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On Solving a Quadratic Diophantine Equation Involving Odd Powers of 17

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Abstract: ***Diophantine Equations named after ancient Greek mathematician Diophantus, plays a vital role not only in number theory but also in several branches of science. In this paper, we will solve one of the quadratic Diophantine equations where the right hand side are odd positive integral powers of 17 and provide its complete solutions. The method adopted to solve the given equation is using the concept of polar form of a particular complex number. This concept can be generalized for solving similar equations.***

Keywords: ***Quadratic Diophantine Equation, Polar Form, Euler's Formula, Positive Integer Solutions.***

I. INTRODUCTION

Diophantine Equations were equations whose solutions must be in integers. Since the solutions are integers and most often positive integers, such equations have more practical applications compared to other equations in mathematics. In this paper, we will solve one of the quadratic Diophantine equations involving odd positive integral powers of 17 in a novel way and present its complete solution in a compact form.

II. QUADRATIC DIOPHANTINE EQUATION

In this paper, we will try to solve the quadratic Diophantine equation $3x^2 + 5y^2 = 17^{2k-1}$ (1), where x, y, k are positive integers. We will try to obtain a general solution of (1) in closed form. For doing this, we will make use of a particular complex number and a fabulous formula proposed by the greatest mathematician of all times, Leonhard Euler.

III. SOLUTIONS TO THE EQUATION

We will try to obtain all positive integer solutions (x, y) satisfying (1) for any given natural number n.

Now, for positive integer n, we will try to determine the polar form of $\left(\sqrt{5}+i2\sqrt{3}\right)^n$

$$\sqrt{5}+i2\sqrt{3} = r\left(\cos\theta + i\sin\theta\right) \Rightarrow r\cos\theta = \sqrt{5}, r\sin\theta = 2\sqrt{3}$$

From this, we obtain

$$r^2 = 5+12 = 17 \Rightarrow r = \sqrt{17}, \theta = \tan^{-1}\left(\frac{2\sqrt{3}}{\sqrt{5}}\right) \quad (2)$$

Hence the polar form of $\left(\sqrt{5}+i2\sqrt{3}\right)^n$ is given by

$$\left(\sqrt{5}+i2\sqrt{3}\right)^n = 17^{n/2} e^{in\tan^{-1}\left(\frac{2\sqrt{3}}{\sqrt{5}}\right)} \quad (3)$$

Now using Euler's Formula in (3), we obtain

$$\left(\sqrt{5}+i2\sqrt{3}\right)^n = 17^{n/2}\left[\cos\left(n\tan^{-1}\left(\frac{2\sqrt{3}}{\sqrt{5}}\right)\right) + i\sin\left(n\tan^{-1}\left(\frac{2\sqrt{3}}{\sqrt{5}}\right)\right)\right] \quad (4)$$

If we now assume $\sqrt{5}y + i\sqrt{3}x = \left(\sqrt{5}+i2\sqrt{3}\right)^n$ (5)

then $\sqrt{5}y - i\sqrt{3}x = \left(\sqrt{5}-i2\sqrt{3}\right)^n$ (6)

Now multiplying (5) and (6), we get

$$\left(\sqrt{5}y+i\sqrt{3}x\right)\times\left(\sqrt{5}y-i\sqrt{3}x\right) = \left(\sqrt{5}+i2\sqrt{3}\right)^n \times \left(\sqrt{5}-i2\sqrt{3}\right)^n$$

Simplifying, we obtain $3x^2 + 5y^2 = 17^n$ which is (1), the original problem if n is odd. Thus the solutions to (1) are given by equating real and imaginary parts of (5). Now using (4) in (5), and for $n \geq 1$ we get

$$\sqrt{3}x = 17^{n/2}\sin\left(n\tan^{-1}\left(\frac{2\sqrt{3}}{\sqrt{5}}\right)\right) \Rightarrow x = \frac{17^{n/2}}{\sqrt{3}}\sin\left(n\tan^{-1}\left(\frac{2\sqrt{3}}{\sqrt{5}}\right)\right) \quad (7)$$

$$\sqrt{5}y = 17^{n/2}\cos\left(n\tan^{-1}\left(\frac{2\sqrt{3}}{\sqrt{5}}\right)\right) \Rightarrow y = \frac{17^{n/2}}{\sqrt{5}}\cos\left(n\tan^{-1}\left(\frac{2\sqrt{3}}{\sqrt{5}}\right)\right) \quad (8)$$

Now from (7) and (8), if we consider $n = 2k - 1$ for $k = 1, 2, 3, 4, 5, \ldots$ then the ordered pairs $\left(|x|, |y|\right)$ would provide all positive integer solutions to the given Quadratic Diophantine Equation $3x^2 + 5y^2 = 17^{2k-1}$ for any natural number k. For more details about solving Diophantine Equations using complex numbers or recurrence relations or by direct proof methods, refer [1 – 16][17][18][19][20][21].

IV. CONCLUSION

Considering a quadratic Diophantine equation $3x^2 + 5y^2 = 17^{2k-1}$ we have used a novel method to solve it completely in this paper. In particular, equations (7) and (8) provide all required positive integer solutions to the given equation. Further, by considering the polar form of a particular complex number, we have obtained nice closed expressions for the given equations.

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In fact, from (7) and (8), we notice that for $k \geq 1$, all positive integer solutions to $3x^2 + 5y^2 = 17^{2k-1}$ are given by

$$x = \frac{17^{(2k-1)/2}}{\sqrt{3}}\left|\sin\left((2k-1)\tan^{-1}\left(\frac{2\sqrt{3}}{\sqrt{5}}\right)\right)\right|, y = \frac{17^{(2k-1)/2}}{\sqrt{5}}\left|\cos\left((2k-1)\tan^{-1}\left(\frac{2\sqrt{3}}{\sqrt{5}}\right)\right)\right| \quad (9)$$

Thus, for *k* = 1, 2, 3, 4, 5, 6, . . . all positive integer solutions to $3x^2 + 5y^2 = 17^{2k-1}$ are given respectively by (2,1); (6,31); (662,145); (7534,6929); (85842,138911); (3379114,57727); . . .

Thus the values of *x* and *y* from expression (9) provides all possible positive integer solutions to the given quadratic Diophantine equation $3x^2 + 5y^2 = 17^{2k-1}$. We can adopt similar methods to solve other types of quadratic Diophantine equations using polar forms of suitable complex numbers.

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Authors Contributions	All authors have equal participation in this article.

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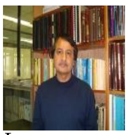

Professor José Luis López-Bonilla is a Teacher and Researcher at National Polytechnic Institute, Mexico city, and his Ph D is in Theoretical Physics: Electrodynamics [motion of classical charged particles], Special and General Relativity [Petrov classification, Lorentz matrix, Lanczos potential, Embedding of Riemannian spaces, Newman-Penrose and 2-spinors formalisms], Quantum Mechanics [matrix elements for several potentials] and Classical Mechanics [constrained Hamiltonian systems]. Besides, he has strong interest in Mathematical Methods Applied to Engineering, [for example, importance of the SVD method in 5G technology] and also in certain topics in Number Theory such as: Partitions, Representations of integers as sums of squares, Combinatorics, Bernoulli and Stirling numbers and Recurrence relations for arithmetical functions.