Riemann Hypothesis disproved - New complex zeros of Zeta Function found off the critical line in the critical strip

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ABSTRACT. In the mid of the 19th Century the world-renowned Mathematician Bernhard Riemann stated in his Riemann hypothesis that all complex zeros would lie on the ½ line which is called the “critical line”. Although trillions of complex zeros have been found using numerical computational methods, till this day, no other complex zero off the critical line have been found. Using the Zeta function derived by Leonhard Euler from the Dirichlet eta function, it is found that there exist at least two other non-trivial zeros which do not lie on the critical line but are included in the critical strip between 0 and 1. These complex zeros have a real part of just slightly smaller than 1. The newly found complex zeros off the critical line provide counter examples to the Riemann hypothesis.

1. State of the art

In this short paper, I would like to present some new evidence which could disprove the Riemann hypothesis established over 160 years ago by famous German Mathematician Bernhard Riemann in 1859 which states that all non-trivial zeros of the (Riemann) zeta function are complex numbers with a real part of 1/2[1]. The confirmation of the Riemann hypothesis would significantly improve our understanding of prime numbers and could allow us to predict their distribution more accurately.

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\dots \tag{1}$$

The formula above shows us the Euler-Riemann zeta function or in short, the Riemann zeta function. It is a function with a complex variable s which is defined for Re(s) > 1 which means that the real part of complex s should be bigger than 1 for the zeta function to converge.

1 cf. Riemann (1859)

But since the zeta function form above is always positive and thus could never be zero which are one of if not the most interesting part of the zeta function, we need to analytically extend the function to a larger domain, which leads to the functional equation of the Riemann zeta function (2) that is defined for the whole complex plane except s=1 where it has a singularity:

$$\zeta(s)=2^{s}\pi^{(s-1)}\sin\left(\frac{(\pi s)}{2}\right)\Gamma(1-s)\zeta(1-s) \qquad (2)$$

Using this functional equation (2) we are able to find the trivial zeros of the Riemann zeta function which exist when variable s is a negative and even number like s = -2, -4, -6, -8, -10 etc. Whereas according to Riemann, the complex zeros should all be lying in the critical strip (0, 1). And for the calculation of the non-trivial zeros there we could use the Riemann-Siegel formula which is often used in combination with the Odlyzko–Schönhage algorithm for efficiency reasons[2].

Below are the first few non-trivial zeros (with 31 decimal places) of the Riemann zeta function found by Andrew Odlyzko[3]:

No	Complex zero
1	0.5 ± 14.1347251417346937904572519835625 **i**
2	0.5 ± 21.0220396387715549926284795938969 **i**
3	0.5 ± 25.0108575801456887632137909925628 **i**
4	0.5 ± 30.4248761258595132103118975305841 **i**
5	0.5 ± 32.9350615877391896906623689640749 **i**

2. Euler zeta function

$$\zeta(s)=\frac{1}{\left(1-2^{(1-s)}\right)}\sum_{n=1}^{\infty}\frac{(-1)^{(n+1)}}{n^{s}}, \textit{where } LHS=\frac{1}{\left(1-2^{(1-s)}\right)}, RHS=\sum_{n=1}^{\infty}\frac{(-1)^{(n+1)}}{n^{s}} \qquad (3)$$

The expression above is an alternative form of the Riemann zeta function that is derived by Leonhard Euler from the Dirichlet eta function (which is an alternating zeta function) about 100 years earlier than the zeta function of Bernhard Riemann and this Euler zeta function is defined for real part of complex s, Re(s), lying between 0 and 1.

2 Gourdon (2004) used the Odlyzko–Schönhage algorithm for calculating the first 10^{13} complex zeros of the Riemann zeta function

3 cf. Plouffe (2018)

When we plug in some of those non-trivial zeros like the first (approximate) zero

s = 0.5 + 14.134725141734693790457251983562 5 **i**

we are able to see that zeta function constructed by Euler (3) could indeed go resp. approach zero, as the whole term equals

-3.7072281416843200017222880498871042005043506420934e-33 +
2.3286790991943695398759987445612225645901146244608e-32 **i**

which is approximately zero (resp. roughly 0 + 0 **i** = 0) [4].

Besides looking at the result of the whole term of the Euler zeta function (3), it is also very interesting to know that the right-hand side (RHS) of (3) would be

3.4005690294717382995710795963142664961025718788764e-33 +
5.5867699048160274131498291646965118358366588415668e-32 **i**

that is also approximately zero while the left-hand side (LHS) results in

0.41125756131649665831188962512245334403511595012077 +
0.091389800453960258945607109723451975517645318913477 **i**

which is not really tending to zero. That means that the Euler zeta function (3) would only be exactly zero (0 ± 0 **i** = 0) if the RHS sum would (exactly) be zero[5]. This reasoning would help us for further analysis and interpretation of the other results concerning the complex zeros of the Riemann resp. Euler zeta function.

4 The term is approximately zero, e.g. when we define that we would not look at the results and numbers below 30 decimal places after zero.

5 If we adjust the value of the 1st complex zero to e.g. s = 0.5 + 14.1347251417346937904572519835624**7** **i** one could see that the right-hand side (RHS) sum would approximately be equal to **-**3.0973590638244729892093158948072846843709796102099112134262960035726 09e-35 **-** 5.0886284772380692626202016702978507627540247507721288176678286281057 01e-34 **i** which means that there exist a change of sign (from positive to negative) for both the real and imaginary part of the RHS sum that provides a strong evidence for the RHS sum to be exactly zero between those 2 complex s variable values. And when that is the case, the whole term would exactly equal to zero (0 ± 0 **i** = 0) as well even though the left-hand side term (LHS) would still be a bit far away from zero (later on in the paper we could use the same logic for further analysis of findings and results).

3. Main results

We have just verified some resp. the first non-trivial zero of the Riemann zeta function by plug in the approximate value of the zero into the equivalent zeta function form provided by Leonhard Euler (3). And this complex zero is lying on the 1/2 line with its real part as it is the case for all the other billions of non-trivial zeros found with computational calculations (which would support the Riemann hypothesis claiming that all non-trivial zeros of the Riemann zeta function have a real part of 1/2). But are those complex zeros on the 1/2 line the only non-trivial ones? The answer is no, because the zeta function provided by Leonhard Euler (3) gives us at least 2 more complex zeros in the critical strip of (0,1) and these complex zeros do not lie on the 0.5 line but its real part is near resp. just slightly smaller than 1 and the first newly found complex zero is (approximately) given as follows:

$\mathbf{s_{1a}}$= 0.9999999999999999999999999999999999 – 9.06472028365438761925536589143333535**i**

And putting this complex number into the right-hand side (RHS) would give us the following result:

-1.326729706702988116820819693492850593905383909624889558431972525583545e-34
- 1.614346628835474294523474991692888126278760161616976857516438756835898e-33 **i**

which is approximately zero (e.g. if we define that all number with more than 30 decimal places of zeros after zero would be considered to be approximately = 0 ± 0 **i** = 0 just like before when we have approximately verified e.g. the first complex zero of the Riemann zeta function with s = 0.5 + 14.1347251417346937904572519835625 **i** by plugging in the zero into the zeta function (3) derived by Euler).

And when we increase the negative imaginary part a little bit, we get:

$\mathbf{s_{1b}}$= 0.9999999999999999999999999999999999 – 9.06472028365438761925536589143333135**i**

and therefore we receive the following approximate result from the Euler zeta function which is equivalent to the Riemann zeta function:

1.7198777517536951164610759689206173182763934377547584263307600162145e-34 +
2.119164625231298708357504806918993387372729230159505817522862274579184e-33 **i**

We could see that the right-hand side (RHS) of (3) is just as small (and approximately zero) as before but both real part and imaginary part has changed the sign from negative to positive which means that between those 2 s (s_{1a}, s_{1b}) there should exist an exact (complex) zero. And when the right hand side (RHS) is exactly zero, no matter what the left-hand side will be – if it does not get infinitely large which is not the case with the newly found complex zero approaching 1 from the left-hand side[6] – then the whole term of the Euler derived zeta function will be exactly zero which disproves the Riemann hypothesis that all complex zeros should be lying on the 1/2 line with its real part.

And if we look more closely, there exists at least one more complex zero off the critical line in the critical strip and similar to the first newly found non-trivial zero, its real part is just slightly smaller than 1 and its imaginary part has the same absolute value but with an opposite (positive) sign now as it is the conjugate of the first non-trivial zero which does not lie on the critical line:

$\mathbf{s_{2a}}$=0.99999999999999999999999999999999999 + 9.0647202836543876192553658914333**5**35**i**

When we put this second complex zero into the Euler zeta function (3) we obtain the following result for the right-hand side of (3) which is approximately zero as well:

-1.326729706702988116820819693492850593905383909624889558431972525583545e-34
+ 1.614346628835474294523474991692888126278760161616976857516438756835898e-33 **i**

6 Although the left-hand side (LHS) of the Euler derived zeta function (3) will get bigger, when the right-hand side (RHS) approaches zero, the LHS will never reach infinity (which will i.a. be the case if s would be exactly equal to 1 and without any imaginary part and that does not apply to our situation with the complex zero having its real part just a bit smaller than 1 and an existing non-zero imaginary part. Even for s=1, while the LHS goes to infinity, the infinite sum on the RHS does not reach anywhere near 0 but is just slightly smaller than 0.7 so in other words, for the right-hand side of (3) to exactly be 0 we can not just set s=1 thus no ambiguous results exist for the whole term LHS*RHS in the Euler zeta function (3) with the RHS only reaching 0 if the LHS is going to infinity which is not even the case for s=1 as demonstrated earlier). And through small adjustments of s with both its real part (going even closer to 1 from the left side than now) and imaginary part, it is possible to get even more exact results when the infinite sum on the right-hand side (RHS) of the Euler zeta function (3) would get (approximately) closer to 0 ± 0 **i** = 0 which is using the same approximation logic when we have verified the first complex zero already found on the critical line of 1/2 in chapter 2. The only difference to our newly found complex zero off the critical line is that the LHS of (3) for the first non-trivial zero on the critical line does not get bigger and the whole term of the Euler zeta function (3) gets smaller when the infinite sum on the right-hand side (RHS) gets smaller as well but the whole term of (3) still never exactly reaches 0 ± 0 **i** = 0 which would only be the case if the infinite sum on the right-hand side (RHS) would exactly be 0 ± 0 **i** = 0. But using approximation techniques and if we define that all numbers smaller than a given threshold (like for example over 30 zero-decimal places after 0) would considered to be 0 then for both the first complex zero on the critical line 1/2 and the newly found non-trivial zero off the critical line (and near 1) the Euler derived zeta function (3) would yield an (approximate) zero which disproves the Riemann hypothesis.

And just like the first newly found complex zero when we decrease the positive imaginary part a little bit, we get:

$\mathbf{s_{2b}}$=0.9999999999999999999999999999999999999999 + 9.06472028365438761925536589143333135**i**

Plug in this zero with adjusted value into the right-hand side (RHS) of the Euler zeta function (3) we obtain the following result from which we could see that both the signs for the real part and imaginary part of the Euler zeta function (RHS) have switched as well:

1.7198777517536951164610759689206173182763934377547584263307600162145e-34
- 2.119164625231298708357504806918993387372729230159505817522862274579184e-33 **i**

And similar to the first newly found complex zero off the critical line (and also to all the existing non-trivial zeros on the critical 1/2 line), the whole Euler zeta function (3) = LHS*RHS decreases exactly to 0, if the infinite sum on the RHS exactly goes to 0 (given that the LHS does not reach infinity which is not the case here unlike e.g. for s=1). Therefore also for the 2nd newly found (approximate) complex zero - off the critical line in the critical strip and with its real part a little bit smaller than 1 - we could once again conclude that between those 2 approximate s (s_{2a}, s_{2b}) lying in a very small interval without any singularities or other discontinuities there should exist an exact complex zero too which provides another counter example to the Riemann hypothesis.

References

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