A Constitutive Model of Ligaments and Tendons Accounting for Fiber-Matrix Interaction

Ratchada Sopakayang, Gerhard A. Holzapfel

***Abstract*—In this study, a new constitutive model is developed to describe the hyperelastic behavior of collagenous tissues with a parallel arrangement of collagen fibers such as ligaments and tendons. The model is formulated using a continuum approach incorporating the structural changes of the main tissue components: collagen fibers, proteoglycan-rich matrix and fiber-matrix interaction. The mechanical contribution of the interaction between the fibers and the matrix is simply expressed by a coupling term. The structural change of the collagen fibers is incorporated in the constitutive model to describe the activation of the fibers under tissue straining. Finally, the constitutive model can easily describe the stress-stretch nonlinearity which occurs when a ligament/tendon is axially stretched. This study shows that the interaction between the fibers and the matrix contributes to the mechanical tissue response. Therefore, the model may lead to a better understanding of the physiological mechanisms of ligaments and tendons under axial loading.

***Keywords*—Hyperelasticity, constitutive model, fiber-matrix interaction, ligament, tendon.

I. Introduction

THE purpose of this work is to develop a simple constitutive model that can describe the hyperelastic behavior of ligaments and tendons. In previous studies, phenomenological and structural models of ligaments and tendons have been formulated by accounting for the physiology of their main structural components, i.e. collagen fibers and proteoglycan-rich matrix [8], [10]. These models are not considering one important contributor to the mechanical behavior of ligaments and tendons which is the interaction between the fibers and the matrix. The fiber-matrix interaction has been identified that it plays a significant role on the elastic and viscoelastic properties of ligaments and tendons [2], [5] and other tissues [1], [3], [7]. Although some previous works incorporated the interaction into their models [2], [5], the mechanism of the coupling between the fibers and the matrix is still unclear and need more advanced studies. Therefore, in the present work, we develop a new mathematical model of ligaments and tendons for describing the tensile response. The model is formulated by using a continuum approach incorporating the structural changes of their main components: the fibers, the matrix and the fiber-matrix interaction. The specification of the work is described here as follows.

R. Sopakayang is with the Department of Mechanical Engineering, Faculty of Engineering, Ubon Ratchathani University, Warinchumrap, Ubon Ratchathani, 34190 Thailand (e-mail: ratchada.s@ubu.ac.th).

G. A. Holzapfel is with the Institute of Biomechanics, Graz University of Technology, Stremayrgasse 16/2, 8010 Graz, Austria and with the Faculty of Engineering Science and Technology, Norwegian University of Science and Technology, Trondheim, Norway.

II. Mathematical Formulation

A. Strain-Energy Function

The strain-energy function is assumed to be composed of three components which are based on the main structure of the tissues and their mechanisms. Therefore, the strain-energy function of ligaments and tendons, say W, are generated from the contributions originating from the collagen fibers, the matrix and the fiber-matrix interaction. Thus,

$$W = \Psi_{\mathrm{M}}(I_1) + P(I_4)\Psi_{\mathrm{F}}(I_4) + (1 - P(I_4))\Psi_{\mathrm{FM}}(I_1, I_4), \quad (1)$$

where Ψ_{M} is the energy stored in the matrix, Ψ_{F} is that of the fibers, Ψ_{FM} is the strain energy of the fiber-matrix interaction and $P(I_4)$ is the cumulative density function of the stretched fibers. In addition, I_1 is the first invariant of the right Cauchy–Green tensor $\mathbf{C}$ and $I_4 = \mathbf{C} : \mathbf{a}_0 \otimes \mathbf{a}_0$ is the fourth invariant of $\mathbf{C}$ and $\mathbf{a}_0$, where $\mathbf{a}_0$ describes the direction of the collagen fibers in the reference configuration, see, e.g., [4].

The matrix is assumed to behave according to a neo-Hookean material so that

$$\Psi_{\mathrm{M}}(I_1) = \frac{\mu}{2}(I_1 - 3), \quad (2)$$

where μ is the shear modulus. The mechanical response of the fibers is modeled as

$$\Psi_{\mathrm{F}}(I_4) = k(I_4 - 1)^2, \quad (3)$$

where k is a material parameter. The mechanical contribution of the interaction between the fibers and the matrix is modeled as

$$\Psi_{\mathrm{FM}}(I_1, I_4) = \frac{\mu}{2}k(I_1 - 3)(I_4 - 1)^2. \quad (4)$$

The collagen fibers are assumed to become straight at different stretches $\lambda_{\mathrm{s}} \geq 1$, defined by the Weibull probability density function [8], [10], i.e.

$$p(\lambda_{\mathrm{s}}) = \frac{\alpha}{\beta}\left(\frac{\lambda_{\mathrm{s}} - 1}{\beta}\right)^{\alpha - 1} \exp\{-[(\lambda_{\mathrm{s}} - 1)/\beta]^{\alpha}\}, \quad (5)$$

where $\alpha > 0$ is the so-called shape parameter and $\beta > 0$ is the so-called scale parameter. Therefore, the cumulative density function P of the stretched fibers follows with (5)

$$P(I_4) = \int_1^{I_4} p(\lambda_{\mathrm{s}})\mathrm{d}\lambda_{\mathrm{s}} = 1 - \exp\{-[(I_4 - 1)/\beta]^{\alpha}\}. \quad (6)$$

Finally, the strain-energy function (1) of the tissue can be rewritten as

$$W = \frac{\mu}{2}(I_1 - 3) + (1 - \exp\{-[(I_4 - 1)/\beta]^{\alpha}\})k(I_4 - 1)^2 + \exp\{-[(I_4 - 1)/\beta]^{\alpha}\}\frac{\mu}{2}k(I_1 - 3)(I_4 - 1)^2. \quad (7)$$

B. Cauchy Stress Tensor

Ligaments and tendons are classified as anisotropic materials with one family of collagen fibers. Therefore, by (just) considering the invariants I_1 and I_4, the Cauchy stress tensor $\mathbf{T}$ of the tissue can be expressed as follows [9]

$$\mathbf{T} = 2(W_1\mathbf{B} + I_4W_4\mathbf{a}\otimes\mathbf{a}) - p\mathbf{I}, \tag{8}$$

where $\mathbf{B}$ is the left Cauchy–Green tensor, $\mathbf{I}$ denotes the second-order unit tensor and p is as an indeterminate Lagrange multiplier which can be identified as a hydrostatic pressure [4]. In addition, in (8) $\mathbf{a} = \mathbf{F}\mathbf{a}_0$ denotes the fiber direction in the deformed configuration, and $\mathbf{F}$ is the deformation gradient. By recalling the strain-energy function (7), the differentiation of W can be written as

$$W_1 = \frac{\partial W}{\partial I_1} = \frac{\mu}{2} + \frac{\mu}{2}k(I_4-1)^2\exp\{-[(I_4-1)/\beta]^\alpha\}, \tag{9}$$

$$\begin{aligned} W_4 = \frac{\partial W}{\partial I_4} &= 2k(I_4-1) \\ &+2k(I_4-1)\exp\{-[(I_4-1)/\beta]^\alpha\}\left[\frac{\mu}{2}(I_1-3)-1\right. \\ &\left.+\frac{\alpha}{2}\left(\frac{I_4-1}{\beta}\right)^\alpha - \frac{\alpha\mu}{4}(I_1-3)\left(\frac{I_4-1}{\beta}\right)^\alpha\right]. \end{aligned} \tag{10}$$

In this study we focus on the description of the tensile response. We assume that the tensile loading, i.e. λ, is applied to the specimen in the x-direction which is also assumed to be the fiber direction, while x_1 and x_2 are the transverse directions. Therefore, $\mathbf{F}$ takes on the matrix form

$$[\mathbf{F}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda \end{bmatrix}. \tag{11}$$

The tissue is assumed to be an incompressible material, therefore the incompressibility constraint must be satisfied, i.e.

$$J = \det[\mathbf{F}] = \lambda_1\lambda_2\lambda = 1. \tag{12}$$

We then obtain

$$\lambda_1 = \lambda_2 = \frac{1}{\sqrt{\lambda}}. \tag{13}$$

Then, the left Cauchy–Green tensor $\mathbf{B}$ takes on the following matrix form

$$[\mathbf{B}] = [\mathbf{F}][\mathbf{F}^{\mathrm{T}}] = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix}, \tag{14}$$

while the identity tensor has the matrix form

$$[\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{15}$$

In the undeformed configuration, the unit fiber orientation vector is

$$[\mathbf{a}_0] = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}. \tag{16}$$

Since the deformed fiber direction $\mathbf{a}$ is related to the undeformed direction $\mathbf{a}_0$ according to $[\mathbf{a}] = [\mathbf{F}][\mathbf{a}_0]$, we obtain the structure tensor $\mathbf{a}\otimes\mathbf{a}$ in the matrix form as

$$[\mathbf{a}\otimes\mathbf{a}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix}. \tag{17}$$

The invariants can then be expressed as

$$I_1 = \mathrm{tr}[\mathbf{B}] = \lambda_1^2 + \lambda_2^2 + \lambda^2 = \frac{2}{\lambda} + \lambda^2, \tag{18}$$

$$I_4 = [\mathbf{a}_0]\cdot[\mathbf{B}][\mathbf{a}_0] = \lambda^2. \tag{19}$$

By substituting Eqs. (13)–(15), (17) and (19) into (8), we obtain the matrix

$$[\mathbf{T}] = \begin{bmatrix} \frac{2}{\lambda}W_1 - p & 0 & 0 \\ 0 & \frac{2}{\lambda}W_1 - p & 0 \\ 0 & 0 & 2\lambda^2W_1 + 2\lambda^4W_4 - p \end{bmatrix}. \tag{20}$$

The boundary condition of the problem is

$$T_{11} = T_{22} = 0. \tag{21}$$

By applying this boundary condition to the Cauchy stress matrix $[\mathbf{T}]$ in (20), we can then find the Lagrange multiplier p as

$$p = \frac{2}{\lambda}W_1 = \frac{\mu}{\lambda} + \frac{\mu}{\lambda}k(I_4-1)^2\exp\{-[(I_4-1)/\beta]^\alpha\}, \tag{22}$$

where (9) has been used. By substituting p into T_{33} of (20), we then obtain

$$\begin{aligned} T_{33} &= \mu\lambda^2 - \frac{\mu}{\lambda} + 4k\lambda^4(I_4-1) \\ &+4k\lambda^4(I_4-1)\exp\{-[(I_4-1)/\beta]^\alpha\} \\ &\left[\frac{\mu}{2}(I_1-3) - 1 + \frac{\alpha}{2}\left(\frac{I_4-1}{\beta}\right)^\alpha\right. \\ &-\frac{\alpha\mu}{4}(I_1-3)\left(\frac{I_4-1}{\beta}\right)^\alpha \\ &\left.+\frac{\mu}{4\lambda^2}(I_4-1) - \frac{\mu}{4\lambda^5}(I_4-1)\right], \end{aligned} \tag{23}$$

where (9) and (10) have been used.

Finally, we substitute the expression of the invariants from $(18)_3$ and $(19)_2$ into T_{33}, then we find an explicit relationship between the Cauchy stress T_{33} and the stretch λ of the tissue as

$$\begin{aligned} T_{33} &= \mu\lambda^2 - \frac{\mu}{\lambda} + 4k\lambda^4(\lambda^2-1) \\ &+4k\lambda^4(\lambda^2-1)\exp\{-[(\lambda^2-1)/\beta]^\alpha\} \\ &\left[\frac{\mu}{2}\left(\frac{2}{\lambda}+\lambda^2-3\right) - 1 + \frac{\alpha}{2}\left(\frac{\lambda^2-1}{\beta}\right)^\alpha\right. \\ &-\frac{\alpha\mu}{4}\left(\frac{2}{\lambda}+\lambda^2-3\right)\left(\frac{\lambda^2-1}{\beta}\right)^\alpha \\ &\left.+\frac{\mu}{4\lambda^2}(\lambda^2-1) - \frac{\mu}{4\lambda^5}(\lambda^2-1)\right]. \end{aligned} \tag{24}$$

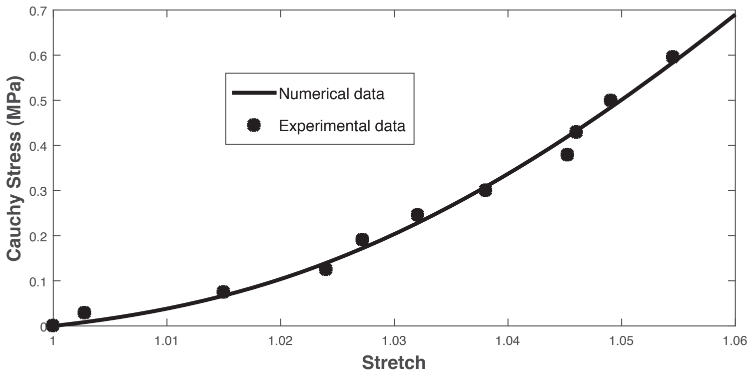

Fig. 1 Nonlinear elastic Cauchy stress-stretch data describing the tensile behavior of a sheep flexor tendon, and model fit with parameters $\mu = k = 1\,\mathrm{MPa}$, $\alpha = 1.431$ and $\beta = 0.143$ ($R^2 \approx 0.9913$)

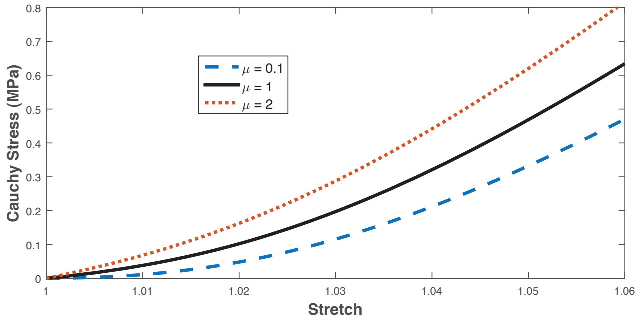

Fig. 2 Influence of the parameter μ (relating to the matrix) on the tensile behavior

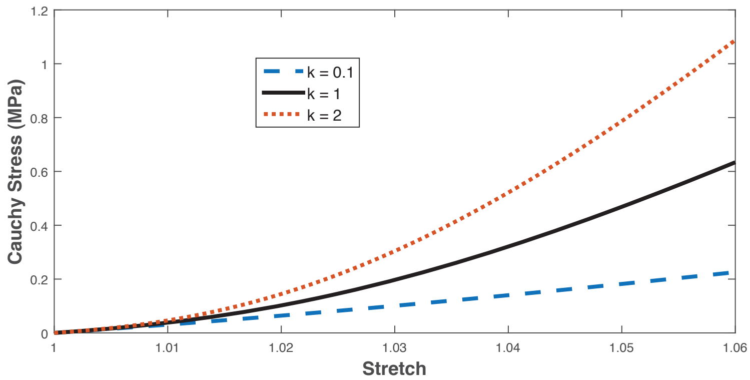

Fig. 3 Influence of the parameter k (relating to the collagen fibers) on the tensile behavior

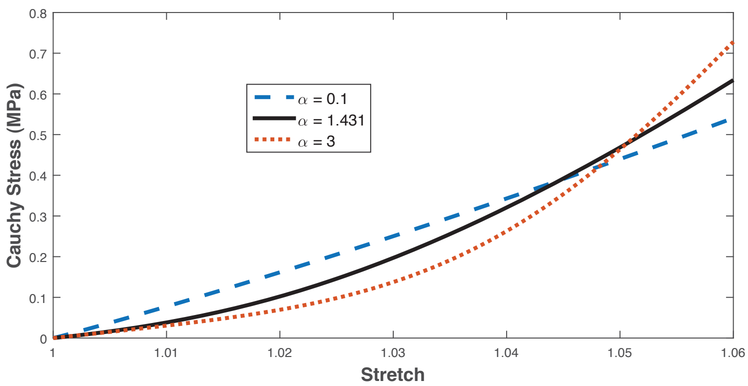

Fig. 4 Influence of the shape parameter α (relating to the Weibull probability density function) on the tensile behavior

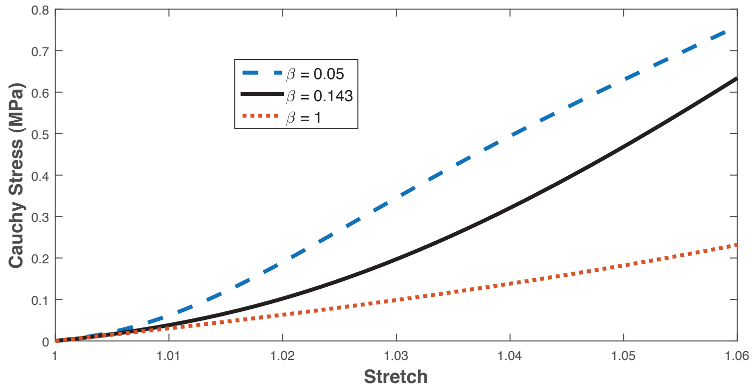

Fig. 5 Influence of the scale parameter β (relating to the Weibull probability density function) on the tensile behavior

III. Results

A. Parameter Estimation

There are four parameters in the model, i.e. (μ, k, α, β), requiring an estimation in order to describe the tensile behavior of ligaments and tendons. In general all parameters in the model can be found by curve fitting the model to the tensile data. Because of the lack of some experimental results in the published papers [2], [6], a complete set of parameters obtained by the curve fitting could not be found. The appropriate values of parameters for any specific type of ligaments and tendons can usually be found by validating the model with the experimental data of some specific tests.

For this work, in order to demonstrate the ability of the proposed model in describing the tensile behavior of ligaments and tendons, the values of the parameters μ and k are assumed to be fixed values of 1 MPa. Then, the model is curve fitted with the tensile experimental data of sheep flexor tendons, as documented in [2], [6]. Hence, the rest of the parameters are estimated from the curve fitting, and are $\alpha = 1.431$ and $\beta = 0.143$ with $R^2 \approx 0.9913$. As shown in Fig. 1, the model has a good fit with the experimental data and can describe the mechanical characterization under tension quite well.

B. Influence of Model Parameters

The influence of the model parameters on the tensile response was studied by varying the parameters as shown in Figs. 2-5. Thereby, each material parameter was varied individually, while the remaining parameters were fixed by the original values obtained by curve fitting of the proposed model to the published stress-strain data [2], [6]. The Figs. 2 and 3 illustrate the influence of the parameters μ and k on the characteristic tensile behavior, respectively. As shown in Fig. 2, an increase of μ causes an increase of the modulus of the ligaments/tendons over the stretch in every stretch region of the tensile behavior. In a similar way, an increase of k leads to an increase of the modulus of the tissues over the stretch, but not so much at the lower stretch domain, as shown in Fig. 3. At the low stretch region of the tensile behavior, the different values of k do not affect the tensile behavior but they play a significant role at the higher stretch domain.

The effects of the shape parameter α, and the scale parameter β of the Weibull probability density function that described the characteristic of the recruitment of collagen fibers are presented in Figs. 4 and 5, respectively. As can be seen from Fig. 4, for smaller values of α, the relationship between the Cauchy stress and the stretch of ligaments/tendons show less nonlinearity. For larger values of α, the curves show more nonlinearity of the modulus of the tissue influenced by the recruitment of fibers. However, the values of the stress of the tissues over stretch for each α are not much different. Therefore, α can mainly influence the nonlinearity of the modulus of the tissue but it does not play the role to increase the modulus over stretch. In another way, the scale parameter β influences both the nonlinearity and the change in the modulus over stretch, as shown in Fig. 5. For larger values of β, the stress-strain curve of the tissue shows less nonlinearity, and smaller values of modulus of the tissues over stretch.

IV. Discussion and Conclusion

A new mathematical model is presented to describe the nonlinear elastic behavior of ligaments and tendons under tensile loading. This mathematical model is formulated by accounting for the mechanical contribution of the main structural units of ligaments and tendons, i.e. the collagen fibers, the matrix and the fiber-matrix interaction. The model has four parameters (μ, k, α, β) representing the mechanical characteristic of the internal structure of the tissue. The parameters μ and k are related to the matrix and the collagen fibers, respectively, while the progression of the fiber recruitment is captured by the Weibull probability distribution function equipped with the shape parameter α and the scale parameter β. According to the results section, it can be seen that the model can describe the typical characteristic of the tensile behavior of ligaments and tendons very well. The curve fitting of the model and the experimental data obtained from the published papers [2], [6] is presented in Fig. 1 with $R^2 \approx 0.9913$. The study of the parameter variation shows that μ and k control the level of the values of the modulus of ligaments and tendons, as shown in Figs. 2 and 3, respectively, while α and β play a role on the nonlinearity of the modulus of the tissue in the tensile characterization, as shown in Figs. 4 and 5, respectively. The model simulations in Figs. 2 and 3 indicate that in a very small stretch region of the tensile behavior, only the matrix is active and responsible for load bearing, while the collagen fibers are embedded in the matrix and still wavy. Therefore, the wavy collagen fibers may move along the matrix at the beginning of the loading but cannot bear the load. The model suggests that only the matrix plays the role of the load bearing element in the very low stretch domain, while in the larger stretch region both collagen fibers and the matrix are load bearing.

In conclusion, a new constitutive model was proposed able to describe the mechanical response of ligaments and tendons under tension. The information about the orientation of the collagen fibers was incorporated into the continuum model, and the nonlinearity of the elastic behavior of the tissue could be demonstrated. The work has shown a simple and straightforward way to formulate an explicit relationship between the Cauchy stress and the stretch λ of the tissue. Certainly, we need more detailed (experimental) information about the fiber-matrix interaction of ligaments and tendons which would be very valuable for refined modeling. The model can easily be extended to describe other connective tissues or other hyperelastic materials with more fiber families and fiber dispersion. The model can also be extended to describe a viscoelastic response such as creep and relaxation along with the related hysteresis which for many materials such as soft tissues, rubbers and polymers is required.

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