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On Fourier Type Integral Transform for a Class of Generalized Quotients

A. S. Issa, S. K. Q. AL-Omari

***Abstract*—** In this paper, we investigate certain spaces of generalized functions for the Fourier and Fourier type integral transforms. We discuss convolution theorems and establish certain spaces of distributions for the considered integrals. The new Fourier type integral is well-defined, linear, one-to-one and continuous with respect to certain types of convergences. Many properties and an inverse problem are also discussed in some details.

***Keyword*—** Fourier type integral, Fourier integral, generalized quotient, Boehmian, distribution

I. Introduction

INTEGRAL transforms had provided a well established method for solving several physical and mathematical problems. Hartley and Fourier transforms are the powerful tools employed in diverse fields of science as spectral analysis, signal and image processing, filtering, encoding, data compression and reconstruction. They also find applications in many different research areas, such as computer science, quantum physics, biomedical and electrical engineering, etc. The Hilbert transform via the Fourier transform of $f(x)$ was defined as [11]

$$f^{\S}(f)(y)=\frac{1}{\pi}\int\limits_0^\infty \left(f^i f(x)\cos(xy)-f^r f(x)\sin(xy)\right)dx$$

where $f^r f(x) = \int_0^\infty f(t)\cos(xt)\,dt$ and $ff(x) = \int_0^\infty f(t)\sin(xt)\,dt$ are respectively the real and imaginary components of the Fourier transform of f, related by $f^t f = f^r f - i f^i f$.

In recent years convolution theorems of various integral transforms such Stieltjes transform [5], Hilbert transform [4], Hankel transform [1], Fourier cosine and sine transforms [3]; Sumudu transform [7]; Fourier cosine and sine transforms [2] were given in many citations. In this section of this paper we define the convolution theorem for $f^{\S}$ as follows.

Theorem 1. Let $f^{\S}f$, $f^{\S}g$ be the $f^{\S}$s of f and g respectively. Then, we have

$$f^{\S}(f\sharp g)(x)=f^{\S}f(x)\,f^{\S}g(x), \qquad (1)$$

where

$$(f\sharp g)(t)=\int\limits_0^\infty\bigg(f(t)\,f^i g(\eta)\cos(x\eta) + f(t)\,f^r g(\eta)\sin(x\eta)\bigg)d\eta.$$

A. S. Issa and S. K. Q. AL-Omari are with the Department of Applied Sciences, Faculty of Engineering Technology, Al-Balqa Applied University, Amman 11134, Jordan (e-mail: ammar82@bau.edu.jo; s.k.q.alomari@fet.edu.jo).

$$\begin{aligned}
&f^{\S}f(x)\,f^{\S}g(x)=\\
&=\int\limits_0^\infty\left(f^i f(\xi)\cos(x\xi)+f^r f(\xi)\sin(x\xi)\right)d\xi\\
&\times\int\limits_0^\infty\left(f^i g(\eta)\cos(x\eta)+f^r g(\eta)\sin(x\eta)\right)d\eta\\
&=\int\limits_0^\infty\left(\int\limits_0^\infty\left(\begin{array}{c} f^i f(\xi)\,f^i(x)\,g(\eta)\cos(x\eta)\\ +f^i g(\xi)\,f^r g(\eta)\sin(x\eta)\end{array}\right)d\eta\right)\cos x\xi d\xi\\
&+\int\limits_0^\infty\left(\int\limits_0^\infty\left(\begin{array}{c} f^r f(\xi)\,f^i(x)\,g(\eta)\cos(x\eta)\\ +f^i f(\xi)\,f^r g(\eta)\sin(x\eta)\end{array}\right)d\eta\right)\sin x\xi d\xi.
\end{aligned}$$

The equation above can be expressed as

$$f^{\S}f(x)\,f^{\S}g(x)=\int\limits_0^\infty\left(\vartheta(\xi)\cos(x\xi)+\partial(\xi)\sin(x\xi)\right)d\xi,$$

where

$$\vartheta(\xi)=\int\limits_0^\infty\bigg(f^i(x)\,f(\xi)\,f^i(x)\,g(\eta)\cos(x\eta) + f^i(x)\,f(\xi)\,f^r g(\eta)\sin(x\eta)\bigg)d\eta$$

and

$$\partial(\xi)=\int\limits_0^\infty\bigg(f^r f(\xi)\,f^i(x)\,g(\eta)\cos(x\eta) + f^i(x)\,f(\xi)\,f^r g(\eta)\sin(x\eta)\bigg)d\eta$$

Therefore, we can write $\vartheta(\xi)$ as

$$\begin{aligned}
\vartheta(\xi) &= \int\limits_0^\infty\bigg(\int\limits_0^\infty\bigg(f(t)\,f^i g(\eta)\cos(x\eta) + f(t)\,f^r g(\eta)\sin(x\eta)\bigg)d\eta\bigg)\sin(t\xi)\,d\xi\\
&= \int\limits_0^\infty (f\sharp g)(t)\sin(t\xi)\,d\xi
\end{aligned}$$

where

$$(f\sharp g)(t)=\int\limits_0^\infty\bigg(f(t)\,f^i g(\eta)\cos(x\eta)$$

$$+f(t)\,f^r g(\eta)\sin(x\eta)\Big)d\eta \qquad (2)$$

Similarly, we proceed to get $\partial(\xi) = f^r(f\sharp g)(\xi)$, where $f\sharp g$ has its usual meaning of (2). Hence the theorem is completely established.

Theorem 2. Let f, g and h be integrable functions over $(0,\infty)$. Then, the following identity holds $f^{\S}(f\sharp g) = f^{\S}(g\sharp f)$.

Proof. Let f, g be integrable functions over $(0,\infty)$. By aid of Theorem 1 we write $f^{\S}(f\sharp g) = f^{\S}ff^{\S}g = f^{\S}gf^{\S}f = f^{\S}(g\sharp f)$.

By considering the inverse transform our theorem follows.

Theorem 3. Let f, g and h be integrable functions over $(0,\infty)$. Then, the following identity holds $f^{\S}((f\sharp g)\sharp h) = f^{\S}(f\sharp(g\sharp h)) = f^{\S}(g\sharp(f\sharp h)) = f^{\S}(h\sharp(f\sharp g))$.

Proof is similar to that of the previous theorem.

This completes the proof of the theorem.

Theorem 4. Let f, g and h be integrable functions over $(0,\infty)$. Then the following identities are truely hold

(i) $f^{\S}(f\sharp(g+h)) = f^{\S}(f\sharp g) + f^{\S}(f\sharp h)$.

(ii) $f^{\S}(f+(g\sharp h)) = f^{\S}((f+g)\sharp(f\sharp h))$.

Proof. Proof of (i). Let f, g Let f, g and h be integrable functions. Then, by taking into account definitions we get

$$\begin{aligned} & f^{\S}(f\sharp(g+h))(x) \\ = & \frac{1}{\pi}\int_0^\infty \begin{pmatrix} f^i(f\sharp(g+h))(y)\cos(xy) \\ +f^r(f\sharp(g+h))(y)\sin(xy) \end{pmatrix} dy \\ = & \frac{1}{\pi}\int_0^\infty \begin{pmatrix} f^i(f\sharp g + f\sharp h)(y)\cos(xy) \\ +f^r(f\sharp g + f\sharp h)(y)\sin(xy) \end{pmatrix} dy. \end{aligned}$$

Hence properties f^i, f^r imply that $f^{\S}(f\sharp(g+h)) = f^{\S}(f\sharp g + f\sharp h)$. Proof of (ii) is analogous to that given for Part (i). The theorem is therefore completely proved. Next is a straightforward corollary of Theorem 2. Proofs are omitted.

Corollary 1. Let f, g and h be integrable functions over $(0,\infty)$. Then, we have

(i) $f\sharp g = g\sharp f$.

(ii) $(f\sharp g)\sharp h = f\sharp(g\sharp h)$

(iii) $f\sharp(g+h) = f\sharp g + f\sharp h$

(iv) $f + (g\sharp h) = (f+g)\sharp(f\sharp h)$.

II. f^t and $f^{\S}$ of the Class of Distributions

The space $\mathcal{D}$ of testing functions consists of all complex valued functions φ that are infinitely smooth and zero outside some finite interval. The set of continuous linear forms on $\mathcal{D}$ defines a distributions space, denoted by $\mathcal{D}'$.

The space of complex valued smooth functions is denoted by $\mathcal{E}$ and its dual space is denoted by $\mathcal{E}'$.

By $\mathcal{S}$ we denote the space of all complex-valued smooth functions φ such that, as $|t|\to\infty$, they and their partial derivatives decay to zero faster than all powers of $|t|^{-1}$. Elements of $\mathcal{S}$ are called testing functions of rapid descents. $\mathcal{S}$ is indeed a linear space. The dual space of $\mathcal{S}$ is called the space of tempered distributions $\mathcal{S}'$.

If $\phi\in\mathcal{S}$, then its partial derivatives are in $\mathcal{S}$. Indeed, $\mathcal{D}$ is dense in $\mathcal{S}$ and $\mathcal{S}$ is dense in $\mathcal{E}$. Moreover, $\mathcal{E}'\subset\mathcal{S}'\subset\mathcal{D}'$, $\mathcal{E}'$ being the space of distributions of compact support.

In this section, we discuss f^t and $f^{\S}$ on the distribution space.

Theorem 5. If f is in $\mathcal{S}$ then $f^t f$ is also in $\mathcal{S}$.

Proof (see [9]).

Corollary 2. If f is in $\mathcal{S}$ then $f^i f$ and $f^r f$ are in $\mathcal{S}$.

Corollary 3. If f is in $\mathcal{S}$ then $f^{\S}f$ is also in $\mathcal{S}$.

Proof. The proof of this corollary follows from the fact that $f^i f, f^r f\in\mathcal{S}$.

Let $f\in\mathcal{S}'$, then, by aid of Corollary 2 and Corollary 3, we define the distributional f^t and $f^{\S}$ transforms as

$$\langle f^t f, \varphi\rangle = \langle f, f^t\varphi\rangle \qquad (3)$$

and

$$\langle f^{\S} f, \varphi\rangle = \langle f, f^{\S}\varphi\rangle. \qquad (4)$$

(3) and (4) are well defined since $f^t\varphi$ and $f^{\S}\varphi$ are in $\mathcal{S}$. Further we have

$$f^t f, f^{\S} f\in\mathcal{S}'$$

for each $f\in\mathcal{S}'$.

Corollary 4. If $\varphi\in\mathcal{S}$ then $f^t\varphi, f^{\S}\varphi\in\mathcal{S}$.

Theorem 6. Let $f\in\mathcal{S}'$. Then $f^t f$ and $f^{\S}f$ are linear mapping from $\mathcal{S}'$ into $\mathcal{S}'$.

Proof. Let $f,\ g\in\mathcal{S}'$ and $\varphi\in\mathcal{S}$, $\alpha\in R$ be arbitrary then

$$\begin{aligned} \langle \alpha f^t(f+g), \varphi\rangle &= \langle \alpha(f+g), f^t\varphi\rangle \\ &= \alpha\langle f, f^t\varphi\rangle + \alpha\langle g, f^t\varphi\rangle \\ &= \alpha\langle f^t f, \varphi\rangle + \alpha\langle f^t g, \varphi\rangle. \end{aligned}$$

Similarly, we proceed for $f^{\S}f$, for all $f\in\mathcal{S}'$.

III. $B_1\left(\mathcal{S}', \mathcal{S}, \Delta, *\right) \approxeq \beta_*^{\S}$ and $B_2\left(f^{\S}\mathcal{S}', f^{\S}\mathcal{S}, f^{\S}\Delta, \dagger\right) \approxeq \beta_\dagger^{\S}$ Spaces

One of the most youngest generalization of functions, and more particularly of distributions, is the theory of Boehmians. The name Boehmian space is given to all objects defined by an abstract construction similar to that of field of quotients. The construction applied to function spaces yields various spaces of generalized functions.

The complete account of Boehmians was given by [6] − [8], [10], [12] − [15] and [16] − [18] and many others.

Let us now consider the convolution theorem requested in defining our quotient spaces of Boehmians $\beta_*^{\S}$ and $\beta_\dagger^{\S}$.

Theorem 7. Let f and g be integrable functions over $(0,\infty)$. Then, we have

$$f^{\S}(f*g) = 2f^{\S}\left(\left(f^t\right)^{-1}\left(\left(f^t f\right)\left(f^t g\right)\right)\right),$$

where $*$ is the convolution product of f and g (see [9]).

Proof. By the definition of $f^{\S}$ we have

$$f^{\S}(f*g)(x) = \int_0^\infty \left(\vartheta^i(\xi)\cos(x\xi) + \vartheta^r(\xi)\sin(x\xi)\right)d\xi. \qquad (5)$$

where $\vartheta^i = f^i(f*g)$ and $\vartheta^r = f^r(f*g)$.

Fubiniz theorem therefore implies

$$\vartheta^i(\xi) = \int_{-\infty}^{\infty} f(z) \int_{-\infty}^{\infty} g(t-z)\sin(t\xi)\,dtdz.$$

The substitution $t - z = y$ and the fact

$$\sin(y+z)\xi = \sin(y\xi)\cos(z\xi) + \cos(y\xi)\sin(z\xi)$$

imply

$$\vartheta^i = f^r f f^i f + f^i f f^r f. \tag{6}$$

Hence, invoking the identities

$$f^r f(\xi) = \frac{f^t f(\xi) + f^t f(-\xi)}{2}, f^i f(\xi) = \frac{f^t f(\xi) - f^t f(-\xi)}{2},$$

$$f^r g(\xi) = \frac{f^t g(\xi) + f^t g(-\xi)}{2},\ f^i g(\xi) = \frac{f^t g(\xi) - f^t g(-\xi)}{2}$$

in (6) and computations yield

$$\begin{aligned} \vartheta^i(\xi) &= \left(f^t f f^t g\right)(\xi) + \left(f^t f f^t g\right)(-\xi) \\ &= f^t\left(\left(f^t\right)^{-1}\left(f^t f f^t g\right)\right)(\xi) \\ &\quad + f^t\left(\left(f^t\right)^{-1}\left(f^t f f^t g\right)\right)(-\xi). \end{aligned} \tag{7}$$

Equivalently,

$$\vartheta^i = 2f^r\left(\left(f^t\right)^{-1}\left(f^t f f^t g\right)\right). \tag{8}$$

Similarly, we proceed to have

$$\vartheta^r = 2f^i\left(\left(f^t\right)^{-1}\left(f^t f f^t g\right)\right). \tag{9}$$

Hence invoking (8) and (9) in (5) completes the proof of our theorem.

Definition 1. Denote by $\beta_*^{\S}$ the Boehmian space with the convolution product $*$ as the operation, the $\mathcal{S}'$ as the group, $\mathcal{S}$ as a subgroup of $\mathcal{S}'$ $\left(\mathcal{S} \text{ dense in } \mathcal{S}'\right)$ and, the set Δ as the collection of delta sequences from $\mathcal{S}$ such that:

Δ_1 $\int \delta_n(x)\,dx = 1$

Δ_2 $\int |\delta_n(x)|\,dx < M$, $0 < M \in R$.

Δ_3 supp $\delta_n(x) \to 0$ as $n \to \infty$.

Let us consider the space $\beta_\dagger^{\S}$ for our next construction.

Denote by $f^{\S}\mathcal{S}'$ the space of $f^{\S}$s of distributions from $\mathcal{S}'$. Indeed, $f^{\S}\mathcal{S}'$ is a subspace of $\mathcal{S}'$, by (4). The member $\varphi_n \in f^{\S}\mathcal{S}'$ is said to converge in $f^{\S}\mathcal{S}'$ to a value φ if there are $\tau_n, \tau \in \mathcal{S}'$ such that τ_n reaches τ for large values of n. Also, denote by $f^{\S}\mathcal{S}$ the set of $f^{\S}$s of test functions from $\mathcal{S}$ then $f^{\S}\mathcal{S}$ is a subspace of $f^{\S}\mathcal{S}'$ by Corollary 8, In similar notations we denote $f^{\S}\Delta$.

Definition 2. Next, let us consider an operation $\dagger : f^{\S}\mathcal{S}' \times f^{\S}\mathcal{S} \to f^{\S}\mathcal{S}'$ defined by

$$\dagger(\varphi,\phi)(x) = 2f^{\S}\left(\left(f^t\right)^{-1}\left(f^t\varphi^* f^t\phi^*\right)\right)(x), \tag{10}$$

for $\varphi = f^{\S}\varphi^*, \phi = f^{\S}\phi^*$.

Theorem 8. Let $\varphi \in f^{\S}\mathcal{S}'$ and $\phi \in f^{\S}\mathcal{S}$. Then for $\varphi = f^{\S}\varphi^*$ and $\phi = f^{\S}\phi^*$, we have

$$\dagger(\varphi,\phi) = f^{\S}(\varphi^* * \phi^*).$$

Proof. For every $\varphi \in f^{\S}\mathcal{S}'$ and$\phi \in f^{\S}\mathcal{S}$, we have

$$\begin{aligned} \dagger(\varphi,\phi)(x) &= 2f^{\S}\left(\left(f^t\right)^{-1}\left(f^t\varphi^* f^t\phi^*\right)\right)(x) \\ &= f^{\S}(\varphi^* * \phi^*)(x). \end{aligned} \tag{11}$$

where $\varphi = f^{\S}\varphi^*, \phi = f^{\S}\phi^*$. This finishes the proof of the theorem.

Theorem 9. Let $\phi_1, \phi_2 \in f^{\S}\mathcal{S}$. Then, we have $\dagger(\phi_1,\phi_2) = \dagger(\phi_1,\phi_2)$.

Proof. Using (9) we get

$$\dagger(\phi_1,\phi_2) = 2f^{\S}\left(\left(f^t\right)^{-1}\left(f^t\phi_1^* f^t\phi_2^*\right)\right),$$

where $\phi_1 = f^{\S}\phi_1^*, \phi_2 = f^{\S}\phi_2^*$.

By (9) and Theorem 8 we obtain

$$\begin{aligned} \dagger(\phi_1,\phi_2)(x) &= f^{\S}(\phi_1^* * \phi_2^*)(x) \\ &= f^{\S}(\phi_2^* * \phi_1^*)(x) \\ &= 2f^{\S}\left(\left(f^t\right)^{-1}\left(f^t\phi_2^* f^t\phi_1^*\right)\right)(x) \\ &= \dagger(\phi_2,\phi_1)(x). \end{aligned}$$

This finishes the proof of the theorem.

Theorem 10. Let $\varphi_1, \varphi_2, \varphi_n, \varphi \in f^{\S}\mathcal{S}'$ and $\phi \in f^{\S}\mathcal{S}$. Then, we have

(i) $\dagger(k\varphi_1,\phi) = \dagger(\varphi_1,k\phi) = k(\dagger(\varphi_1,\phi)), k \in R$.

(ii) $\dagger(\varphi_1+\varphi_2,\phi) = \dagger(\varphi_1,\phi) + \dagger(\varphi_2,\phi)$.

(iii) $\dagger(\varphi_n,\phi) \to \dagger(\varphi,\phi)$ as $n \to \infty$.

Proof. Proof of (i). Linearity of $f^{\S}$s and f^t which are obvious by properties of the integral operators and (9) suggest to have

$$\begin{aligned} \dagger(k\varphi,\phi)(x) &= 2f^{\S}\left(\left(f^t\right)^{-1}\left(kf^t\varphi^* f^t\phi^*\right)\right)(x) \\ &= 2f^{\S}\left(\left(f^t\right)^{-1}\left(f^t\varphi^*\left(kf^t\phi^*\right)\right)\right)(x) \\ &= 2f^{\S}\left(\left(f^t\right)^{-1}\left(f^t\varphi^* f^t\left(k\phi^*\right)\right)\right)(x) \\ &= \dagger(\varphi,k\phi)(x). \end{aligned}$$

Similarly,

$$\dagger(k\varphi,\phi) = k(\dagger(\varphi_1,\phi)).$$

Proof of (ii) and (iii) follows from simple computations. This finishes the proof of the theorem.

Theorem 11 Let $(\alpha_n), (\varepsilon_n) \in f^{\S}\Delta$. Then, we have $\dagger(\alpha_n,\varepsilon_n) \in f^{\S}\Delta$.

Proof. For $(\alpha_n), (\varepsilon_n) \in f^{\S}\Delta$, we have

$$\begin{aligned} \dagger(\alpha_n,\varepsilon_n)(x) &= 2f^{\S}\left(\left(f^t\right)^{-1}\left(f^t\alpha_n^* f^t\varepsilon_n^*\right)\right)(x) \\ &= f^{\S}(\alpha_n^* * \varepsilon_n^*)(x). \end{aligned}$$

Since $\alpha_n^* * \varepsilon_n^* \in \Delta$ we get

$$\dagger(\alpha_n,\varepsilon_n)(x) \in f^{\S}\Delta.$$

This finishes the proof of the theorem.

Theorem 12 Let $\varphi \in f^{\S}\mathcal{S}', \phi_1, \phi_2 \in f^{\S}\mathcal{S}$. Then, we have

$$\dagger(\dagger(\varphi,\phi_1),\phi_2) = \dagger(\varphi,\dagger(\phi_1,\phi_2)).$$

Proof. Follows from similar computations to that used for the above theorem. In details, for $\phi_1 = f^{\S}\phi_1^*, \phi_2 = f^{\S}\phi_2^*$ and $\varphi = f^{\S}\varphi^*$ we see that

$$\begin{aligned} \dagger(\dagger(\varphi,\phi_1),\phi_2)(x) &= f^{\S}\left(\dagger(\varphi,\phi_1)^* * \phi_2^*\right)(x) \\ &= f^{\S}\left(\left(f^{\S}(\varphi^* * \phi_1^*)\right)^* * \phi_2^*\right)(x) \\ &= f^{\S}\left((\varphi^* * \phi_1^*) * \phi_2^*\right)(x) \\ &= f^{\S}\left(\varphi^* * (\phi_1^* * \phi_2^*)\right)(x) \\ &= f^{\S}\left(\varphi^* * (\phi_1^* * \phi_2^*)\right)(x) \\ &= \dagger(\varphi,\dagger(\phi_1,\phi_2))(x). \end{aligned}$$

Hence our theorem is completely proved.

Theorem 13. Let $\varphi_1, \varphi_2 \in f^{\S}\mathcal{S}'$ and $(\delta_n) \in f^{\S}\Delta$ and $\dagger(\varphi_1,\delta_n) = \dagger(\varphi_2,\delta_n)$, Then $\varphi_1 = \varphi_2$.

Proof. Assume $\dagger(\varphi_1,\delta_n)(x) = \dagger(\varphi_2,\delta_n)(x)$. Then, we have

$$2f^{\S}\left(\left(f^t\right)^{-1}\left(f^t\varphi_1^* f^t\delta_n^*\right)\right)(x) = 2f^{\S}\left(f^t\varphi_2^* f^t\delta_n^*\right)(x).$$

Hence, $f^{\S}(\varphi_1^* * \delta_n^*)(x) = f^{\S}(\varphi_2^* * \delta_n^*)(x)$. Allowing $n \to \infty$ gives $f^{\S}(\varphi_1^*) = f^{\S}(\varphi_2^*)$. Hence $\varphi_1 = \varphi_2$. This finishes the proof of the theorem.

Theorem 14. Let $(\delta_n) \in f^{\S}\Delta$ and $\varphi \in f^{\S}\mathcal{S}'$. Then, we have

$$\dagger(\varphi,\delta_n) \to \varphi \text{ as } n \to \infty.$$

Proof. Since $\varphi \in f^{\S}\mathcal{S}', (\delta_n) \in f^{\S}\Delta$ there are $\varphi^* \in \mathcal{S}, \delta_n^* \in \Delta$ such that $f^{\S}\varphi^* = \varphi$ and $\delta_n = f^{\S}\delta_n^*$. Hence

$$\begin{aligned} \dagger(\varphi,\delta_n)(x) &= 2f^{\S}\left(\left(f^t\right)^{-1}\left(f^t\varphi^* f^t\delta_n^*\right)\right)(x) \\ &= f^{\S}(\varphi^* * \delta_n^*)(x) \to f^{\S}\varphi^* = \varphi \end{aligned}$$

as $n \to \infty$. This finishes the proof of the theorem.

The Boehmian space $\beta_{\dagger}^{\S}$ is completely established.

A typical element in $\beta_{\dagger}^{\S}$ is given as $\left[\frac{f^{\S}f_n}{f^{\S}\phi_n}\right]$. Concept of quotients of sequences is justified by the computation,

$$\begin{aligned} \dagger\left(f^{\S}f_n, f^{\S}\phi_m\right) &= 2f^{\S}\left(\left(f^t\right)^{-1}\left(f^t f_n f^t\phi_m\right)\right) \\ &= f^{\S}(f_n * \phi_m) \\ &= f^{\S}(f_m * \phi_n) \\ &= f^{\S}\left(\left(f^t\right)^{-1}\left(f^t f_m f^t\phi_n\right)\right) \\ &= \dagger\left(f^{\S}f_m, f^{\S}\phi_n\right). \end{aligned}$$

Hence, $\dagger\left(f^{\S}f_n, f^{\S}\phi_m\right) = \dagger\left(f^{\S}f_m, f^{\S}\phi_n\right)$.

Two quotients $\dfrac{f^{\S}f_n}{f^{\S}\phi_n}$ and $\dfrac{f^{\S}g_n}{f^{\S}\tau_n}$ are said to be equivalent in the sense of $\beta_{\dagger}^{\S}$ if $\dagger\left(f^{\S}f_n, f^{\S}\tau_m\right) = \dagger\left(f^{\S}g_m, f^{\S}\phi_n\right)$.

Sum and multiplication by a scalar of two Boehmians can be defined in a natural way

$$\left[\frac{f^{\S}f_n}{f^{\S}\phi_n}\right] + \left[\frac{f^{\S}g_n}{f^{\S}\tau_n}\right] = \left[\frac{f^{\S}f_n \dagger f^{\S}\tau_n + f^{\S}g_n \dagger f^{\S}\phi_n}{f^{\S}\phi_n \dagger f^{\S}\tau_n}\right]$$

and

$$\alpha\left[\frac{f^{\S}f_n}{f^{\S}\phi_n}\right] = \left[\frac{\alpha f^{\S}f_n}{f^{\S}\phi_n}\right],\ \alpha \text{ being a complex number.}$$

The operation $\dagger$ and differentiation are defined by

$$\left[\frac{f^{\S}f_n}{f^{\S}\phi_n}\right] \dagger \left[\frac{f^{\S}g_n}{f^{\S}\tau_n}\right] = \left[\frac{f^{\S}f_n \dagger f^{\S}g_n}{f^{\S}\phi_n \dagger f^{\S}\tau_n}\right]$$

and

$$\mathcal{D}^{\alpha}\left[\frac{f^{\S}f_n}{f^{\S}\phi_n}\right] = \left[\frac{\mathcal{D}^{\alpha}f^{\S}f_n}{f^{\S}\phi_n}\right].$$

IV. $f^{\S e}$ OF GENERALIZED QUOTIENTS (BOEHMIANS)

Let us define the $f^{\S e}$ of a Boehmian $\left[\dfrac{f^{\S}f_n}{f^{\S}\phi_n}\right] \in \beta_*^{\S}$ by

$$f^{\S e}\left[\frac{f^{\S}f_n}{f^{\S}\phi_n}\right] = \left[\frac{f^{\S}f_n}{f^{\S}\phi_n}\right] \in \beta_{\dagger}^{\S}. \tag{12}$$

The operator $f^{\S e} : \beta_*^{\S} \to \beta_{\dagger}^{\S}$ is clearly well-defined.

We state without proof the following two theorems.

Theorem 15. $f^{\S e} : \beta_*^{\S} \to \beta_{\dagger}^{\S}$ is linear.

Theorem 16. $f^{\S e} : \beta_*^{\S} \to \beta_{\dagger}^{\S}$ is one-one.

Theorem 17. $f^{\S e} : \beta_*^{\S} \to \beta_{\dagger}^{\S}$ is continuous with respect to δ convergence.

Proof. Let $\beta_n \xrightarrow{\delta} \beta$ in $\beta_*^{\S}$ as $n \to \infty$. We show that $f^{\S e}\beta_n \to f^{\S e}\beta$ in $\beta_{\dagger}^{\S}$ as $n \to \infty$.

For each $\beta_n, \beta \in \beta_*^{\S}$ we, can find $f_{n,k}, f_k \in \mathcal{S}'$ such that

$$\beta_n = \left[\frac{f_{n,k}}{\phi_k}\right]$$

and $\beta = \left[\dfrac{f_k}{\phi_k}\right]$ and $f_{n,k} \to f_k$as $n \to \infty, \forall k \in N$.

Continuity of the transforms $f^{\S}$ implies

$$f^{\S}f_{n,k} \to f^{\S}f_k \text{ as } n \to \infty \text{ in } f^{\S}\mathcal{S}',$$

and, hence,

$$\frac{f^{\S}f_{n,k}}{f^{\S}\phi_k} \sim \frac{f^{\S}f_k}{f^{\S}\phi_k}.$$

Thus

$$\beta_n = \left[\frac{f^{\S}f_{n,k}}{f^{\S}\phi_k}\right] \to \beta\left[\frac{f^{\S}f_k}{f^{\S}\phi_k}\right] \text{ as } n \to \infty \text{ in } \beta_{\dagger}^{\S}.$$

This finishes the proof of the theorem.

Theorem 18. $f^{\S e}$ is continuous with respect to Δ convergence.

Proof. Let $\beta_n \xrightarrow{\Delta} \beta$ in $\beta_*^{\S}$ as $n \to \infty$. Then there is $f_n \in \mathcal{S}'$ and $\phi_n \in \Delta$ such that

$$(\beta_n - \beta) * \phi_n = \left[\frac{f_n * \phi_k}{\phi_k}\right]$$

and $f_n \to 0$ as $n \to \infty$. Hence by Theorem 7,

$$\begin{aligned} f^{\S}((\beta_n - \beta) * \phi_n) &= f^{\S}\left[\frac{f_n * \phi_k}{\phi_k}\right] \\ &= \left[\frac{f^{\S}(f_n * \phi_k)}{f^{\S}\phi_k}\right] \simeq f^{\S}f_n \to \infty.. \end{aligned}$$

as $n \to \infty$. This finishes the proof of the theorem.

Remark 1. Let $\beta = \left[\dfrac{f^{\S}f_n}{f^{\S}\delta_n}\right] \in \beta_{\dagger}^{\S}$. Then, we define the inverse transform $\left(f^{\S e}\right)^{-1} : \beta_{\dagger}^{\S} \to \beta_*^{\S}$ of $f^{\S e}$ as

$$\left(f^{\S e}\right)^{-1}\beta = \left[\frac{f_n}{\delta_n}\right]$$

which belongs *to the space* $\beta_*^{\S}$.

Properties of transform $\left(f^{\S e}\right)^{-1}$ can similarly obtained by techniques similar to that used for $f^{\S e}$. We prefer to omit the details.

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