**JOURNAL OF THEORETICAL
AND APPLIED MECHANICS**
55, 1, pp. 177-188, Warsaw 2017
DOI: 10.15632/jtam-pl.55.1.177

LARGE DEFORMATION AND STABILITY ANALYSIS OF A CYLINDRICAL RUBBER TUBE UNDER INTERNAL PRESSURE

Jianbing Sang, Sufang Xing, Haitao Liu, Xiaolei Li, Jingyuan Wang, Yinlai Lv
School of Mechanical Engineering, Hebei University of Technology, Tianjin, China
e-mail: sangjianbing@126.com

Rubber tubes under pressure can undergo large deformations and exhibit a particular nonlinear elastic behavior. In order to reveal mechanical properties of rubber tubes subjected to internal pressure, large deformation analysis and stability analysis have been proposed in this paper by utilizing a modified Gent's strain energy function. Based on the nonlinear elastic theory, by establishing the theoretical model of a rubber tube under internal pressure, the relationship between the internal pressure and circumferential principal stretch has been deduced. Meanwhile stability analysis of the rubber tube has also been proposed and the relationship between the internal pressure and the internal volume ratio has been achieved. The effects on the deformation by different parameters and the failure reasons of the rubber tube have been discussed, which provided a reasonable reference for the design of rubber tubes.

Keywords: large deformation analysis, stability analysis, rubber tube, nonlinear elastic theory

1. Introduction

Cylindrical tube structures have been a subject of interest in the recent years due to their applicability in numerous fields. In many engineering applications, cylindrical tubes are subject to internal pressures and as a result undergo large deformations (Bertram, 1982, 1987). In the past, the analysis of this problem was based on small deformations and on the assumption that the material was linear elastic, but this led to prediction results not inaccurate for large deformation. It is well known that rubber-like materials exhibit highly nonlinear behavior character. In the case of nonlinear rubber tube structures undergoing large deformations, the problem is even more acute due to geometric and material nonlinearities (Antman, 1995; Bharatha, 1967; Green and Zerna, 1968; Ogden, 1984), and we can not utilize typical Hooke's law to describe the relationship between stress and strain.

From the point of view of mechanics perspective, the vital problem that should be solved is to select the reasonable and practical strain energy density function that describes the mechanical property of a rubber-like material. It follows from the fundamental representation theory in continuum mechanics that the strain-energy function of an isotropic rubber-like material can be represented in terms of either the principal invariants or principal stretches.

The pioneering work of Mooney, Rivlin and others on the nonlinear theory of elasticity sets up the basis for the analysis of rubber-like materials under large deformations.

In 1948, Rivlin put forward the strain energy function model to isotropic hyper elastic materials (Rivlin, 1948)

$$W = \sum_{i,j=0}^{\infty} C_{ij}(I_1-3)^i(I_2-3)^j \tag{1.1}$$

in which C_{ij} stands for the material constant; I_1 and I_2 are, respectively, the first and second invariants of the left Cuachy-Green deformation tensor.

Taking the linear combination of the Rivlin model, we can get the Mooney-Rivlin material (Mooney, 1940), the strain energy density function may be written as

$$W = C_1(I_1 - 3) + C_2(I_2 - 3) = C_1[(I_1 - 3) + \alpha(I_2 - 3)] \tag{1.2}$$

in which, C_1 and C_2 are material constants, and $\alpha = C_2/C_1$.

To simplify, the first of the Rivlin model can be used and it is a neo-Hookean material (Treloar, 1976), which can be expressed as follows

$$W(I_1) = \frac{1}{2}nkT(I_1 - 3) \tag{1.3}$$

A generalized neo-Hookean model widely used in the domain of biomechanics is a two--parameter exponential strain-energy named by Fung and Demiray (Fung, 1967)

$$W = \frac{\mu}{2b}\{\exp[b(I_1 - 3)] - 1\} \tag{1.4}$$

in which b is a positive dimensionless material parameter which can display the degree of strain--stiffening. In soft tissues, the value of b is in the range $1 \leqslant b \leqslant 5.5$.

Another well-known model of this type is the three parameter Knowles power law model (Knowles, 1977) as follows

$$W = \frac{\mu}{2b}\Big[\Big(1 + \frac{b}{n}(I_1 - 3)\Big)^n - 1\Big] \tag{1.5}$$

Gent (1996) proposed a new strain energy function for the non-linear elastic behavior of rubber-like materials. Because of its formal simplicity, this model has been widely applied to large elastic deformations of solids. The energy density function proposed by Gent for incompressible, isotropic, hyper elastic materials is shown as

$$W = -\frac{\mu}{2}J_m \ln\Big(1 - \frac{I_1 - 3}{J_m}\Big) \tag{1.6}$$

where μ is the shear modulus and J_m is the constant limiting value for $I_1 - 3$. Since W depends on the only first invariant of B, the Gent model belongs to the class of the generalized neo-Hookean materials.

Based on Gent's constitutive model, a modified model by Gent has been proposed to describe the mechanical property of an arterial wall in (Sang *et al.*, 2014), whose modified strain energy function is expressed as

$$W = -\frac{\mu J_m}{2} \ln\Big(1 - \frac{I_1^n - 3^n}{J_m}\Big) \tag{1.7}$$

where n is the material parameter.

From constitutive model (1.7), we can see that it can be transformed to the Gent model when $n = 1$. If $n = 1$ and$J_m \to \infty$, constitutive model (1.7) can be transformed to the neo-Hookean model.

The developments of analysis of rubber tubes have continually been accompanied by discussions. Zhu *et al.* (2008, 2010) analyzed the finite axisymmetric deformation of a thick-walled circular cylindrical elastic tube subject to pressure on its external lateral boundaries and zero displacement on its ends. Meanwhile, they considered bifurcation from a circular cylindrical deformed configuration of a thick-walled circular cylindrical tube of an incompressible isotropic elastic material subject to combined axial loading and external pressure. Research on the physical behavior of compressible nonlinear elastic materials for the problem of inflation of a thin-walled pressurized torus was developed by Papargyri-Pegiou and Stavrakakis (2000). Gent (2005) analyzed a inflating cylindrical rubber tube in terms of simple strain energy functions using Rivlin's theory of large elastic deformations. Mangan and Destrade (2015) used the 3-parameter Mooney and Gent-Gent (GG) phenomenological models to explain the stretch-strain curve of typical inflation. Based on the strain energy function by Gent, a thorough discussion (Feng *et al.*, 2010; Hariharaputhiran and Saravanan, 2016; Horgan, 2015; Horgan and Saccomandi, 2002; Pucci and Saccomandi, 2002; Rickaby and Scott, 2015) was given on molecular models and their relation to deformation of rubber-like materials.

Akyüz and Ertepinar (1999) investigated cylindrical shells of arbitrary wall thickness subjected to uniform radial tensile or compressive dead-load traction. By using the theory of small deformations superposed on large elastic deformations, the stability of the finitely deformed state and small, free, radial vibrations about this state are investigated. Akyüz and Ertepinar (2001) also investigated the stability of homogeneous, isotropic, compressible, hyperelastic, thick spherical shells subjected to external dead-load traction and gave the critical values of stress and deformation for a foam rubber, slightly compressible rubber and a nearly incompressible rubber. Alexander (1971), by using the non-linear analysis, predicted that the axial load had a significant effect on the value of tensile instability pressure. With thin-walled tubes of latex rubber, experiments were performed and the results were according with the results of the non-linear analysis in stable regions where the membrane retained its cylindrical shape. Based on the theory of large elastic deformations, Ertepinar (1977) investigated finite breathing motions of multi-layered, long, circular cylindrical shells of arbitrary wall thickness. And a tube consisting of two layers of neo-Hookean materials was solved both by exact and approximate methods, which was observed as an excellent agreement between the two sets of results. Bifurcation of inflated circular cylinders of elastic materials under axial loading was researched by Haughton and Ogden (1979a,b), who proposed that bifurcation might occur before the inflating pressure reached the maximum. A combination of the two mode interpreted in terms of bending for a tube under axial compression was discussed in terms of the length to radius ratio of the tube. At the same time, prismatic, axisymmetric and asymmetric bifurcations for axial tension and compression combined with internal or external pressure was discussed and presented for a general form of incompressible isotropic elastic strain energy function. Haughton and Ogden (1980) put a research on the deformation of a circular cylindrical elastic tube of finite wall thickness rotating about its axis, and achieved a range of values of the axial extension for which no bifurcation could occur during rotation. Jiang and Ogden (2000) proposed the axial shear deformation of a thick-walled right circular cylindrical tube of the compressible isotropic elastic material and discussed explicit solutions for several forms of the strain-energy function. Jiang and Ogden (2000) also analyzed the plane strain character of the finite azimuthal shear of a circular cylindrical annulus of a compressible isotropic elastic material by utilizing the strain energy as a function of two independent deformation invariants. Merodio and Ogden (2015) proposed a new example of the solution to the finite deformation boundary-value problem for a residually stressed elastic body and combined extension, inflation and torsion of a circular cylindrical tube subject to radial and circumferential residual stresses.

Based on Gent's constitutive model, a modified model has been proposed to describe incompressible rubber-like materials. The inductive material parameter n can reflect the hardening character of rubber-like materials. With the modified model, mechanical properties of rubber tubes subjected to internal pressure has been revealed and large deformation analysis and stability analysis has been proposed by utilizing Gent's modified strain energy function. Based on the nonlinear elastic theory, by establishing the theoretical model of rubber tubes under internal pressure, the relationship between the internal pressure and circumferential principal stretch has been deduced. Meanwhile, stability analysis of rubber tube has also been proposed and the relationship between the internal pressure and internal volume ratio has been achieved. The results show that the constitutive parameter n has a major impact on mechanical properties of the rubber tube, and when $n \leqslant 1$, the rubber tube becomes softening. The instability phenomenon in the rubber tube will appear only when n is less than 1.5. For different values of n, the range of the value of J_m which leads to instability also changes.

2. Finite deformation analysis

Based on the elastic finite deformation theory, the left Cauchy–Green tensor can be denoted by $\mathbf{B} = \mathbf{F}\cdot\mathbf{F}^{\mathrm{T}}$, where $\mathbf{F}$ is the gradient of the deformation and λ_1, λ_2, λ_3 are the principal stretches, then, for an isotropic material, W is a function of the strain invariants as follows

$$
\begin{aligned}
I_1 &= \operatorname{tr}\mathbf{B} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\
I_2 &= \frac{1}{2}[(\operatorname{tr}\mathbf{B}^2 - \operatorname{tr}(\mathbf{B}^2)] = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \\
I_3 &= \det\mathbf{B} = \lambda_1^2\lambda_2^2\lambda_3^2
\end{aligned}
\tag{2.1}
$$

By utilizing strain energy function (1.7), the Cauchy stress tensor can be expressed as

$$
\boldsymbol{\sigma} = -p\mathbf{I} + \frac{n\mu J_m}{J_m - (I_1^n - 3^n)} I_1^{n-1}\mathbf{B} \tag{2.2}
$$

in which I_1 is the first invariant of and p is the undetermined scalar function that justifies the incompressible internal constraint conditions.

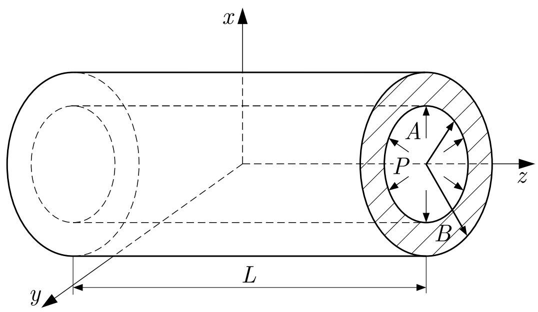

Fig. 1. Rubber tube under pressure

Consider a cylindrical rubber tube under uniform pressure, which is illustrated in Fig. 1. If (R, Θ, Z) and (r, θ, z) are the coordinates of the rubber tube before deformation and after deformation respectively, then the deformation pattern of the rubber tube can be expressed as

$$
r = f(R) \qquad \theta = \Theta \qquad z = \lambda_z Z \tag{2.3}
$$

The deformation gradient tensor $\mathbf{F}$ can be expressed as

$$
\mathbf{F} = \mathbf{F}^{\mathrm{T}} = \begin{bmatrix} \dfrac{dr}{dR} & 0 & 0 \\ 0 & \dfrac{r}{R} & 0 \\ 0 & 0 & \lambda_z \end{bmatrix} = \begin{bmatrix} \lambda_r & 0 & 0 \\ 0 & \lambda_\theta & 0 \\ 0 & 0 & \lambda_z \end{bmatrix} \tag{2.4}
$$

in which, λ_r, λ_θ and λ_z are the principal stretch in the radial, circumferential and axial direction of the cylinder membrane. It can be expressed as

$$
\lambda_r = \frac{dr}{dR} = (\lambda\lambda_z)^{-1} \qquad \lambda_\theta = \frac{r}{R} = \lambda \qquad \lambda_z = \lambda_z \tag{2.5}
$$

The left Cuachy-Green deformation tensor $\mathbf{B}$ can be shown as follows

$$\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}} = \begin{bmatrix} \lambda_r^2 & 0 & 0 \\ 0 & \lambda_\theta^2 & 0 \\ 0 & 0 & \lambda_z^2 \end{bmatrix} = \begin{bmatrix} (\lambda\lambda_z)^{-2} & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda_z^2 \end{bmatrix} \tag{2.6}$$

And the first invariants of the left Cuachy-Green deformation tensor $\mathbf{B}$ can be expressed as

$$I_1 = \operatorname{tr}\mathbf{B} = (\lambda\lambda_z)^{-2} + \lambda^2 + \lambda_z^2 \tag{2.7}$$

Substituting (2.7) and (2.4) into (2.2), we get

$$\sigma_{rr} = -p + 2(\lambda\lambda_z)^{-2}\frac{\partial W}{\partial I_1} \qquad \sigma_{\theta\theta} = -p + 2\lambda^2\frac{\partial W}{\partial I_1} \qquad \sigma_{zz} = -p + 2\lambda_z^2\frac{\partial W}{\partial I_1} \tag{2.8}$$

in which

$$\frac{\partial W}{\partial I_1} = \frac{\mu}{2}\frac{nJ_m}{J_m - (I_1^n - 3^n)}I_1^{n-1}$$

and p is the Lagrange multiplier associated with hydrostatic pressure.

In the absence of body forces, the equilibrium equation of the axial symmetry in the current configuration can be achieved as

$$\frac{d\sigma_{rr}}{dr} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0 \tag{2.9}$$

For the cylinder rubber tube under internal pressure, it should be satisfied with that the radical stress is zero outside of the rubber tube and the radical stress is equal to the internal pressure, which can be expressed as

$$\sigma_{rr}(a) = -P \qquad \sigma_{rr}(b) = 0 \tag{2.10}$$

From (2.9) and (2.10), we can get

$$\int\limits_{-P}^{0} d\sigma_{rr} = \int\limits_{a}^{b} \frac{1}{r}(\sigma_{\theta\theta} - \sigma_{rr})\, dr = \int\limits_{a}^{b} \frac{1}{r}\frac{\mu n J_m}{J_m - (I_1^n - 3^n)}I_1^{n-1}[\lambda^2 - (\lambda\lambda_z)^{-2}]\, dr \tag{2.11}$$

in which, $a = f(A)$, $b = f(B)$, a and b are the internal and external radii of the cylinder rubber tube after deformation. A and B are the internal and external radii of the cylinder rubber tube before deformation.

By utilizing the expression $\lambda = r/R$, we can arrive at the following expression

$$dr = \frac{R}{1 - \lambda^2\lambda_z}d\lambda \tag{2.12}$$

Substituting (2.12) into (2.11), we get

$$\begin{aligned} P &= \int\limits_{\lambda_a}^{\lambda_b} \frac{1}{\lambda}\frac{\partial W}{\partial I_1}[\lambda^2 - (\lambda\lambda_z)^{-2}]\frac{1}{1 - \lambda^2\lambda_z}\, d\lambda \\ &= \int\limits_{\lambda_a}^{\lambda_b} \frac{1}{\lambda}\frac{\mu n J_m}{J_m - (I_1^n - 3^n)}I_1^{n-1}[\lambda^2 - (\lambda\lambda_z)^{-2}]\frac{1}{1 - \lambda^2\lambda_z}\, d\lambda \end{aligned} \tag{2.13}$$

in which, $\lambda_a = a/A$, $\lambda_b = b/B$.

Taking into account the incompressibility of rubber-like materials, the following equations can be achieved

$$(r^2 - a^2)\lambda_z = R^2 - A^2 \qquad R^2(\lambda^2\lambda_z - 1) = \lambda_z a^2 - A^2 \tag{2.14}$$

Equation (2.14) can be transformed into

$$\lambda_a^2\lambda_z - 1 = (\varepsilon + 1)^2(\lambda_b^2\lambda_z - 1) \tag{2.15}$$

where $\varepsilon = (B - A)/A$. For a thin-walled cylinder rubber tube, wall thickness is far less than the mean radius, so the value of ε is far less than 1. Removing the high-order term of ε, we can get

$$\lambda_a^2\lambda_z - 1 = \lambda_b^2\lambda_z - 1 + 2\varepsilon(\lambda_b^2\lambda_z - 1) \tag{2.16}$$

By utilizing the expressions $\lambda_a + \lambda_b = 2\lambda$, $\lambda_b = \lambda$, Eq. (2.16) can be transformed into

$$\lambda_a - \lambda_b = \frac{\varepsilon}{\lambda\lambda_z}(\lambda^2\lambda_z - 1) \tag{2.17}$$

From (2.17), a simplified equation from (2.13) can be expressed as

$$P = \frac{\mu n J_m}{J_m - (I_1^n - 3^n)} I_1^{n-1}[\lambda^2 - (\lambda\lambda_z)^{-2}]\frac{\varepsilon}{\lambda^2\lambda_z} \tag{2.18}$$

In order to discuss the effect of constitutive parameters J_m and n on the mechanical properties of the rubber tube under pressure, non-dimensional stress is introduced. From Eq. (2.18), we can get

$$P^{\#} = \frac{n J_m}{J_m - (I_1^n - 3^n)} I_1^{n-1}[\lambda^2 - (\lambda\lambda_z)^{-2}]\frac{1}{\lambda^2\lambda_z} \tag{2.19}$$

where $P^{\#} = P/(\mu\varepsilon)$.

In order to study the effect on the rubber tube under pressure by the constitutive parameters J_m and n, three circumstances are considered. Firstly, when J_m and λ_z is fixed, the distribution between the internal pressure and circumferential principal stretch with the change of n has been researched. Secondly, when n and λ_z is fixed, the distribution between the internal pressure and circumferential principal stretch with the change of J_m has also been researched. Thirdly, we simultaneously investigate the distribution between the internal pressure and circumferential principal stretch with the change of λ_z when J_m and n is fixed.

Figures 2a-2c show distribution curves between the internal pressure $P^{\#}$ and circumferential principal stretch λ according to the above three circumstances.

As shown in Fig. 2a, for fixed material parameters $J_m = 2.3$ and $\lambda_z = 1$, when the material parameter n increases, the circumferential principal stretch increases in accordance with the internal pressure. It can also be seen in Fig. 2a that the effect of the constitutive parameter n has a major impact on the mechanical properties of the rubber tube. When the material parameter n takes higher values, the range of the circumferential principal stretch is larger, which means that the rubber tube has strong inflation capability and good elasticity. On the other hand, when the material parameter n takes a lesser value, the range of the circumferential principal stretch is smaller, which means that the rubber inflation capability tube is weak. Especially when $n \leqslant 1$, the rubber tube starts softening and the material becomes unstable, which means the stability analysis is necessary.

As can be noted in Fig. 2b, the material parameter J_m has also a certain influence on the circumferential principal stretch of the rubber tube. As the value of J_m increases, the circumferential principal stretch increases in accordance with the internal pressure. When the material

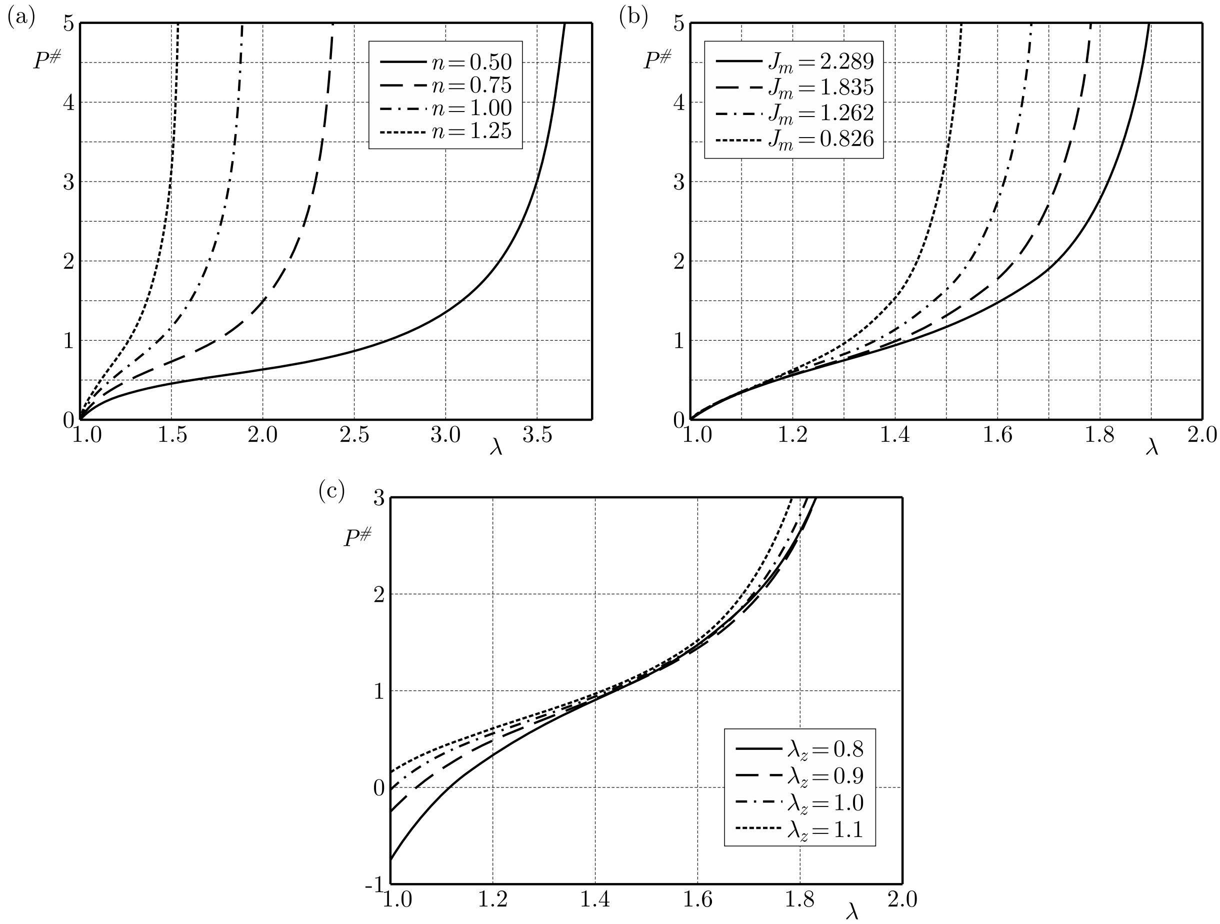

Fig. 2. Distribution curve between $P^{\#}$ and λ with the effect of the material parameter: (a) n ($J_m = 2.3$, $\lambda_z = 1$), (b) J_m ($n = 1$, $\lambda_z = 1$), (c) λ_z ($n = 1$, $J_m = 2.3$)

parameter J_m takes higher values, the range of the circumferential principal stretch is larger, which means that the rubber tube has strong inflation capability and good elasticity. On the other hand, when the material parameter J_m takes a lesser value, the range of the circumferential principal stretch is smaller, which means that the inflation capability of the tube is weak.

Figure 2c displays the relation between the internal pressure and circumferential principal stretch. From that we can see when the material parameters J_m and n are fixed, the circumferential principal stretch decreases as the axial principal stretch increases, which means that the rubber tube is incompressible. We also can infer that the axial principal stretch has a minor impact on the mechanical properties of the rubber tube.

3. Stability analysis

According with the membrane hypothesis, $\sigma_{rr} = 0$. From (2.8), we can get

$$p = \frac{\mu n J_m}{J_m - (I_1^n - 3^n)} I_1^{n-1} (\lambda\lambda_z)^{-2} \tag{3.1}$$

Substituting (3.1) into (2.8), we get

$$\begin{aligned} \sigma_{\theta\theta} &= 2\frac{\partial W}{\partial I_1}[\lambda^2 - (\lambda\lambda_z)^{-2}] = \frac{\mu n J_m}{J_m - (I_1^n - 3^n)} I_1^{n-1}[\lambda^2 - (\lambda\lambda_z)^{-2}] \\ \sigma_{zz} &= 2\frac{\partial W}{\partial I_1}[\lambda_z^2 - (\lambda\lambda_z)^{-2}] = \frac{\mu n J_m}{J_m - (I_1^n - 3^n)} I_1^{n-1}[\lambda_z^2 - (\lambda\lambda_z)^{-2}] \end{aligned} \tag{3.2}$$

For an incompressible rubber tube under pressure, when its two sides are closed, there is no constraint along the length direction, from which the following expression can be achieved

$$\sigma_{\theta\theta} = \frac{Pr_0}{h} \qquad \sigma_{zz} = \frac{Pr_0}{2h} \tag{3.3}$$

where P is the internal pressure of the cylinder membrane, r_0 is the mean radius after deformation and h is the thickness of the rubber membrane after deformation.

Considering the incompressibility of the rubber membrane, we can get

$$\sigma_{\theta\theta} = \frac{Pr_0}{h} = \frac{P\lambda^2\lambda_z R_0}{H} \qquad \sigma_{zz} = \frac{Pr_0}{2h} = \frac{P\lambda^2\lambda_z R_0}{2H} \tag{3.4}$$

in which, R_0 is the mean radius before deformation and H is the thickness of the rubber membrane before deformation.

From (3.2) and (3.4), the following equation can be formulated

$$\begin{aligned} \frac{P\lambda^2\lambda_z R_0}{H} &= \frac{\mu n J_m}{J_m - (I_1^n - 3^n)} I_1^{n-1}[\lambda^2 - (\lambda\lambda_z)^{-2}] \\ P^{\#} &= \frac{1}{\lambda^2\lambda_z}[\lambda^2 - (\lambda\lambda_z)^{-2}] \frac{nJ_m}{J_m - (I_1^n - 3^n)} I_1^{n-1} \end{aligned} \tag{3.5}$$

where $P^{\#} = PR_0/H$.

From (3.4), we get

$$\sigma_{\theta\theta} = 2\sigma_{zz} \tag{3.6}$$

Substituting (3.6) into (3.2), the following expression can be found

$$\lambda_z^3 = \frac{(\lambda^2\lambda_z)^2 + 1}{2\lambda^2\lambda_z} \tag{3.7}$$

Substituting (3.7) into (2.5), the principal stretch in the radial and axial direction of the cylinder membrane can be expressed as

$$\lambda = \nu^{\frac{1}{2}}\Big(\frac{2\nu}{\nu^2+1}\Big)^{\frac{1}{6}} \qquad \lambda_z = \Big(\frac{\nu^2+1}{2\nu}\Big)^{\frac{1}{3}} \tag{3.8}$$

where $\nu = \lambda^2\lambda_z$, which can reflect the volume expansion ratio, i.e., the ratio of the internal volume of the cylinder membrane in the deformed state to that in the undeformed state.

Substituting (3.8) into $(3.5)_2$, we get

$$P^{\#} = \frac{\nu^2-1}{\nu^2}\Big(\frac{2\nu}{\nu^2+1}\Big)^{\frac{1}{3}} \frac{nJ_m}{J_m - (I_1^n - 3^n)} I_1^{n-1} \tag{3.9}$$

In order to examine stability of the rubber cylinder membrane, the stationary point of $P^{\#}$ should be determined first.

When $J_m \to \infty$, Eq. (1.7) can be transformed into the strain energy function proposed by Gao (1990) as follows

$$W = A(I_1^n - 3^n) \tag{3.10}$$

where $A = \mu/2$.

Based on strain energy function (3.10), Eq. (3.9) can be transformed as

$$P_\infty^{\#} = \frac{\nu^2-1}{\nu^2}\Big(\frac{2\nu}{\nu^2+1}\Big)^{\frac{1}{3}} nI_1^{n-1} \tag{3.11}$$

When the material parameter $n = 1$, the neo-Hookean constitutive equation can be achieved from (3.10). Then, we get the following expression from (3.11)

$$P^{\#} = \frac{\nu^2-1}{\nu^2}\Big(\frac{2\nu}{\nu^2+1}\Big)^{\frac{1}{3}} \tag{3.12}$$

4. Discussion

As shown in Fig. 3, when $J_m \to \infty$ and $n = 1$, we obtain the turning point $\nu^* = 2.930$. For the volume expansion ratio $\nu \leqslant \nu^*$, the inflation curve is monotonically increasing. But for the volume expansion ratio $\nu \geqslant \nu^*$, the inflation curve is decreasing.

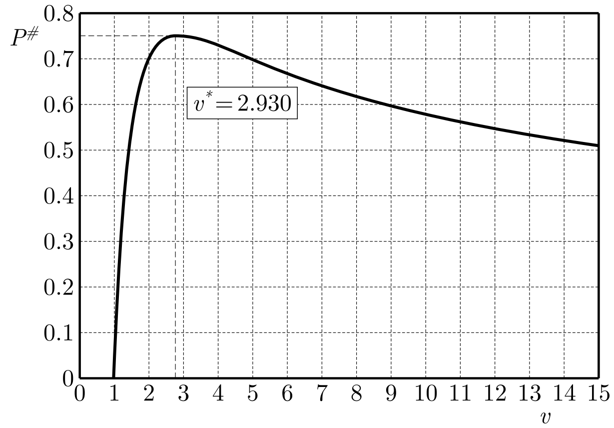

Fig. 3. Distribution curve between $P^{\#}$ and ν in the rubber tube inflation ($J_m \to \infty$ and $n = 1$)

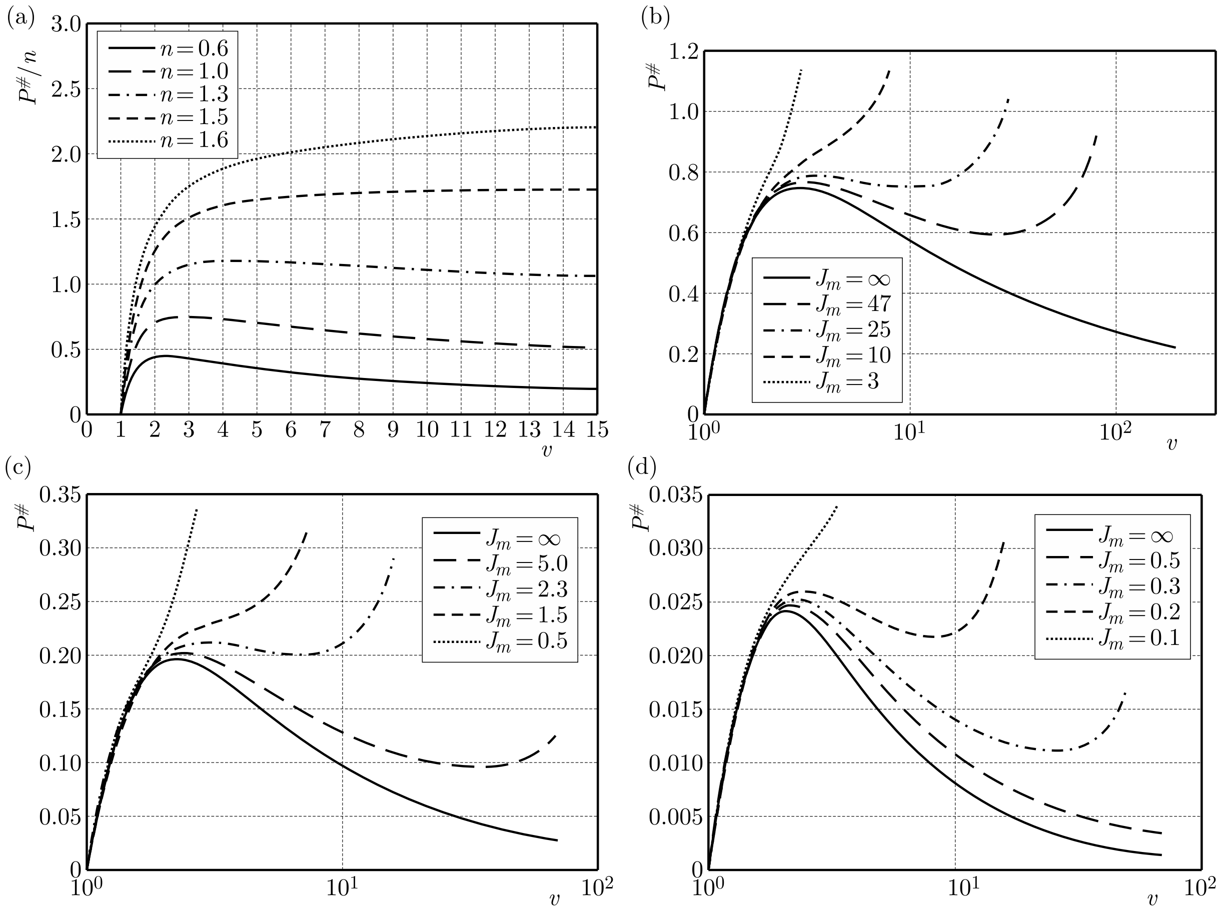

Fig. 4. Distribution curve between $P^{\#}$ and ν with the effect of: (a) n ($J_m \to \infty$), (b) J_m ($n = 1$), (c) J_m ($n = 0.5$), (d)J_m ($n = 0.1$)

In order to discuss the effect of the material parameter n on the rubber tube inflation, the distribution between the internal pressure and volume expansion ratio with the change of n has been investigated when $J_m \to \infty$. Figure 4a displays the relation between the internal pressure and volume expansion ratio when $n = 0.6$, 1.0, 1.3, 1.5 and 1.6, respectively. We can see that the inflation curve of the rubber tube has no limit point when $n = 1.6$, which means that there is no instability in the rubber tube. Only if $n \leqslant 1.5$, instability of the rubber tube under pressure occurs.

As can be seen in Figs. 4b to 4d, the distribution between the internal pressure and volume expansion ratio with the change of J_m when $n = 1$, $n = 0.5$ and $n = 0.1$, respectively. In Fig. 4b, we can see when $n = 1$, the constitutive parameter J_m has obviously the effect on the stability of the rubber tube. The inflating pressure is seen to pass through a maximum when $J_m \geqslant 25$, which means that instability of the rubber tube under pressure will occur. The results are consistent with the results by Gent (2005). It can be seen in Fig. 4c that the instability of the rubber tube under pressure occurs when $J_m \geqslant 2.3$ with the material parameter $n = 0.5$. And we also can see in Fig. 4d that the instability occurs when $J_m \geqslant 0.5$ with the material parameter $n = 0.1$.

5. Conclusion

A modified Gent's strain energy function has been utilized to examine the large deformation problem and the stability problem of the rubber tube subjected to internal pressure. By establishing the theoretical model of the rubber tube under internal pressure, the relationship between internal pressure and circumferential principal stretch has been deduced with the change of the constitutive parameters J_m and n, from which we can conclude that the constitutive parameter n has a major impact on the mechanical properties of the rubber tube. When $n \leqslant 1$, the rubber tube becomes softening and the material becomes unstable, which means thst the stability analysis is necessary. For a cylinder rubber tube closed at two sides, the relationship between the internal pressure and internal volume ratio has also been deduced and the effect of the two constitutive parameters n and J_m on the stability of the rubber tube has been invesigated. Accordingly, the instability phenomenon appears only when n is less than 1.5. For different values of n, the range of the value of J_m leading to the instability also changes.

Acknowledgement

This paper has been supported by Hebei National Nature Science Foundation (Grant No. A2017202076), Scientific Research Key Project of Hebei Province Education Department (Grand No. ZD20131019 and No. ZD2016083) and Scientific Research Project of Hebei Province Education Department (Grand No. QN2014111).

References

1. Akyüz U., Ertepinar A., 1999, Stability and asymmetric vibrations of pressurized compressible hyperelastic cylindrical shells, *International Journal of Non-Linear Mechanics*, **34**, 391-404
2. Akyüz U., A. Ertepinar A., 2001, Stability and breathing motions of pressurized compressible hyperelastic spherical shells *Journal of Sond Vibrations*, **247**, 293-304
3. Alexander H., 1971, The tensile instability of an inflated cylindrical membrane as affected by an axial load, *International Journal of Mechanical Sciences*, **13**, 87-95
4. Antman S.S., 1995, *Nonlinear Problems of Elasticity*, Springer-Verlag, NewYork-Budapest
5. Bertram C.D., 1982, Two modes of instability in a thick-walled collapsible tube conveying a flow, *Journal of Biomechanics*, **15**, 223-224
6. Bertram C.D., 1987, The effects of wall thickness, axial strain and end proximity on the pressure-area relation of collapsible tubes, *Journal of Biomechanics*, **20**, 863-876

7. Bharatha S., 1967, Cylindrically symmetrical deformations of Mooney materials, *Archiwum Mechaniki Stosowanej (Archives of Mechanics)*, **19**, 6, 857-865

8. Ertepinar A., 1977, Large amplitude radial oscillations of layered thick-walled cylindrical shells, *International Journal of Solids and Structure*, **13**, 717-723

9. Feng Z., Renauda C., Cros J., Zhang H., Guan Z., 2010, A finite element finite-strain formulation for modeling colliding blocks of Gent materials, *International Journal of Solids and Structure*, **47**, 2215-2222

10. Fung Y.C., 1967, Elasticity of soft tissues in simple elongation, *American Journal of Physiology*, **213**, 1532-1544

11. Gao Y.C., 1990, Elastostatic crack tip behavior for a rubber like material, *Theoretical and Applied Fracture Mechanics*, **14**, 219-231

12. Gent A.N., 1996, A new constitutive ralation for rubber, *Rubber Chemistry and Technology*, **69**, 1, 59-61

13. Gent A.N., 2005, Elastic instabilities in rubber, *International Journal of Non-Linear Mechanics*, **40**, 165-175

14. Green A.E., Zerna W., 1968, *Non-linear Elastic Deformations*, Horwood, Chichester

15. Hariharaputhiran H., Saravanan U., 2016, A new set of biaxial and uniaxial experiments on vulcanized rubber and attempts at modeling it using classical hyperelastic models, *Mechanics of Materials*, **92**, 211-222

16. Haughton D.M., Ogden R.W., 1979a, Bifurcation of inflated circular cylinders of elastic material under axial loading – I. Membrane theory for thin-walled tubes, *Journal of the Mechanics and Physics of Solids*, **27**, 179-212

17. Haughton D.M., Ogden R.W.,1979b, Bifurcation of inflated circular cylinders of elastic material under axial loading – II. Exact theory for thick-walled tubes, *Journal of the Mechanics and Physics of Solids*, **27**, 489-512

18. Haughton D.M., Ogden R.W., 1980, Bifurcation of rotating thick-walled elastic tubes, *Journal of the Mechanics and Physics of Solids*, **28**, 59-74

19. Horgan C.O., 2015, The remarkable Gent constitutive model for hyperelastic materials, *International Journal of Non-Linear Mechanics*, **68**, 9-16

20. Horgan C.O., Saccomandi G.A., 2002, Molecular-statistical basis for the gent constitutive model of rubber elasticity, *Journal of Elasticity*, **68**, 1, 167-176

21. Jiang X., Ogden R.W., 2000, Some new solutions for the axial shear of a circular cylindrical tube of compressible elastic material, *International Journal of Non-Linear Mechanics*, **35**, 361-369

22. Knowles J.K., 1977, The finite anti-plane field near the tip of a crack of incompressible elastic solids, *International Journal of Fracture*, **13**, 4, 611-639

23. Mangan R., Destrade M., 2015, Gent models for the inflation of spherical balloons, *International Journal of Non-Linear Mechanics*, **68**, 52-58

24. Merodio J., Ogden R.W., 2015, Extension, inflation and torsion of a residually stressed circular cylindrical tube, *Continuum Mechanics and Thermodynamics*, **28**, 157-174

25. Mooney M.A., (1940), A theory of large elastic deformation, *Journal of Applied Physics*, **11**, 582-592

26. Ogden R.W., 1984, *Non-linear Elastic Deformations*, Ellis Horwood, Chichester

27. Papargyri-Pegiou S., Stavrakakis E., 2000, Axisymmetric numerical solutions of a thin-walled pressurized torus of incompressible nonlinear elastic materials, *Computers and Structures*, **77**, 747-757

28. Pucci E., Saccomandi G.A., 2002, A note on the Gent model for rubber-like materials, *Rubber Chemistry and Technology*, **75**, 5, 839-851

29. Rickaby S.R., Scott N.H., A comparison of limited-stretch models of rubber elasticity, *International Journal of Non-Linear Mechanics*, **68**, 71-86

30. Rivlin R.S., 1948, Large elastic deformations of isotropic materials: I Fundametal concepts, II.Some uniqueness theories for homogenous deformation, *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, London*, **240**, 459-508

31. Sang J.B., Sun L.F., S.F.Xing, et al., 2014, Mechanical properties of polymer rubber materials based on a new constitutive model, *Polymers and Polymer Composites*, **22**, 8, 693-698

32. Treloar L.R.G., 1976, The mechanics of rubber elasticity, *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, **351**, 301-330

33. Zhu Y., Luo X.Y., Ogden R.W., 2008, Asymmetric bifurcations of thick-walled circular cylindrical elastic tubes under axial loading and external pressure, *International Journal of Solids and Structure*, **45**, 3410-3429

34. Zhu Y., Luo X.Y., Ogden R.W., 2010, Nonlinear axisymmetric deformations of an elastic tube under external pressure, *European Journal of Mechanics – A/Solid*, **29**, 216-229

Manuscript received January 29, 2016; accepted for print July 5, 2016