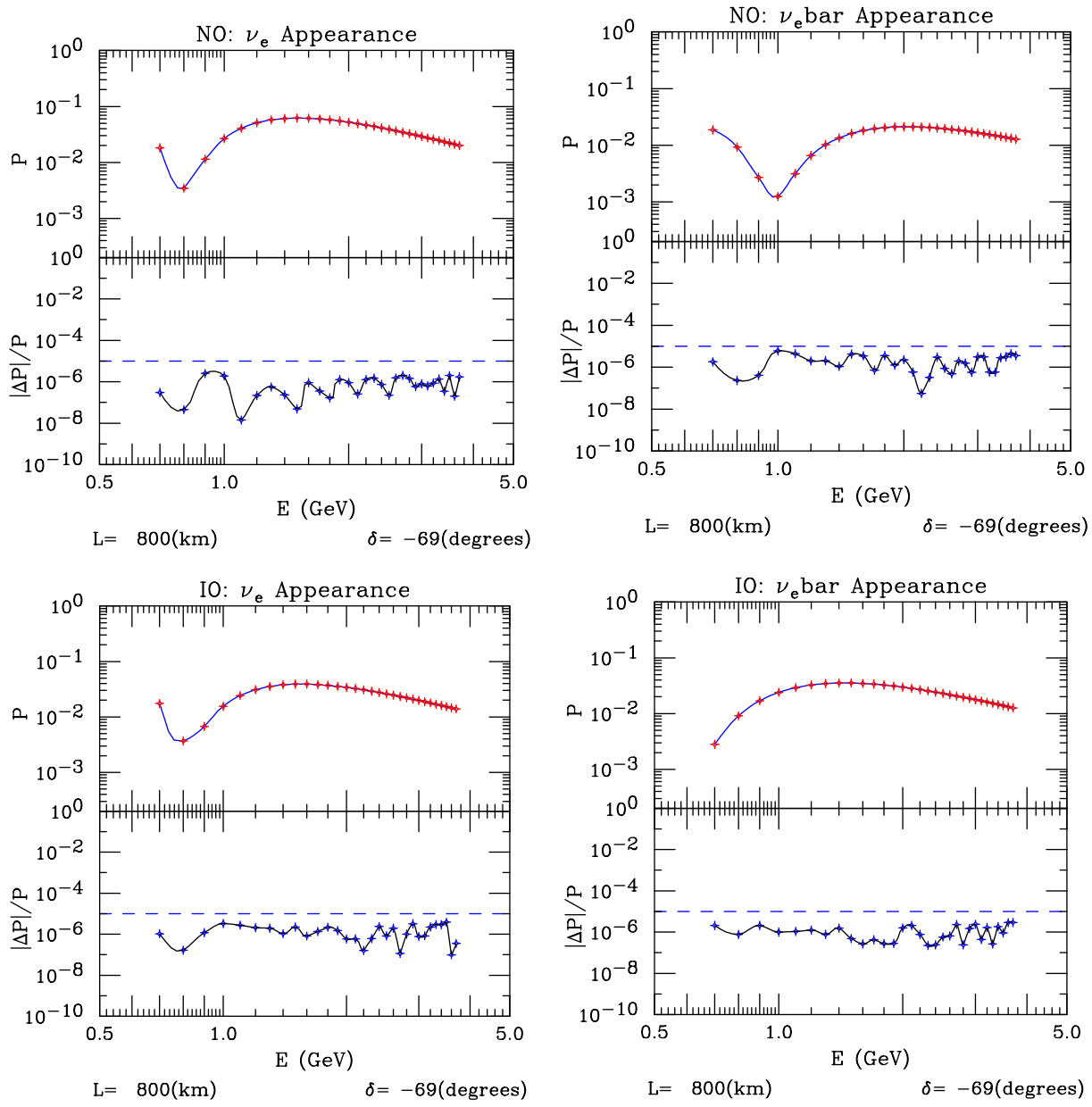

FIG. 2: Appearance: : $P(\nu_\mu \to \nu_e)$ and $P(\bar{\nu}_\mu \to \bar{\nu}_e)$ for NO (first row) and IO (second row).

III. A SIMPLE, ACCURATE METHOD FOR CALCULATE OSCILLATION PROBABILITIES IN MATTER

In this section, a simple and accurate way to evaluate oscillation probabilities, recently shown in [4], is given.[2] Details as to the why's and how's of this method are contained in this paper.

[2] In this note ϕ, ψ and $\Delta\lambda_{jk}$ of [4], are replaced with the more meaningful notation $\widetilde{\theta}_{13}$ and $\widetilde{\theta}_{12}$ and $\Delta\widetilde{m^2}_{jk}$ respectively.

The mixing angles in matter, which we denote by a $\widetilde{\theta}_{13}$ and $\widetilde{\theta}_{12}$ here, can also be calculated in the following way, using $\Delta m^2_{ee} \equiv \cos^2\theta_{12}\Delta m^2_{31} + \sin^2\theta_{12}\Delta m^2_{32}$, as follows[3]:

$$\cos 2\widetilde{\theta}_{13} \;=\; \frac{(\cos 2\theta_{13} - a/\Delta m^2_{ee})}{\sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}}, \tag{3}$$

where $a \equiv 2\sqrt{2}G_F N_e E$ is the standard matter potential, and

$$\cos 2\widetilde{\theta}_{12} \;=\; \frac{(\cos 2\theta_{12} - a'/\Delta m^2_{21})}{\sqrt{(\cos 2\theta_{12} - a'/\Delta m^2_{21})^2 \;+\; \sin^2 2\theta_{12}\cos^2(\widetilde{\theta}_{13} - \theta_{13})}}, \tag{4}$$

where $a' \equiv a\,\cos^2\widetilde{\theta}_{13} + \Delta m^2_{ee}\sin^2(\widetilde{\theta}_{13} - \theta_{13})$ is the θ_{13}-modified matter potential for the 1-2 sector. In these two flavor rotations, both $\widetilde{\theta}_{13}$ and $\widetilde{\theta}_{12}$ are in range $[0, \pi/2]$.

θ_{23} and δ are unchanged in matter for this approximation.

The neutrino mass squared differences in matter, i.e. the Δm^2_{jk} in matter, which we denote by $\Delta\widetilde{m^2}_{jk}$, are given by

$$\begin{aligned}
\Delta\widetilde{m^2}_{21} \;&=\; \Delta m^2_{21}\sqrt{(\cos 2\theta_{12} - a'/\Delta m^2_{21})^2 \;+\; \sin^2 2\theta_{12}\cos^2(\widetilde{\theta}_{13} - \theta_{13})}\;\;, \\
\Delta\widetilde{m^2}_{31} \;&=\; \Delta m^2_{31} + \frac{1}{2}\left(\, 2\,a - 3\,a' + \,\Delta\widetilde{m^2}_{21} - \Delta m^2_{21}\,\right), \\
\Delta\widetilde{m^2}_{32} \;&=\; \Delta\widetilde{m^2}_{31} - \Delta\widetilde{m^2}_{21} = \Delta m^2_{32} + \frac{1}{2}\left(\, 2\,a - 3\,a' - \,\Delta\widetilde{m^2}_{21} + \Delta m^2_{21}\,\right).
\end{aligned} \tag{5}$$

Note that the same square root[4] appears in both $\Delta\widetilde{m^2}_{21}$ and $\widetilde{\theta}_{12}$. To see that the $\Delta\widetilde{m^2}_{31}$ and $\Delta\widetilde{m^2}_{32}$ have the right asymptotic forms, use the fact that $(\Delta\widetilde{m^2}_{21} - \Delta m^2_{21}) = |a'| + \mathcal{O}(\Delta m^2_{21})$, for $|a| \gg \Delta m^2_{21}$.

To calculate the oscillation probabilities, to 0th order, use the above $\Delta\widetilde{m^2}_{jk}$ instead of Δm^2_{jk} and replace the vacuum PMNS matrix as follows

$$U^0_{PMNS} \equiv U_{23}(\theta_{23})\,U_{13}(\theta_{13},\delta)\,U_{12}(\theta_{12}) \;\Rightarrow\; U^M_{PMNS} \equiv U_{23}(\theta_{23})\,U_{13}(\,\widetilde{\theta}_{13},\delta)\,U_{12}(\widetilde{\theta}_{12}). \tag{6}$$

That is, replace

$$\begin{aligned}
\Delta m^2_{jk} \;&\to\; \Delta\widetilde{m^2}_{jk} \\
\theta_{13} \;&\to\; \widetilde{\theta}_{13} \\
\theta_{12} \;&\to\; \widetilde{\theta}_{12},
\end{aligned} \tag{7}$$

it is that simple. θ_{23} and δ remain unchanged.

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[3] Vacuum values to be used in calculating Δm^2_{ee}.

[4] If $a = 0$, then $\widetilde{\theta}_{13} = \theta_{13}$ and since $a' = 0$ then $\widetilde{\theta}_{12} = \theta_{12}$ and both $\sqrt{\cdots} = 1$, also $\Delta\widetilde{m^2}_{jk} = \Delta m^2_{jk}$ for all (j,k) as required. The identity $s^2_\theta = (1 - \cos 2\theta)/2$ is useful for calculating both s_θ and c_θ.

These expressions are valid for both NO, $\Delta m^2_{31} > 0$ and IO, $\Delta m^2_{31} < 0$. For anti-neutrinos, just change the sign of a and δ. Our expansion parameter is $\left|\sin(\widetilde{\theta}_{13} - \theta_{13})\ s_{12}c_{12}\ \frac{\Delta m^2_{21}}{\Delta m^2_{ee}}\right| \leq 0.015$, which is small and vanishes in vacuum, so that our perturbation theory reproduces the vacuum oscillation probabilities exactly.

In Fig. 3 & 4 we have compared the exact oscillation probability with our approximation.. One sees that the 0th order oscillation probabilities, relevant for the NOvA experiment, have a difference, from the exact calculation, $\Delta P < 1 \times 10^{-4}$, which is unobservable in the NOvA experiment.

A. Higher Orders

If the 0th order is not accurate enough, going to 1st order is simple and gives another two orders of magnitude in accuracy. First the $\Delta\,\widetilde{m^2}_{jk}$ remain unchanged but the mixing matrix is modified by

$$U^M_{PMNS} \Rightarrow V \equiv U^M_{PMNS}(1 + W_1), \tag{8}$$

where the matrix W_1 is given by

$$W_1 = \sin(\widetilde{\theta}_{13} - \theta_{13})\ s_{12}c_{12}\ \Delta m^2_{21} \begin{pmatrix} 0 & 0 & -\widetilde{s}_{12}\, e^{-i\delta}/\Delta\,\widetilde{m^2}_{31} \\ 0 & 0 & +\widetilde{c}_{12}\, e^{-i\delta}/\ \Delta\,\widetilde{m^2}_{32} \\ +\widetilde{s}_{12}\, e^{+i\delta}/\Delta\,\widetilde{m^2}_{31} & -\widetilde{c}_{12}\, e^{+i\delta}/\Delta\,\widetilde{m^2}_{32} & 0 \end{pmatrix}.$$

where $\widetilde{s}_{12} = \sin\widetilde{\theta}_{12}$ and $\widetilde{c}_{12} = \cos\widetilde{\theta}_{12}$. The $\Delta\,\widetilde{m^2}_{jk}$ and the V-mixing matrix can be used to calculate the oscillation probabilities and improve the accuracy so that $\Delta P < 10^{-6}$. The next highest order is also discussed in [4].

IV. ACKNOWLEDGEMENTS

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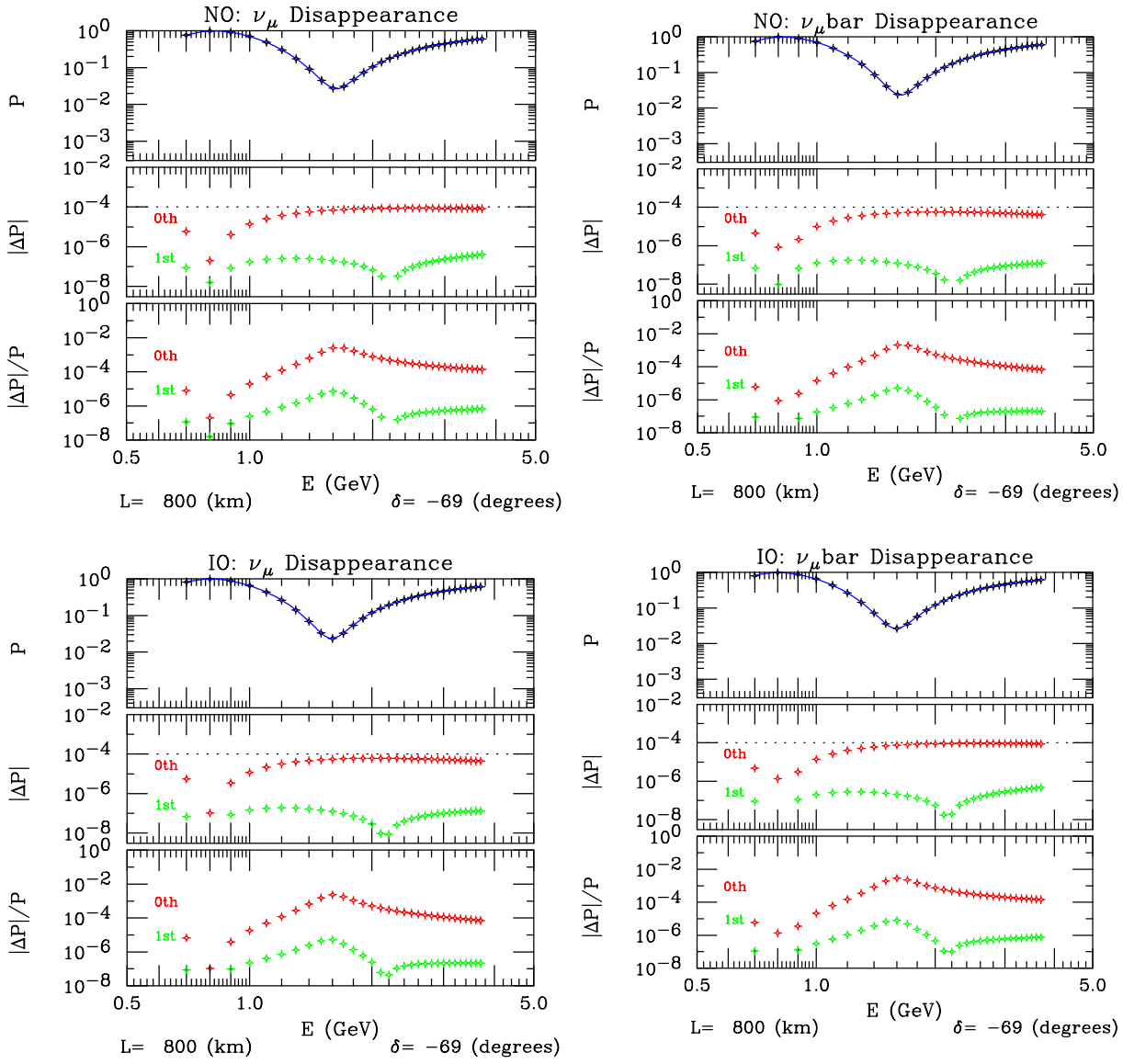

FIG. 3: Disappearance: $P(\nu_\mu \to \nu_\mu)$ and $P(\bar{\nu}_\mu \to \bar{\nu}_\mu)$ for NO (first row) and IO (second row). The comparison is between the exact calculation of Zaglauer et al, [3], and 0th order of the simple approximation of Denton et al [4] in red (green is the same comparison with 1st order).

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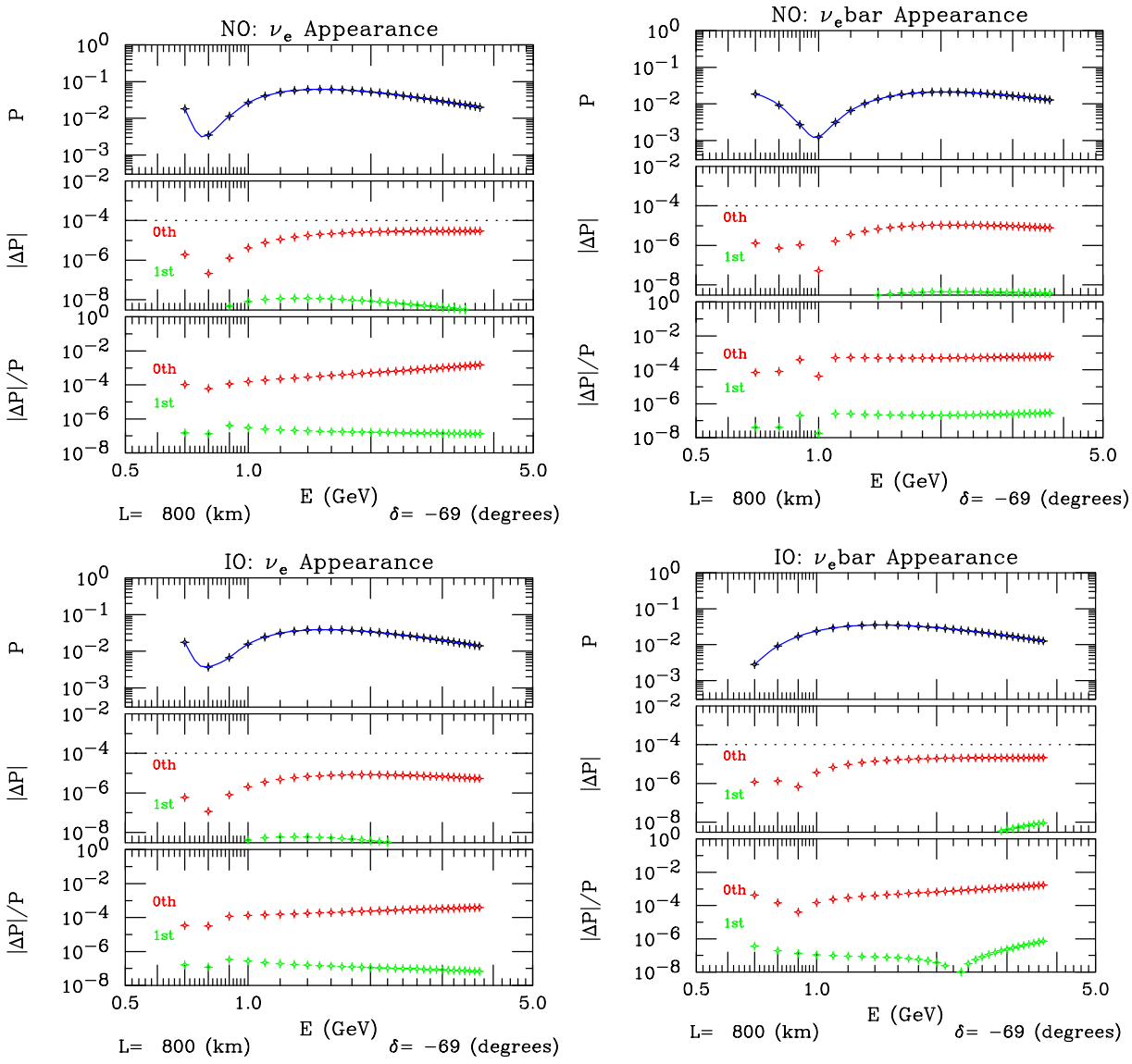

FIG. 4: Appearance: $P(\nu_\mu \to \nu_e)$ and $P(\bar{\nu}_\mu \to \bar{\nu}_e)$ for NO (first row) and IO (second row). The comparison is between the exact calculation of Zaglauer et al, [3], and 0th order of the simple approximation of Denton et al [4] in red (green is the same comparison with 1st order).

[3] H. W. Zaglauer and K. H. Schwarzer, "The Mixing Angles in Matter for Three Generations of Neutrinos and the MSW Mechanism," Z. Phys. C **40**, 273 (1988). doi:10.1007/BF01555889

[4] P. B. Denton, H. Minakata and S. J. Parke, "Compact Perturbative Expressions For Neutrino Oscillations in Matter," JHEP **1606**, 051 (2016) doi:10.1007/JHEP06(2016)051 [arXiv:1604.08167 [hep-ph]]. See https://github.com/PeterDenton/Nu-Pert

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