Ptolemy's table of chords: Implications considered and discussed

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Abstract

Ptolemy's table of *chord lengths* from the *Almagest* converted into *decimal* values and recalculated. Presentation and discussion of Ptolemy's method of calculation and *sexagesimal* values in comparison to calculations by trigonometric functions with regard to several editors. A significant relationship between the *difference values* of the two methods and *angle number* was found: The higher the expansion angle, the greater the *overestimation* of chord lengths. Implications for early scientific astronomy in context with the development of mathematical methodology are discussed.

Introduction

„[ANTONINUS.AUG.PIUS.PP.IMP II.] Sous le règne de ce sage Empereur, l'astronomie prit une nouvelle face. Ptolemée en rassembla les principes [...] dans un ouvrage excellent auquel il donna le titre de Composition Mathématique.“, (Halma, 1813, preface, c.f. Cassini, 1693, p. 14).

Under the reign of this wise Emperor, astronomy took on a new face. *Ptolemy* brought together the principles [...] in an excellent work to which he gave the title *Mathematical Composition*.

The famous astronomer and mathematician *Claudius Ptolemæus* and his main work from the 2nd century AD have often been described; a very appropriate introductory overview is given, for example, by Manitius (1912):

„Zu den größten Geisteswerken des Altertums, die uns in tadelloser Fassung erhalten geblieben sind, gehört das Handbuch der Astronomie, welches Claudius Ptolemäus in Alexandria um die Mitte der Regierung des Kaisers Antoninus Pius (138—161 n. Chr.) [...] verfaßt hat. Die Bedeutung des [...] Werkes wird wesentlich dadurch erhöht, daß es auf den Forschungen und Beobachtungen des Hipparchus von Nizäa beruht, des ‚Vaters der Astronomie‘ [...] Die Akademie in Alexandria ging ihrem Verfall entgegen, als [...] die [...] nestorianischen Christen [...] der Sitz einer Gelehrsamkeit wurden, welche sich [...] auch die Schätze der griechischen Literatur durch syrische Übersetzungen zugänglich machte. Von der Reichskirche verfolgt, fanden die Nestorianer zuvorkommende Aufnahme im Perserreich [...] unter Chosru I. Nuschirwan (532—579) [...] entfalteten sie als Übersetzer der geschätztesten griechischen Werke in die Landessprache eine rege Tätigkeit.“ (Manitius, 1912, p. 3 ff.).

One of the *greatest* intellectual works of antiquity that has been preserved to us in an impeccable version is the *Handbook of Astronomy*, which Claudius Ptolemy wrote in Alexandria around the middle of the reign of Emperor Antoninus Pius [...] The importance of the work is significantly increased by the fact that it is based on the research and observations of Hipparchus of Nicæa, the „father of astronomy” [...] The Academy in Alexandria was approaching its decline when [...] the [...] Nestorian Christians [...] became the seat of a scholarship that [...] also made the treasures of Greek literature accessible through *Syrian* translations. Persecuted by the imperial church, the Nestorians found a courteous reception in the Persian Empire [...] under Khosrow I [...] they developed a lively activity as translators of the most valued Greek works into the national language.

Here the important *table* of *chord lengths* according to Ptolemy's *Mathematical Composition* or *Almagest* (1515, fol. 7r ff.), converted into *decimal* values and calculated in *comparison* using the *sine* function with its significance for determining measurements, especially in astronomy, is discussed and displayed (c.f. Fig. 2, CHORD application, Schrausser, 2024a). For the *chord tables* itself in (a) *Latin* see also Giunta (1528, fol. 6v ff.), in (b) *French* see Halma (1813, p. 38 ff.) and in (c) Greek see Heiberg (1898, p. 48 ff.), whose work is based on the *Greek* manuscripts *Codex Parisinus Græcus 2389* and *2390*, *Marcianus Græcus 310* and *313* and *Vaticanus Græcus 180* and *1594* (c.f. Ræder, 1928 or Toomer, 1984, p. 3), where he places particular emphasis on authenticity and proximity to the sources (see Tab. 1):

„Horum codicum ope uerba Ptolemæi talia restitui posse confido, qualia a uiris doctis Alexandriæ anno circiter 500 legerentur.", (Heiberg 1898, p. V—VI).

With the help of these codices I trust that the words of Ptolemy can be *restored* as they were read by the learned men of Alexandria in about the *year 500*.

C.f. further the notable (d) *German* translation of Heiberg's Greek transcription from Manitius (1912, p. 37 ff.), as well as the (e) *English* translations from Taliaferro, (1952, p. 21 ff.) and Toomer (1984, p. 57 ff., 1998, res.).

Regarding the (here primarily considered) printed *Latin* version by Petrus Lichtenstein from 1515, the following should be noted, since a higher *reliability*, especially in relation to numerical values was found early on in the medieval *Arabic transcription line*:

„ [...] ex arabico translationem suscepit & absoluit GERARDUS CREMONENSIS, quæ manuscripta extat in Bibliotheca Laurentiana Norimbergensi, ubi eandem anno 1727 me uidisse memini.", (Weidler, 1741, p. 178 ff.).

[...] a translation from the *Arabic*, undertaken and completed by *Gerardus Cremonensis*, which manuscript is in the Laurentian Library at Nuremberg, where I remember seeing it in 1727.

See moreover Ideler, referring to the astronomer Jérôme de Lalande (1771):

„Diese Uebersetzung gilt in Ansehung der Zahlen für die richtigste. Für mich ist sie [...] sehr wichtig gewesen, und ich citire sie desshalb häufig. Sie ist übrigens sehr selten. LaLande versichert [...] nur Ein Exemplar davon gesehn zu haben. Ein anderes besitzt die hiesige königliche Bibliothek.", (Ideler, 1809, p. LXVIII).

This translation is the *most correct* in terms of the numbers. For me it was [...] very important, and that's why I quote it often. By the way, it is *very rare*. La Lande assures [...] that he has only seen *one* copy of it. Another one is owned by the local royal library.

Lalande (also c.f. 1777, 1795) continues here:

„369. Le texte Grec de Ptolomée ne fut connu en Europe que dans le quinzième siécle : jusqu'alors on avoir employé les exemplaires Arabes pour traduire l'Almageste [...] ", (De Lalande, 1771, p. 157).

Ptolemy's *Greek* text was *not known* in Europe until the 15th century: until then, *Arabic* copies were used to *translate* the Almagest.

Important *manuscripts* of the Almagest *containing* the *chord table* are given in (a) *Greek* by e.g. Ptolemæus (9th c., fol. 20v ff., 10th c., fol. 18v ff., res.) and (b) *Arabic* by al-Ḥajjāj (828, fol. 8v ff.) and Isḥāq & Thābit (9th c., fol. 10r ff., c.f. e.g. Ideler, 1809, p. 67; Manitius, 1912, p. 12; Cooper, 2007; Grupe, 2019 and also Bellver, 2021, 2024 or Pouyan, 2023a, b).

See especially the (c) *Latin* translations from Cremona (1175, fol. 10v ff., c.f. Juste, 2024; Haskins, 1912) and Trebizond (1451, fol. 13v ff.). Not to forget the *editio princeps* from Regiomontanus (1496, 1550, res.) based on Greek Manuscripts (c.f. Manitius, 1912, p. 17 ff.), which, however, does *not contain* a chord table.

For Āryabhata's *famous* table of 24 *sine* values from c. 510 AD (Chatterjee, 1975) in Sanskrit see Kern (1874, p. 16 ff.) or in English see Clark (1930, p. 19); see furthermore Gersonides' sine tables in *De sinibus, chordis et arcubus* from 1342 (c.f. Curtze, 1898, p. 101 ff.; Simonson, 2000a, b; Wagner et al., 2016). *Detailed* sine tables are given by Peuerbach (1541) based on John of Gmunden's work from 1437 (Busard, 1971; Juste, 2022), for an overview see e.g. Roegel (2021).

Table 1. Timeline and *names* of the most important historical protagonists and sources from 600 BC to 1600, including *ab urbe condita* or *anno urbis conditæ AVC* and *Anno Hegiræ AH*.

BC	*AD*	*AVC*	*AH*	*name*	*from*	*to*		*tr. and Arabic transcription line*
601		*153*		Πυθαγόρας ὁ Σάμιος	570	495	BC	Pythagoras of Samos
551		*203*						
501		*253*						
451		*303*						
401		*353*		Εὐκλείδης	360	-	BC	Euclid of Alexandria
351		*403*		Ἀρίσταρχος	310	230	BC	Aristarchus of Samos
301		*453*		Ἐρᾰτοσθένης	276	194	BC	Eratosthenes of Cyrene
251		*503*						
201		*553*		Ἵππᾰρχος	190	120	BC	Hipparchus of Nicäa
151		*603*						
101		*653*						
51		*703*						
1	*0*	*753*						
	50	*803*		ANTONINVS PIVS	86	161	AD	Titus Ælius Hadrianus Antoninus Augustus Pius
	100	*853*		Κλαύδιος Πτολεμαῖος	100	170	AD	Claudius Ptolemæus
	150	*903*						
	200	*953*						
	250	*1003*		Πάππος	290	350	AD	Pappus of Alexandria
	300	*1053*						
	350	*1103*						
	400	*1153*						
	450	*1203*						
	500			خسرو انوشيروان	512	579	AD	Khosrow I Anushirvan
	550							
	600							
	650		*30*					
	700		*81*					
	750		*133*	بن يوسف بن مطر الحجاج	786	833	AD	al-Haddschādsch Yūsuf ibn Matar
	800		*184*	إسحاق بن حنين	830	910	AD	Abū Yaʿqūb Isḥāq ibn Ḥunayn
	850		*236*					
	900		*288*					
	950		*339*					
	1000		*391*					
	1050		*442*					
	1100		*494*	Gerardus Cremonensis	1114	1187		Gerard of Cremona
	1150		*545*					
	1200		*597*					
	1250		*648*					
	1300		*700*					
	1350		*751*	Γεώργιος Τραπεζούντιος	1395	1486		George of Trebizond
	1400		*803*	Regiomontanus	1436	1476		Johannes Müller von Königsberg
	1450		*854*	Petrus Liechtenstein	1497	1547		
	1500		*906*					
	1550		*957*					
	1600		*1009*					

Method

Chord lengths were *derived* according to Ptolemy's *Theorema II* (c.f. Trebizond, 1451, fol. 11r) within the relation between four sides and two diagonals of a cyclic quadrilateral (c.f. Fig. 1):

„Sit circulo .a.b.g.d. inscriptu[m] quadrilateru[m] .a.b.g.d. cuius diametri .a.g. [et] .b.d. Dico quod sit ex .b.d. in .a.g. esse equale duob[us] que fiunt ex .a.d. in .b.g. [et] ex .a.b. in .d.g. rectangulis.", (Regiomontanus, 1496, fol. a7v).

„Let there be a circle with an arbitrary quadrilateral ABGD inscribed in it. Join AG and BD. We must prove that AG.BD = AB.DG + AD.BG.", (Toomer, 1984, p. 50).

Where expressed in current notation as $AC{\cdot}BD = AB{\cdot}CD + BC{\cdot}AD$:

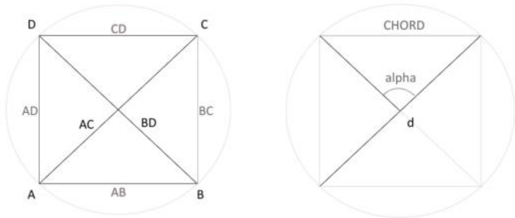


Figure 1. Cyclic quadrilateral with chord length representation.

Chord lengths l_0 are given in fractional parts of sexagesimal numerals; a basic description of the *chord tables* is given by Ptolemy:

„Et quonia[m] necesse est nobis scire numerum partium chordarum [et] quantitatem earum: [et] vt sint preparate: faciam tabulas [...] Et describam in prima tabula numerum partium arcuum superfluentium medietate partis [et] medietate partis. Et in tabula secunda numerum partium chordarum [et] minutorum partium [et] secundo[rum] earum: que subtendantur arcubus consequenter et latere. Itaq[ue] queq[ue] chorda suum consequatur arcum [...] ", (Ptolemæus, 1515, fol. 6v—7r).

Since it is necessary for us to know the number of parts and the quantity of the chords, I will *prepare tables* [...] And I will describe in the *first table* the number of the parts of the *arcs*, in half steps. And in the *second table* the number of the parts of the *chords* with minutes and seconds, assigned to the arcs in sequence and on the side [...] .

With $x = parts\ (p)$, $y = minutes\ (min)$ and $z = seconds\ (sec)$, decimal values l_1 from l_0 are calculated by

$$l_1 = x + \frac{y}{60} + \frac{z}{60^2}. \tag{1}$$

Sixtieth i_0, also given in sexagesimal numbers, is the average *interpolation* number to be added to length l_0 or l_1 each time angle increases by one minute of arc, that is $n = 30$ times per half angle degree α:

„Pars tricesima superflui: quod est inter o[mn]es duas chordas. [et] est portio arcus vnius minuti.", (Ptolemæus, 1515, fol. 7r).

The *thirtieth* part of the excess, which is between every two chords, is a *portion* of *one minute* of arc.

Decimal values i_1 with $z_0 = seconds\ (sec)$ and $z_1 = thirds\ ('''')$ are calculated by

$$i_1 = x + \frac{y}{60} + \frac{z_0}{60^2} + \frac{z_1}{60^3}, \tag{2}$$

where arithmetical interpolation

$$i_2 = \frac{l_{1[\alpha_n]} - l_{1[\alpha_{n-1}]}}{30}. \tag{3}$$

Chord lengths l_2 to given arcus α and diameter d are calculated using the *sine* function by

$$l_2 = d \cdot \sin\frac{\alpha \cdot \pi}{360}, \tag{4}$$

what is equivalent in terms of content to distance s or radius r determination (being of primary practical relevance) via *angular diameter* V (corresponding to *chord* l) with

$$s = r \cdot \left(\tan\frac{V}{2}\right)^{-1} \rightarrow \; r = s \cdot \tan\frac{V}{2} \; \rightarrow V = 2 \cdot \tan^{-1}\frac{r}{s}. \tag{5}$$

One should consider here, that it was only in the 16th century that the *change* in mathematics from *geometric* to *algebraic* representation took place, what expanded the methodological apparatus significantly to *abstract* levels (see Bochner, 1978; Anglin & Lambek, 1995; Malet, 2006; Alten et al., 2014).

For the *origin* of *trigonometric series* (6) of *tangents* (5) and *sine* (4) see Jyesthadeva (1530), Whish (1834), Gupta (1974) or Divakaran (2007), for the European reinvention in the 17th c. see Gregory cited in a letter by Collins (1671), the development from 1673 to 1676 and publication from 1682 of Leibniz (1682, 2012), above all Gregory (1668a, b), Newton (1669, 1711) and Taylor (1715, 1717, p. 108 ff.), defining the *Taylor series* of *sine* as infinite sum, where

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots - (-1^n) \cdot \frac{x^{2 \cdot n+1}}{(2 \cdot n + 1)!}, \tag{6}$$

or in sigma notation as

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1^n)}{(2 \cdot n + 1)!} \cdot x^{2 \cdot n+1}. \tag{7}$$

See also Gregory & Collins (1939), Boyer (1968, p. 422 ff.), Feigenbaum (1985) or Merzbach & Boyer (2011).

Euler (1748a, b) was mostly responsible for *establishing* the analytic treatment of trigonometric functions in Europe, also defining them as infinite series and presenting *Euler's formula*

$$\mathrm{e}^{ix} = \cos x + i \sin x \tag{8}$$

as the relationship between trigonometric and *complex* exponential functions, with *Euler's number*

$$\mathrm{e} = \sum_{n=0}^{\infty} \frac{1}{n!} \tag{9}$$

as base of the natural logarithm and the imaginary unit i, as well as the abbreviations *sin*, *cos* etc.; <u>*there*</u> he lays the foundations of *modern* mathematical analysis (c.f. Finkel, 1897; Walter, 1982; Koyama & Kurokawa, 2005; Calinger, 2016).

In the *absence* of *trigonometric functions* (7) and (8), however, in the 2nd century AD no chord *calculation* was possible with distance s (5) or arcus α (4) parameters, but rather tabularized values from previous model calculations with given $d = 120$ by means of the most fundamental *Pythagorean theorem* in Euclidean geometry, $a^2 + b^2 = c^2$, where

$$\left(\frac{d}{2}\right)^2 = s^2 + \left(\frac{l}{2}\right)^2 \rightarrow l = 2 \cdot \left[\left(\frac{d}{2}\right)^2 - s^2\right]^{\frac{1}{2}} \tag{10}$$

were used and *interpolated* to the corresponding angle values of expansion V, l, respectively. See in this context Euclid's *Elementa Geometriæ, Propositio 46*, first printed version by Ratdolt (1482) from the oldest surviving 12th c. translation by Adelard of Bath from an Arabic version (Russel, 1946, 2004, res.; Shloming, 1970). Of certain origin during the Old Babylonian period from the 20th to 16th c. BC (see e.g. Neugebauer, 1969, p. 36; Robson, 2008, p. 109), named after the *Pythagorean* school (e.g. Wolf, 1581, p. 810 ff.) from the 6th c. BC, see also e.g. Adelard & Campano (1491), for the first english translation Billingslay (1570) as well as Simson (1756, 1838), Heiberg & Menge (1883), Heath (1908a, b, c) and Fitzpatrick (2008):

„IN omni triangulo rectangulo quadratum q[uo]d a latere recto angulo opposito in semetipso ducto describit[ur] equ[al]u[m] e[st] duobus quadratis que ex duob[us] reliquis lateribus conscribuntur.“, (Ratdolt, 1482, fol. a11r).

In every *right-angled* triangle, the square drawn *opposite* the right angle is equal to the two squares drawn from the remaining two sides.

Corresponding statements (10) in Ptolemy's *Almagest* are given in a more specific context:

„ […] Et quonia[m] per penultima[m] primi quadratu[m] .b.z. est equale duobus quadratis .b.d. [et] .d.z. [et] .b.d. est latus hexagoni: [et] .d.z. latus decagoni.“, (Regiomontanus, 1496, fol. a7r).

„ […] and, in the right-angled triangle BDZ, the square on BZ equals the sum of the squares on BD, which is the side of the hexagon, and on DZ, which is the side of the decagon.“, (Toomer, 1984, p. 49).

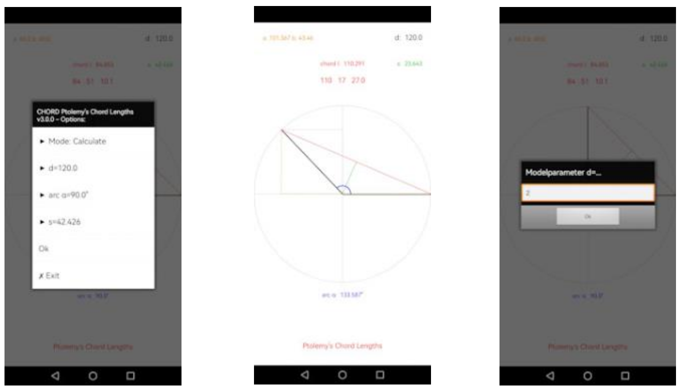


Figure 2. Screenshots from CHORD application.

Chord parameters $l_{(120)}$ can then be adapted to empirical $l_{(d)}$ *proportions* by transforming the model parameter with

$$l_{(d)} = l_{(120)} \cdot \frac{d}{120}. \tag{11}$$

Chord length values $l_{(e)}$ corresponding to *empirical* distances $s_{(e)}$ can be expressed by multiplying $l_{(d)}$ (11) with a ratio factor $\delta = \frac{s_{(e)}}{s_{(d)}}$ as $l_{(e)} = l_{(d)} \cdot \delta$ to given angle α, where according to *Pythagoras* (10)

$$\delta = s_{(e)} \cdot \left[\left(\frac{d}{2}\right)^2 - \left(\frac{l_{(d)}}{2}\right)^2 \right]^{-\frac{1}{2}}, \tag{12}$$

see Fig. 2, CHORD application (Schrausser, 2024a).

To *validate* the chord table itself, the following *parameters* were collected: Differences *diff* show the difference between (a) the calculation types of *chord lengths* l_1 (1) and l_2 (4), where

$$diff_{l_2} = l_2 - l_1 \tag{13}$$

as well as the difference between (b) *sixtieth* i_1 (2) and arithmetical interpolation i_2 (3) by $diff_{i_2} = i_2 - i_1$. To characterize the mean similarities, ratio factors

$$rf_{l_2} = \frac{l_2}{l_1} \text{ and } rf_{i_2} = \frac{i_2}{i_1} \tag{14}$$

were calculated (c.f. Schrausser, 2024a, tables).

The *mean* differences $\overline{diff_{l_2}}$, $\overline{diff_{i_2}}$, mean *absolute* values $\overline{|diff_{l_2}|}$, $\overline{|diff_{i_2}|}$ and mean *ratio* factors $\overline{rf_{l_2}}$, $\overline{rf_{i_2}}$ were calculated, tables of *Cremona* (1175) C, *Ptolemey* (1515) P, *Manitius* (1912) M and *Toomer* (1984) T were compared.

Results

Regarding $\overline{diff_{l_2}}$, T showed the highest *conformity* with calculated values, closely followed by C and M, only P revealed higher aberrations (c.f. Tab. 2). Mostly equivalent results were obtained for $\overline{diff_{i_2}}$, where C displays an almost perfect conformity, again only P shows a *noticeable* deviation.

Similar results were found for *absolute* values, where again C consistently reveals the best congruence with the true chord and sixtieth values. Since all ratio factors are close to $rf = \frac{1}{1}$, deviations from the true values are on average very small and thus resulting measurement inaccuracies (by means of the chord tables) have certainly been of rather little *practical* effect.

In general, it was found that variations *between* the table *versions* are caused by (some obviously) *erroneous modified* values, especially in the case of P ($n_l = 2$, $P_l = 0.56\%$; $n_i = 5$, $P_i = 1.39\%$) with a rate of positive deviations $P_l^+ = 50\%$ compared to C, where

$$P = \frac{n}{360} \cdot 100, \; P^+ = \frac{k^+}{n} \cdot 100, \tag{15}$$

with number of total deviations n and *positive* deviations k^+, where a positive deviation indicates an *increased* inaccuracy.

As far as M from the *Greek manuscript line* is concerned, there is a higher *modification and error* rate ($n_l = 22$, $P_l = 6.11\%$; $n_i = 189$, $P_i = 52.50\%$) compared to C from the *Arabic* *transcription line* with $P_l^+ = 63.64\%$ (15). *Overall* the Cremona table C displays the highest concordance with the *true* chord and (particularly) interpolation values l_2 and i_2, res., where the *largest* notable inconsistency can merely be found at $\alpha = 49°$ with $l_0\ sec_C = 49$ (for detailed tables see Schrausser, 2024a).

Table 2. Chord difference mean $\overline{diff_{l_2}}$ and sixtieth difference mean $\overline{diff_{i_2}}$, mean of absolute values $\overline{|diff_{l_2}|}$, $\overline{|diff_{i_2}|}$ and mean ratios $\overline{rf}_{l_2}$ and $\overline{rf}_{i_2}$ with standard deviation σ by *table* C, P, M, T.

	$\overline{diff}$	σ	$\overline{\|diff\|}$	σ	$\overline{rf}$	σ
Chords						
C	-0.0000749	<u>0.0001116</u>	<u>0.0001073</u>	<u>0.0000810</u>	0.9999990	0.0000073
P	-0.0002593	0.0035165	0.0002917	0.0035140	0.9999966	0.0000476
M	0.0000972	0.0026949	0.0002758	0.0026826	1.0000013	0.0000332
T	<u>-0.0000703</u>	0.0001165	0.0001085	0.0000822	<u>0.9999991</u>	0.0000073
Sixtieth						
C	<u>0.0000000</u>[1]	<u>0.0000007</u>	<u>0.0000001</u>	<u>0.0000007</u>	<u>1.0000000</u>	<u>0.0000000</u>[2]
P	0.0001392	0.0015262	0.0001513	0.0015229	1.1769587	1.9277621
M	-0.0000006	0.0001271	0.0000143	0.0000030	0.9995343	0.0126237
T	-0.0000005	0.0000044	0.0000030	0.0000032	0.9995392	0.0067149

[1]) 1.1E-19.
[2]) 2.1E-13.

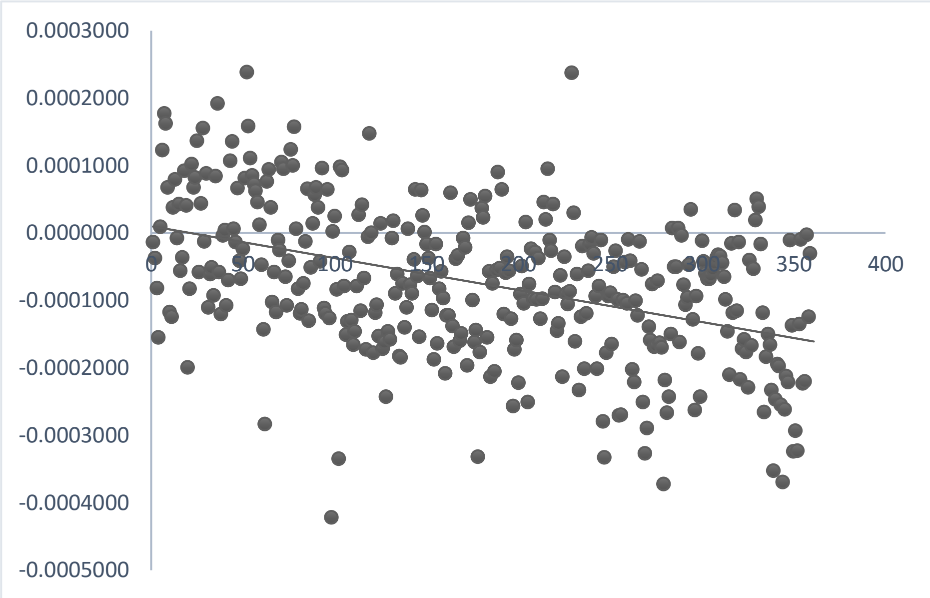
Pearson correlation r between arcus α and $diff_{l_2}$ was significant for C and T ($r_C = -0.434$, $p < .001$; $r_T = -0.433$, $p < .001$, res.) with redundancy $det_C = 18.85\%$ and $det_T = 18.73\%$ (c.f. Fig. 3). The larger α, the more difference $diff_{l_2}$ tends into the *negative* range; or, with increasing angle, the *true* chord lengths l_2 (calculated via the sine function) were *increasingly overestimated* by l_0, l_1, respectively.

Considering P and M, this effect is superimposed by error values that manifest themselves as *strong outliers* at $\alpha = 77°$ ($min_P = 46$ vs. $min_C = 42$), at $\alpha = 143°$ ($sec_M = 26$ vs. $sec_T = 56$) and $\alpha = 88.5°$ ($min_M = 41$ vs. $min_T = 44$); after correcting these extreme values, the same relations are evident for P and M too ($r_P = -0.445$, $p < .001$; $r_M = -0.385$, $p < .001$, res.).

The higher the expansion angle, the greater the *overestimation* of chord lengths may be due to the increasing *inaccuracy* of the *model* for determining the *empirical values* with *increasing angle sizes*, possibly combined with a factor of *fatigue* during repeated measurements.

Correlation between α and $diff_{i_2}$ showed no effect, no trend whatsoever could be identified at *sixtieth* since the significant correlation at P is due to (obvious) incorrect *outliers* at $\alpha = 0.5°$ to $\alpha = 1.5°$ ($min_P = 0$ vs. $min_C = 1, r_P = 0.000$, $p < .997$). For detailed results and tables c.f. Schrausser (2024a, chords_tab.md).

Figure 3. Arcus α × chord difference $diff_{l_2}$ correlation $r = -0.434, p < .001$ for the Cremona table C.



Discussion

It is certain that since *Aristarchus* (c.f. Commandino, 1572; Berggren & Sidoli, 2007), but certainly since *Hipparchus* (c.f. Swerdlow, 1969; Toomer 1974; Carman, 2020) at the latest, there has been a *reasonable*, scientifically *objective* astronomical worldview (see Tab. 1). This is *particularly* evident in Ptolemæus' statements in which the Earth is defined as a (*mathematical*) *point*[1] in relation to the star sphere:

„**M**Aius quo scitur q[ue] terra s[ecundu]m sensum qua[n]tu[m] ad spaciu[m] q[uo]d peruenit a centro totius ad orbe[m] stellaru[m] fixaru[m] sit sicut punctum: est q[ue] magnitudines quantitatum stellaru[m]: [et] intervallor[um] que inter eas existunt: videntur in omnibus plagis celi vbicu[m]q[ue] terraru[m] in eadem bora equales [et] similes. que[m]admodum inuenimus considerationes que sunt in diuersis climatibus non diuersas neq[ue] in aliquo decipientes.“, (Ptolemæus, 1515, fol. 4r).

„ [...] the earth has [...] the ratio of a point to the distance of [...] the fixed stars. A strong indication of this is the fact that the sizes and distances of the stars [...] appear equal and the same from all parts of the earth everywhere, [...] “, (Toomer, 1984, p. 43).

Using the *chord table* along with methods for *parallax* determination (c.f. Ptolemæus, 1515, fol. 53r ff.), the Alexandrian astronomers were indeed *able* to determine e.g. Moon's distance ($d_L = 59\ R_E$) and radius ($R_L = \left[3\frac{2}{5}\right]^{-1} \approx 0.294\ R_E$), $\frac{R}{d} = \frac{1}{200.6}$ quite accurate (Williams, 2024b):

„Iam ergo aggregatum est nobis: vt cum fuerit medietas diametri terre pars vna: erit s[ecundu]m illam quantitate[m] longitudo lune quide[m] media in applicationibus .59. partes. [et] longitudo solis 1210. partes. [...] S[e]c[un]d[u]m quantitatem igitur qua erit diameter lune pars vna: erit diameter terre tres partes [et] due quinte fere.“, (Ptolemæus, 1515, fol. 56r).

„Therefore we have calculated that where the earth's radius is 1 the mean distance of the moon [...] is 59 the distance of the sun is 1210 [...] Therefore where the moon's diameter is 1, the earth's diameter will be about 3 $^{2}/_{5}$ [...] “, (Toomer, 1984, p. 257).

[1] „**P**Unctus est cuius p[ar]s no[n] est.“, (Ratdolt, 1482, fol. 2r). A dimensionless geometric object having *no* properties except location.

Or as described in Ptolemy's *Planetary Hypotheses*:

„We have explained in the Almagest [...] that the least distance of the Moon is 33 earth radii, and its greatest distance 64 earth radii [...] ", (Goldstein, 1967, p. 7).

So, according to *Eratosthenis*, (a) $C_E^{st} = 250000$ stadia for the Earth's circumference C_E, which at that time arguably defined the reference value for that parameter (Ziegler, 1891) are given:

„Totus itaq[ue] circulus ducentorum quinquaginta millium erit. Atque hæc Eratosthenis ratio est.", (Balfour, 1605, p. 55).

Thus the whole circle will be *two hundred and fifty thousand*. This is the ratio given by Eratosthenes.

It is furthermore given (b) that 1 stadion st ranges between about $st \geqq 145\ m$ and $st \leqq 185\ m$ (c.f. e.g. Engels, 1985; Gulbekian, 1987; Walkup, 2010 also Dicks, 1960 or Berggren & Jones, 2000) and that (c) π, according to Weierstraß (1894, p. 53) *„bekanntlich"* defined by

$$\frac{\pi}{2} = \int_0^\infty \frac{d\lambda}{1+\lambda^2} \tag{16}$$

is approximated by

$$\pi' = 3\,\frac{17}{120} = 3.141\overline{6}, \tag{17}$$

as noted in the *Almagest* (c.f. Archimedes & Eutocius, 1544, also Heiberg, 1972, p. 237 or Arndt & Haenel, 2001, res.):

„ [...] (oder 3 $^{17}/_{120}$ = 3,14166 . . . : 1), was das Verhältnis des Kreisumfangs zum Durchmesser ist. Dieses Verhältnis liegt nämlich ohne beträchtlichen Fehler in der Mitte zwischen den Werten 3 $^{1}/_{7}$ (oder 3,14285) und 3 $^{10}/_{71}$ (oder 3,14084), welche Archimedes schlechthin nebeneinander angewendet hat.", (Manitius, 1912, p. 385).

„ [...] we assumed that the ratio of the circumference to the diameter is 3;8,30 : 1, since this ratio is about halfway between 3 $\frac{1}{7}$: 1 and 3 $\frac{10}{71}$: 1, which Archimedes used as rough [bounds].", (Toomer, 1984, p. 302).

Calculating the radius r in km by

$$r = \frac{\frac{C_E^{st}}{\pi'} \cdot st}{2}, \tag{18}$$

it follows that (a) the Earth's *radius*[2] was estimated between c. $R_E \geqq 5769\ km$ and $R_E \leqq 7361\ km$ and thus (b) the *distance* to the Moon[3] d_L calculated by Ptolemæus ranges between $d_L \geqq 340385\ km$ and $d_L \leqq 434284\ km$ with (c) Lunar *radius* R_L between $R_L \geqq 1697\ km$ and $R_L \leqq 2165\ km$.

Since (a) the *true* astronomical parameters are known today (c.f. Williams, 2024a) and (b) assuming that Eratosthenes measured exactly, what can be considered feasible due to the state of mathematical knowledge and ability, since Hipparchus at the latest, one can alternatively *derive* a *corresponding* conversion factor st' to Stadia, where

[2] Equatorial radius $R_{E_e} = 6378.137\ km$, polar radius $R_{E_p} = 6356.752\ km$, volumetric mean radius $R_{E_V} = 6371.000\ km$, model radius defined to be $R_E = 6378\ km$ (c.f. Williams, 2024a). This *calculates* the actual circumference of the Earth by $C_E = 2 \cdot R_E \cdot \pi$ as $C_E = 40074.156\ km$.

[3] Semimajor axis $d_L = 384400\ km$ where perigee $d_{L_P} = 363300\ km$ and apogee $d_{L_A} = 405500\ km$. Equatorial radius $R_{L_e} = 1738.1\ km$, polar radius $R_{L_p} = 1736.0\ km$ and volumetric mean radius $R_{L_V} = 1737.4\ km$, with a $\frac{R}{d}$ ratio ranging from $\frac{1}{209}$ to $\frac{1}{234}$ (c.f. Williams, 2024b).

$$st' = \frac{C_E}{C_E^{st}} \tag{19}$$

results in a factor of $st' = \frac{40074.156}{250000}$ or about $st' \approx 160\,m$ per stade (c.f. Engels, 1985 or Shcheglov, 2018):

„ [...] ancient geographers knew perfectly well the length of the stade they used, [...] the ‚variable quantities' of ancient stades are more often the result of modern confusion and not ancient measurements. It will be [...] argued that Eratosthenes used the Attic stade of 184.98 m. (606 ft. 10 in.) based on 600 Attic feet of 308.3 mm. apiece, which was also the standard unit of measurement of the Greco-Roman geographical tradition. [...] The [...] alternative is the stade of c. 157.5 m. in length [...] ", (Engels, 1985, p. 298, 304).

According to the *Itinerary stade* of $st = 157.5\,m$, one calculates an Earth *radius* of $R_E = 6267\,km$ (18) and from that (a) a Lunar *distance* of $d_L = 369728\,km$ and (b) a Lunar *radius* of $R_L = 1843\,km$. These parameters are with ratio factors rf of

$$rf_{R_E} = \frac{1}{1.02},\ rf_{d_L} = \frac{1}{1.04} \text{ and } rf_{R_{Le}} = \frac{1.06}{1} \tag{20}$$

in remarkable consistency with *actual* astronomical measurements (c.f. Williams, 2024a, b).

„L' itinéraire qu'on attribuë à l' empereur Antonin [...] n'est en effet [...] un recueïl des distances qui avoient esté mesurées dans toute détendue de l' empire Romain.", (Cassini, 1693, p. 14).

The *itinerary* attributed to Emperor Antoninus [...] is in fact [...] a collection of the distances which had been measured throughout the *entire* Roman Empire.

Further related works in this context are given e.g. by Ptolemæus et al. (1562), Petavius (1630, p. 71 ff.), Halma (1820, 1830), Heiberg (1895, 1903, 1907), Boll & Boer (1940, 1957), Lammert & Boer (1952), Jones (2005), Jones & Duke (2005), Ptolemæus (2018), Rubino (2021) or Dufossé (2023). Recalculations and validations of Ptolemy's *table of chords* are given by Glowatzki et al. (1976), Zieme (2023) or Schrausser (2024a).

In context of the acquisition of *cosmological and astronomical knowledge* up to the *actual* status, for the (a) *astral plane* see Pythagoras' doctrine of *musica universalis* (c.f. Davis et al., 1901; Rackham, 1967), Anaxagoras and his concepts of the *primum mobile* and *nous* (c.f. Ferchius, 1646), the *Speculum Naturale* (De Beauvais, 1264, 1494, fol. i85r), based on the *Historia Naturalis* of Plinius Secundus (1250) where he *refers* to musica universalis as concept that regards proportions in the movements of celestial bodies as a form of music, originating in *Pythagoreanism* (Kircher, 1650; Mathiesen, 2002):

„ [...] Sed Pythagoras interdum ex musica r[ati]one appellat Tonum: q[an]tum absit a terra luna. Ab ea ad Mercurium spatu eius dimidiu[m]. Et ab eo ad Venerem. A qua ad Solem sescuplum. A Sole ad Martem Tonum id est qua[n]tum ad Lunam a terra. Ab eo ad Iouem dimidium. Et ab eo ad Saturnu[m] dimidium. Et inde sescuplum ad Signiferu[m]. Ita. vii. Tonos effici. qua[m] Diapason harmoniam uocant. hoc est uniuersitatem co[n]centus [...] ", (Plinius Secundus, 1250, fol. 22r).

„XX. But occasionally Pythagoras draws on the Tone theory of music, and designates the distance between the earth and the moon as a whole tone, that between the moon and Mercury a semitone, between Mercury and Venus the same, between her and the sun a tone and a half, between the sun and Mars a tone (the same as the distance between the earth and the moon), between Mars and Jupiter half a tone, between Jupiter and Saturn half a tone, between Saturn and the zodiac a tone and a half: the seven tones thus producing the so-called diapason, i.e. a universal harmony [...] ", (Rackham, 1967, p. 227 ff.).

Plato's *Book X* of his *Republic*, the *Myth of Er* (c.f. Böckh, 1852; Bergren, 2017) deals with the *fifth etheric element*, the *quintessence* what the stars *and* the human psyche are meant to consist of[4]:

„As the eyes, said I, seem formed for studying astronomy, so do the ears seem formed for harmonious motions: and these seem to be twin sciences to one another, as also the Pythagoreans say.", (Davis et al., 1901, p. 252).

See further the concept of *Hylomorphism*, conceiving every physical entity as a compound of *matter* (hyle) and *form* (morphē) in *Metaphysics, Book VII* of Aristotle (c.f. e.g. Aristoteles, 1325; Aristoteles & Clichtoveus, 1510 or Aristoteles & Theophrastus, 1608) and Pseudo[5]-Dionysius' *De Cælesti Hierarchia* (c.f. Dionysius, 1350; Ficinus, 1503 or Corderius, 1644, res.); c.f. also the distinction between the celestial and the elemental orbits, *Distinct[i]o orbitu[m] tam celestiu[m] q[uonia]m elementarium*, by Schedel (1493):

„Distantia predictorum orbium et planetarum hec est. A terra vsq[ue] ad lunam sunt miliaria .xv.dc.xxv. miliaria. hec sunt stadia .cxxvi.", (Schedel, 1493, fol. VIr).

This is the distance of the *predicted* orbs and planets. From the Earth to the Moon there are 15625 miles, where miles are 126 stades[6].

For the (b) *scientific approach* to astronomy see the *Phenomena* and *Enoptron* by Eudoxus of Cnidus (Goldstein & Bowen, 1983; Schrausser, 2024b), Euclid's *Elementa Geometriæ* (Ratdolt, 1482), Aristarchus (Berggren & Sidoli, 2007), Hipparchus (Swerdlow, 1969; Toomer 1974), above all in addition to the *Almagest*, Ptolemy's *Planetary Hypotheses* (Goldstein, 1967) and Bartholomeu Velho's adaptation of it in his *Cosmographia* (Velho, 1568; Meirinhos, 2022; Schrausser, 2024c), additionally, after the *introduction of the telescope* in 1608 (c.f. Wolf, 1870), the foundations of contemporary astronomy with the works of Messier (1784), Herschel (1786) and Dreyer (1888), further astronomical *simulation software* of e.g. Ochsenbein et al. (2000), the Strasbourg astronomical Data Center (2023) or from the author (Schrausser, 2023):

„ [...] largely responsible for turning astronomy into a mathematical science [...] was Eudoxus of Cnidus (ca. 390-337 B.C.). [...] we propose that Eudoxus was influenced by cosmological speculation, particularly that of the Pythagoreans and Plato. For, in their view, the circular motions of the heavenly bodies manifested a moral order that was ultimately analyzable by means of [...] whole-number ratios as melodious sound.", (Goldstein & Bowen, 1983, p. 332—333).

„The Ptolemaic System is the name usually given to the world picture, current in the Middle Ages and the Renaissance, according to which the planetary spheres are nested to fill exactly the space between the highest sublunary element, fire, and the fixed stars. There is [...] no trace of it in Ptolemy's Almagest [...] The *Planetary Hypotheses* was included by Heiberg in his edition of Ptolemy's minor astronomical works. There one finds [...] that the Ptolemaic System is indeed the creation of Ptolemy [...] ", (Goldstein, 1967, p. 3).

An overview of the History of *ancient* mathematics can be found in Neugebauer (1969) or Robson (2008). For the History of *Greek* Mathematics in special see Heath (1921a, b) or of course Pappus' *Mathematicae Collection* (trans. Commandino & Concordia, 1588, Commandino, 1660, res.), for the *Middle Ages* and western Europe see e.g. Suter (1887) and especially Bertrand Russell (1946, 2004, respectively).

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[4] By that circumstance the *influence* of the stars on *human fate* was considered to be explained in context with *astrology*.
[5] 5th – 6th c. AD, portraying himself as Dionysius the Areopagite (1st c. AD), an Athenian judge at the Areopagus Court, who converted to Christianity.
[6] Where $d_L = 364179\ km$, with Attic stades $st = 184.98\ m$; from this, one further calculates the total size up to the *firmament* to be approximately $d_F = 2525855\ km$, which, interestingly, corresponds largely to the size of the *Saturn system* up to *Iapetus, Saturn VIII* (see Williams, 2023).

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