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ON THE MINIMUM NUMBER OF MONOCHROMATIC 2-DIMENSIONAL SCHUR TRIPLES

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Abstract

Let χ be a 2-coloring of $[1,n]\times[1,n]$ and define $M_\chi(n)$ to be the number of monochromatic solutions to $\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{z}$ with $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in[1,n]\times[1,n]$ under χ. Let $M(n)$ be the minimum value of $M_\chi(n)$ over all χ. We show that $\frac{n^4}{209}(1+o(1)) < M(n) < \frac{n^4}{124}(1+o(1))$.

1. Introduction and Statement of Result

While attempting to prove Fermat's Last Theorem, Schur [9] proved the following result, which has since been named Schur's Theorem. We use integer interval notation in its statement and throughout the paper; that is, $[1,n]=\{1,2\ldots,n\}$

Schur's Theorem. *For every $r\in\mathbb{Z}^+$, there exists a minimal integer $S(r)$ such that any r-coloring of $[1,S(r)]$ admits a monochromatic solution to $x+y=z$.*

We call a solution to $x+y=z$ a *Schur triple*. Simultaneously and independently, this author along with Zeilberger [7] and Schoen [8] proved that the minimum number of monochromatic Schur triples over all 2-colorings of $[1,n]$ is $\frac{n^2}{22}(1+o(1))$. Later, Datskovsky [3] gave a nice proof of this result.

Recently, Balaji, Lott, and Rice [2] extended Schur's Theorem to hold over the two-dimensional integer lattice. Now, clearly if we 2-color $[1,S(r)]\times[1,S(r)]$ and restrict our attention only to those points on $y=x$, we have, by Schur's Theorem, a monochromatic solution to $\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{z}$ with $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ in $[1,S(r)]\times[1,S(r)]$ (where addition is component-wise). Their result is stronger. We state their result only for Schur triples and 2 dimensions (the full result concerns monochromatic generalized Schur triples and higher dimensional lattices; see [2], [1], [6] for more information).

Theorem 1 (Bilaji, Lott, and Rice [2]). *For every $r\in\mathbb{Z}^+$, there exists a minimal integer $S_2(r)$ such that for any $n\geq S_2(r)$, every r-coloring of $[1,n]\times[1,n]$ admits a monochromatic solution to $\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{z}$ with $\{\boldsymbol{x},\boldsymbol{y}\}$ linearly independent (over $\mathbb{Q}$).*

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The proof is an ingenious application of the Vandermonde matrix. Note that the monochromatic solutions to $\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{z}$ given before Theorem 1 clearly do not satisfy the linearly independent criterium.

Lott gave a very nice presentation of this result at the Integers 2023 Conference held in Athens, Georgia on May 18, 2023. Naturally, the subject piqued this author's interest, especially since the extremal colorings presented in the talk appeared to have a nice structure (which will make an appearance in this paper).

This article investigates the minimum number of monochromatic solutions to $\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{z}$ that must occur in any 2-coloring of $[1,n]\times[1,n]$. We will not be concerned about whether or not the linearly independent criterium is satisfied and will refer to any solution to $\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{z}$ with $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in[1,n]\times[1,n]$ as a *2-dimensional Schur triple*. To wit, we have the following result.

Theorem 2. *The minimum number, $M(n)$, of monochromatic 2-dimensional Schur triples in any 2-coloring of $[1,n]\times[1,n]$ satisfies*

$$\frac{n^4}{209}(1+o(1)) < M(n) < \frac{n^4}{124}(1+o(1)).$$

More accurately (but at the expense of beauty) we have

$$.0047989675n^4 \lessapprox M(n) \lessapprox .008002212n^4(1+o(1)).$$

Remark. In a roundabout way, Theorem 2 provides an alternative proof of Theorem 1 since the number of linearly dependent solutions is $o(n^4)$. To see this, let $(a,b)+(c,d)=(e,f)$ with $a\le c$ and assume (a,b) and (c,d) are linearly dependent, so that there exists $k\in\mathbb{Q}^+$ such that $(c,d)=k(a,b)$. For a fixed m, if $k=\frac{\ell}{m}$ then we must have $m\mid a$ and $m\mid b$. This gives us at most $\frac{n}{m}\cdot\frac{n}{m}=\frac{n^2}{m^2}$ possible choices for (a,b) and, since we must have $ka\le n$, there are at most $\ell\le nm$ possibilities for the value of k (with m fixed). Summing over the possible values of m we have at most

$$\sum_{m=1}^{n}\frac{n^2}{m^2}\cdot nm < n^3(\ln(n)+1)=o(n^4)$$

2-dimensional Schur triples of the form $((a,b),k(a,b),(k+1)(a,b))$.

2. Lower Bound

We will be counting unique solutions to $\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{z}$ so that each solution triple is viewed as a set and not an ordered triple. In this section it will be useful to double-count the unique solutions by viewing them as ordered triples so that both $\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{z}$ and $\boldsymbol{y}+\boldsymbol{x}=\boldsymbol{z}$ are counted. To clarify this, we make the following ordering of the points in $\mathbb{Z}^+\times\mathbb{Z}^+$.

Definition 3. Let $(a,b),(c,d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. We say that (a,b) is less than (c,d), and write $(a,b) \prec (c,d)$, if $a < b$ or $a = b$ and $c < d$. When enumerating 2-dimensional Schur triples, we will be counting solutions to $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{z}$ with $\boldsymbol{x} \prec \boldsymbol{y}$ and refer to $\boldsymbol{x}$ as the smallest value.

Remark. Note that in the above definition we do not consider $\boldsymbol{x} = \boldsymbol{y}$. We may safely ignore these solutions in our asymptotic calculations as the number of such solutions is $O(n^2)$.

Expanding the (well-established) approach used by Datskovsky in [3] to 2 dimensions, let χ be an arbitrary 2-coloring of $[1,n] \times [1,n]$ and let R and B be the red and blue points, respectively, under χ. Define

$$r(x,y) = \sum_{(s,t)\in R} e^{2\pi i(sx+ty)} \quad \text{and} \quad b(x,y) = \sum_{(s,t)\in B} e^{2\pi i(sx+ty)}.$$

Letting $N(\chi)$ be the number of monochromatic 2-dimensional Schur triples under χ and accounting for the aforementioned double-counting, by the orthogonality of characters (that is; we have $\int_0^1 e^{2\pi i x}\,dx$ equal to 1 for $x = 0$ and equals 0 for all other x; see page 5 of [5] for a short development of this fact) we have

$$2N(\chi) = \int_0^1 \int_0^1 \left((r(x,y))^2\, \overline{r(x,y)} + (b(x,y))^2\, \overline{b(x,y)} \right) dx\, dy,$$

where $\overline{r(x,y)} = r(-x,-y)$ and $\overline{b(x,y)} = b(-x,-y)$ (i.e., the complex conjugates). Using the same algebraic rearrangement as found in [3], and letting

$$f(x,y) = \left(r(x,y)\overline{b(x,y)} + \overline{r(x,y)}b(x,y) \right) \quad \text{and} \quad g(x,y) = r(x,y)b(x,y)(\overline{r(x,y)} + \overline{b(x,y)})$$

we see that

$$2N(\chi) = \int_0^1 \int_0^1 \left((r(x,y)+b(x,y))^2(\overline{r(x,y)} + \overline{b(x,y)}) - f(x,y) - g(x,y) \right) dx\, dy.$$

It is easy to see that

$$\int_0^1 \int_0^1 \left((r(x,y)+b(x,y))^2(\overline{r(x,y)} + \overline{b(x,y)}) \right) dx\, dy$$

is equal to $|\{(x_1,y_1),(x_2,y_2),(x_3,y_3) \in [1,n]\times[1,n] : (x_1,y_1)+(x_2,y_2) = (x_3,y_3)\}|$ and that this is equal to

$$\sum_{x_1=1}^{n} \sum_{y_1=1}^{n} \sum_{x_2=1}^{n-x_1} \sum_{y_2=1}^{n-y_1} 1 = \frac{n^4}{4} + O(n^3).$$

(Note that this informs us that the expected number of monochromatic 2-dimensional Schur triples under a random 2-coloring of $[1,n]\times[1,n]$ is $\frac{n^4}{32}(1+o(1))$.)

To find a lower bound for $N(\chi)$ we define

$$S(\chi) = \int_0^1 \int_0^1 f(x,y)\,dx\,dy \quad \text{and} \quad T(\chi) = \int_0^1 \int_0^1 g(x,y)\,dx\,dy$$

so that providing an upper bound on $S(\chi)+T(\chi)$ will give us a lower bound (up to $O(n^3)$) on $2N(\chi) = \frac{n^4}{4} - (S(\chi)+T(\chi))$.

Translating $S(\chi)$ and $T(\chi)$ into sets we find that

$$S(\chi) = \left|\left\{(x_1,y_1),(x_2,y_2) \in [1,n]^2 \text{ of different colors} : 1 \le x_2 - x_1, y_2 - y_1 \le n\right\}\right|$$

and

$$T(\chi) = \left|\{(x_1,y_1) \in R, (x_2,y_2) \in B : 1 \le x_1 + x_2, y_1 + y_2 \le n\}\right|.$$

This author was unable to determine the maximum of $S(\chi)+T(\chi)$ over all χ, so effective approximation is the next option. To this end, we partition $[1,n]\times[1,n]$ into squares of side length $\frac{n}{L}$, where L is a parameter to be chosen later:

$$I(i,j) = \left\{(x,y) \in [1,n]\times[1,n] : \frac{(i-1)n}{L} + 1 \le x \le \frac{in}{L}, \frac{(j-1)n}{L} + 1 \le y \le \frac{jn}{L}\right\}$$

for $1 \le i,j \le L$ and let $r_{i,j} = |R \cap I(i,j)|$ and $b_{i,j} = |B \cap I(i,j)|$. By overcounting elements in each of the sets underlying $S(\chi)$ and $T(\chi)$ we have

$$S(\chi) \le \sum_{i=1}^{L}\sum_{j=1}^{L}\sum_{k=i}^{L}\sum_{\ell=j}^{L}(r_{i,j}\,b_{k,\ell} + b_{i,j}\,r_{k,\ell}) \text{ and } T(\chi) \le \sum_{i=1}^{L}\sum_{j=1}^{L}\sum_{k=1}^{L-i+1}\sum_{\ell=1}^{L-j+1} r_{i,j}\,b_{k,\ell}.$$

It turns out that we achieve better results by replacing the overcounted squares with their total maximum possible value (e.g., $r_{i,j}\,b_{L-i+1,L-j+1} \le \frac{n^2}{2L^2}\cdot\frac{n^2}{2L^2}$) and allow the variables to be free from the places where overcounting occurs. Hence, we will use

$$S(\chi) \le \frac{3n^4}{8L} - \frac{7n^4}{32L} + \sum_{i=1}^{L}\sum_{j=1}^{L}\sum_{k=i+1}^{L}\sum_{\ell=j+1}^{L}(r_{i,j}\,b_{k,\ell} + b_{i,j}\,r_{k,\ell}) + O(n^3)$$

and

$$T(\chi) \le \frac{n^4}{2L} + \sum_{i=1}^{L-1}\sum_{j=1}^{L-1}\sum_{k=1}^{L-i}\sum_{\ell=1}^{L-j} r_{i,j}\,b_{k,\ell} + O(n^3)$$

so that, using $2N(\chi) = \frac{n^4}{4} - (S(\chi)+T(\chi))$, we have

$$2N(\chi) \ge \frac{n^4}{4} - \frac{7n^4}{8L} + \frac{7n^4}{32L^2} - A(\chi) + O(n^3),$$

where

$$A(\chi) = \sum_{i=1}^{L}\sum_{j=1}^{L}\sum_{k=i+1}^{L}\sum_{\ell=j+1}^{L}(r_{i,j}\,b_{k,\ell} + b_{i,j}\,r_{k,\ell}) + \sum_{i=1}^{L-1}\sum_{j=1}^{L-1}\sum_{k=1}^{L-i}\sum_{\ell=1}^{L-j} r_{i,j}\,b_{k,\ell}.$$

Next, as was done in [4], we let $r_{i,j} = (1 + x_{i,j})\frac{n^2}{2L^2}$ and $b_{i,j} = (1 - x_{i,j})\frac{n^2}{2L^2}$. Doing so, and relying on Maple for the simplification, we obtain (up to $O(n^3)$) the following:

$$A(\chi) = \frac{3n^4}{16} - \frac{3n^4}{8L} + \frac{3n^4}{16L^2} - \frac{n^4}{4L^4}\left(\sum_{i=1}^{L}\sum_{j=1}^{L}\sum_{k=i+1}^{L}\sum_{\ell=j+1}^{L} 2x_{i,j}\,x_{k,\ell} + \sum_{i=1}^{L-1}\sum_{j=1}^{L-1}\sum_{k=1}^{L-i}\sum_{\ell=1}^{L-j} x_{i,j}\,x_{k,\ell}\right).$$

with $-1 \leq x_{i,j} \leq 1$ for all $(i,j) \in [1,L] \times [1,L]$.

Remark. Not visible in the above simplification is the not-so-obvious fact that $\sum_{i=1}^{L-1}\sum_{j=1}^{L-1}\sum_{k=1}^{L-i}\sum_{\ell=1}^{L-j}(x_{i,j} - x_{k,\ell}) = 0$, which is quite fortunate as we have produced a quadratic form $A(\chi)$ with no linear term, which is needed for us to apply a very useful lemma (Lemma 4, below).

We now have

$$2N(\chi) \geq \frac{n^4}{16} - \frac{n^4}{2L} + \frac{n^4}{32L^2} + \frac{n^4}{4L^4}B(\boldsymbol{x}) + O(n^3), \tag{1}$$

where $B(\boldsymbol{x})$ is the quadratic form

$$B(\boldsymbol{x}) = \sum_{i=1}^{L}\sum_{j=1}^{L}\sum_{k=i+1}^{L}\sum_{\ell=j+1}^{L} 2x_{i,j}\,x_{k,\ell} + \sum_{i=1}^{L-1}\sum_{j=1}^{L-1}\sum_{k=1}^{L-i}\sum_{\ell=1}^{L-j} x_{i,j}\,x_{k,\ell}.$$

The last step before turning to the computer is to map $x_{i,j} \mapsto x_{(i-1)L+j}$ and writing

$$B(\boldsymbol{x}) = \boldsymbol{x}^T M \boldsymbol{x}, \quad \boldsymbol{x} = (x_1, x_2, \ldots, x_{L^2}), \tag{2}$$

where M is an $L^2 \times L^2$ matrix. We use the computer to create M; the Matlab code is in the Appendix.

Our first attempt at maximizing our quadratic form was done by Maple's QPSolve routine with $L = 16$. Unfortunately, the resulting bound was worse than the trivial bound $N(\chi) \geq 0$. However, it did give an interesting structure to the coloring that minimized $B(\boldsymbol{x})$, as seen in Figure 1 (disregard the lines at this point of the article). This will be useful in the next section.

Since it is now clear that we need a more computationally efficient algorithm, we need to translate the problem into one that can be solved with known semidefinite optimization programs. We will rely on the following well-known lemma (we include the proof for completeness).

Lemma 4. *Let A be an $m \times m$ matrix and let $D = (d_1, d_2, \ldots, d_m)$ be a diagonal $m \times m$ matrix. If $A + D$ is positive semidefinite, then $\boldsymbol{x}^T A \boldsymbol{x} \geq -\mathrm{trace}(D)$ for any $\boldsymbol{x} \in [-1,1]^m$.*

Proof. Since $A + D$ is positive semidefinite we have $\boldsymbol{x}^T(A + D)\boldsymbol{x} \geq 0$ for any $\boldsymbol{x}$. Hence,

$$\boldsymbol{x}^T A \boldsymbol{x} \geq -\boldsymbol{x}^T D \boldsymbol{x} = -\sum_{i=1}^{m} d_i x_i^2 \geq -\sum_{i=1}^{m} d_i = -\mathrm{trace}(D). \qquad \square$$

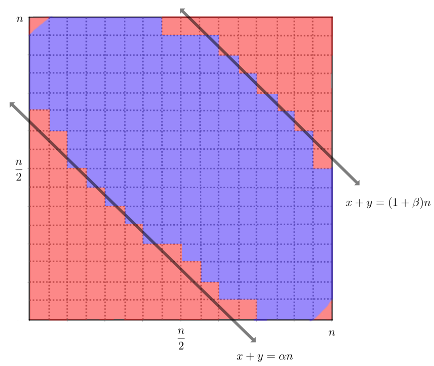

Figure 1: Coloring produced by QPSolve minimizing $B(\boldsymbol{x})$ with $L = 16$. The diagonal structure is similar to the colorings presented by Lott at the Integers 2023 conference.

Using this lemma, our goal is to find a diagonal matrix D with minimal trace such that $M + D$ is positive definite. We turn to the Matlab add-on cvx and used its SeDuMi engine option to run the optimization; see the Appendix for the cvx code.

Once such a D is found, using Inequality (1), we have

$$N(\chi) \geq \frac{n^4}{32} - \frac{n^4}{4L} + \frac{n^4}{64L^2} - \frac{n^4}{8L^4}\text{trace}(D) + O(n^3). \tag{3}$$

The best result obtained (using a single Apple Macbook Pro with an i7 2.3GHz chip and 32G of RAM) used $L = 100$ (and, according to Matlab over 50 million variables) and 4 days of time. Based on this, and some rounding to rationals, a "small" trace of 19,162,076 was discovered. Using Inequality (3) we can state that

$$N(\chi) \geq .0047989675n^4(1 + o(1)).$$

With more time and better resources, the method above will surely produce a better lower bound for $L > 100$.

3. Upper Bound

Taking our cue from the coloring produced by Maple's QPSolve routine given in Figure 1, we let $\frac{1}{2} \leq \beta \leq \alpha < 1$ and consider 2-colorings of $[1, n] \times [1, n]$ given by coloring all points between $x + y = \alpha n$ and $x + y = (1 + \beta)n$ blue and the other points red. Denote such a coloring by $\gamma(\alpha, \beta)$. We present the case $\frac{2+\beta}{4} \leq \alpha \leq \frac{2-\beta}{2}$ as this situation produced our smallest discovered bound.

Figure 2 will help explain our method of counting monochromatic 2-dimensional Schur triples under $\gamma(\alpha, \beta)$. (Regions A through G are not the same under different assumptions on α and β.)

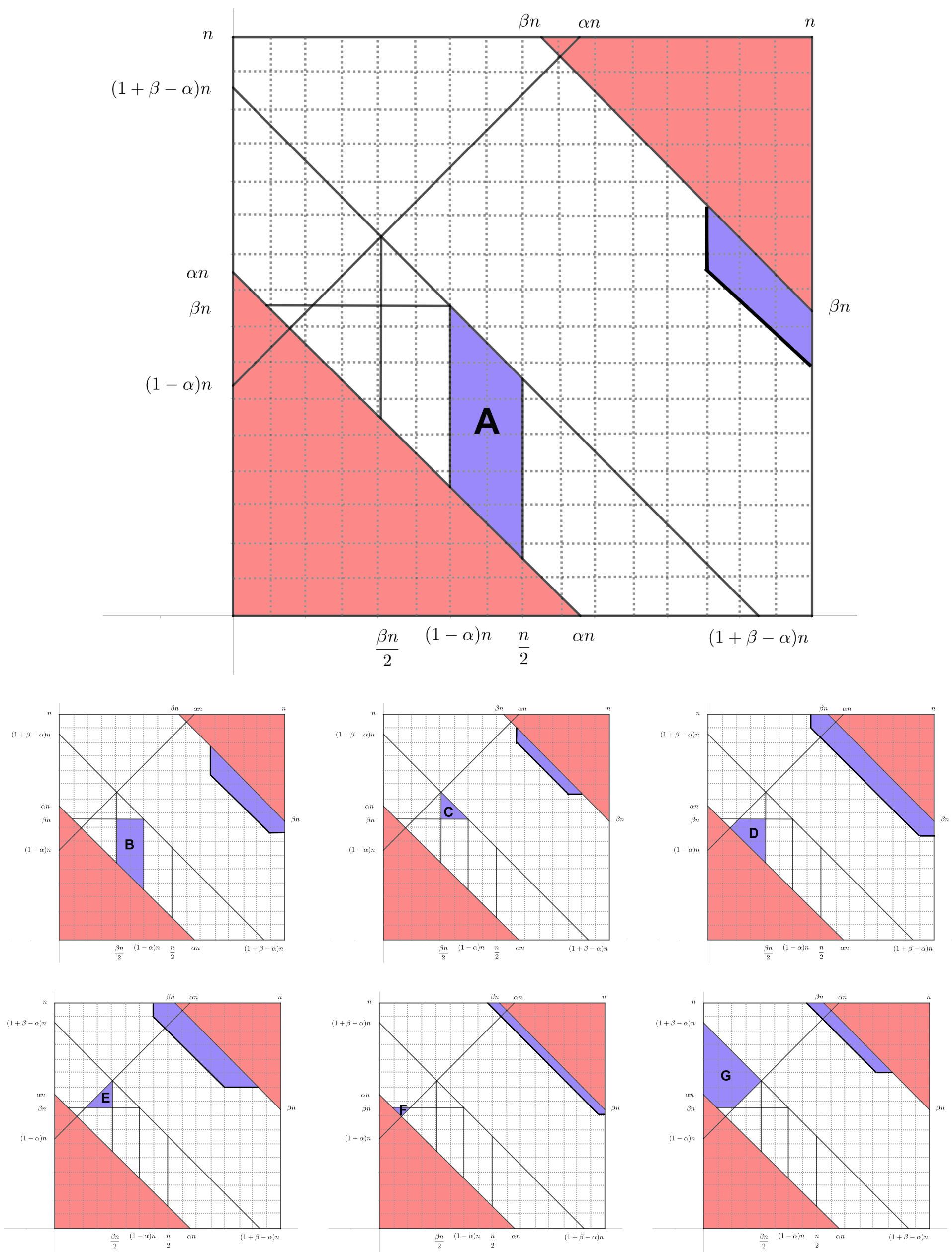

Figure 2: Counting monochromatic 2-dimensional Schur triples under $\gamma(\alpha, \beta)$ with (a, b) in the labeled region. The axes' labels across all cases are identical (Case A is larger for ease of reading these labels).

Considering the smallest blue point (a,b) of the Schur triple to be in each of the regions A through G, all points in the upper right area (bordering the line $x+y=(1+\beta)n$) can be reached by some blue point (c,d) with $(a,b) \prec (c,d)$. Summing these areas over all possible (a,b) in each region produces the following formulas (up to $O(n^3)$):

A: $$\sum_{a=(1-\alpha)n}^{n/2} \sum_{b=\alpha n-a}^{(1+\beta-\alpha)n-a} (n-2a)\left((1+\beta-\alpha)n-a-b\right)$$

B: $$\sum_{a=\beta n/2}^{(1-\alpha)n} \sum_{b=\alpha n-a}^{\beta n} \left((n-2a)\left((1+\beta-\alpha)n-a-b\right)-\frac{((1-\alpha)n-a)^2}{2}\right)$$

C: $$\sum_{a=\beta n/2}^{(1-\alpha)n} \sum_{b=\beta n}^{(1+\beta-a)n-a} \left(((1+\beta)n-2a-b)((1+\beta-\alpha)n-a-b)-\frac{((1+\beta-\alpha)n-a-b)^2}{2}\right)$$

D: $$\sum_{a=(\alpha-1/2)n}^{\beta n/2} \sum_{b=\alpha n-a}^{n/2} \left((n-b)((1+\beta-\alpha)n-a-b)-\frac{((1-\alpha)n-a+b)^2}{2}-\frac{(\beta n-b)^2}{2}\right)$$
$$+\sum_{b=n/2}^{\beta n} \sum_{a=b-(1-\alpha)n}^{\beta n/2} \left((n-b)((1+\beta-\alpha)n-a-b)-\frac{((1-\alpha)n-a+b)^2}{2}-\frac{(\beta n-b)^2}{2}\right)$$

E: $$\sum_{a=(\alpha+\beta-1)n}^{\beta n/2} \sum_{b=\beta n}^{(1-\alpha)n+a} \left((n-b)((1+\beta-\alpha)n-a-b)-\frac{((1-\alpha)n+a-b)^2}{2}\right)$$

F: $$\sum_{a=(\alpha-\beta)n}^{(\alpha-1/2)n} \sum_{b=\alpha n-a}^{\beta n} \left((n-b)((1+\beta-\alpha)n-a-b)-\frac{(\beta n-b)^2}{2}\right)$$
$$+\sum_{a=(\alpha-1/2)n}^{(\alpha+\beta-1)n} \sum_{b=(1-\alpha)n+a}^{\beta n} \left((n-b)((1+\beta-\alpha)n-a-b)-\frac{(\beta n-b)^2}{2}\right)$$

G: $$\sum_{b=\beta n}^{(1-\alpha+\beta/2)n} \sum_{a=\alpha n-b}^{b-(1-\alpha)n} ((n-b)((1+\beta-\alpha)n-a-b))$$
$$+\sum_{b=(1-\alpha+\beta/2)n}^{\alpha n} \sum_{a=\alpha n-b}^{(1+\beta-\alpha)n-b} ((n-b)((1+\beta-\alpha)n-a-b))$$
$$+\sum_{b=\alpha n}^{(1+\beta-\alpha)n} \sum_{a=1}^{(1+\beta-\alpha)n-b} ((n-b)((1+\beta-\alpha)n-a-b))$$

Using `Maple` we evaluate each of the above sums and discard all terms less than quartic powers of n. Summing all of these accounts for the blue 2-dimensional Schur triples. For the red 2-dimensional Schur triples, it is easy to find that the number of red 2-dimensional Schur triples entirely in the lower-left red triangle is $\frac{\alpha^4}{48}n^4(1+o(1))$ and that the only other red 2-dimensional Schur triples must have only their smallest term in the lower-left red triangle, of which there are $\frac{(1-\beta)^4}{24}n^4(1+o(1))$ such red triples. Having a formula for the number of monochromatic 2-dimensional Schur triples under $\gamma(\alpha,\beta)$, we next minimize the result as a function of α and β, checking critical points against all boundaries. We find that $\alpha \approx .6500298027$ and $\beta \approx .602220070$ (the actual values are functions of a root of $63x^3+240x^2-438x+166$) gives our best-discovered upper bound of $.008002212n^4(1+o(1))$.

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Appendix

First we have the `cvx` code (run in Matlab) for the optimization problem minimizing the trace of the diagonal matrix in Inequality 3:

```
cvx_begin sdp
  variable D(L^2,L^2) diagonal
  minimize(trace(D));
  subject to
  M+D>=0;
cvx_end
```

Below is the `Matlab` code to produce matrix M in Equation (2).

```
L=100
M=zeros(L^2,L^2);
for i=1:L/2,
for j=1:L/2,
M(L*(i-1)+j,L*(i-1)+j)=1;
end
end
for i=1:L,
for j=1:L,
for k=i+1:L,
for l=j+1:L,
M(L*(i-1)+j,L*(k-1)+l)=1;
M(L*(k-1)+l,L*(i-1)+j)=1;
end
end
end
end
for i=1:L/2,
for j=1:L-1,
for l=j+1:L-j,
M(L*(i-1)+j,L*(i-1)+l)=M(L*(i-1)+j,L*(i-1)+l)+1;
M(L*(i-1)+l,L*(i-1)+j)=M(L*(i-1)+l,L*(i-1)+j)+1;
end
for k=i+1:L-i,
for l=1:L-j,
M(L*(i-1)+j,L*(k-1)+l)=M(L*(i-1)+j,L*(k-1)+l)+1;
M(L*(k-1)+l,L*(i-1)+j)=M(L*(k-1)+l,L*(i-1)+j)+1;
end
end
end
end
```