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On Some Relations Involving the Ramanujan's Tau Function

R. Sivaraman, J. López-Bonilla, S. Vidal Beltrán

***Abstract:** It is known a recurrence relation for the Ramanujan's tau-function involving the sum of divisors function$\sigma(n)$, whose solution gives a closed formula for $\tau(n)$ in terms ofcomplete Bell polynomials, and a determinantal expression for $\sigma(m)$ where participate the values $\tau(k)$.*

***Keywords:** Sum of divisors function, Recurrence relations, Ramanujan's function $\tau(n)$, Bell polynomials, Chebyshev polynomials of the second kind.*

I. INTRODUCTION

We know the following recurrence relation for the Ramanujan's tau-function [1, 2]:

$$n\,\tau(n+1) = -24\sum_{j=1}^{n}\sigma(j)\,\tau(n+1-j), \quad n\ge 1, \qquad (1)$$

which allows an easy recursive manner to calculate the values of $\tau(m)$: 1, -24, 252, -1472, 4830, - 6048,…, that is, the sequence A000594 [3]. Besides, this function verifies interesting properties if p is a prime number [1, 2, 4-8]:

$$\tau(p^{n+2}) = \tau(p)\,\tau(p^{n+1}) - p^{11}\,\tau(p^n), \qquad n\ge 0, \qquad (2)$$

$$|\tau(m)| \le m^{\frac{11}{2}}\,d(m) \;\therefore\; |\tau(p)| \le 2\,p^{\frac{11}{2}}, \qquad (3)$$

where $d(m)$is the number of divisors of m.

In Sec. 2 we show that (1) gives two options: To write $\tau(n)$ in terms of $\sigma(m)$ via the complete Bell polynomials [9-15][21][22], or to express $\sigma(n)$ as a determinant whose entries are values of the tau function. In Sec. 3we use (2), (3) and the Chebyshev polynomials [16] to obtain a formula of Ramanujan [1][18] for$\tau(p^n)$.

II. EXPLICIT SOLUTIONS OF (1)

From (1) it is immediate a closed expression for the Ramanujan's tau-function in terms of the complete Bell polynomials [15]:

$$\tau(n+1) = \frac{1}{n!}B_n\big(-24\cdot 0!\,\sigma(1), -24\cdot 1!\,\sigma(2), -24\cdot 2!\,\sigma(3), \ldots, -24\cdot(n-1)!\,\sigma(n)\big), \quad n\ge 0, \qquad (4)$$

which also allows reproduce the sequence of integers A000594., or equivalently:

$$\tau(n+1) = \sum_{k=0}^{n}\frac{(-24)^k}{k!}C_{n-k}^{(k)}, \quad C_r^{(0)} = \delta_{0r}, \quad C_r^{(1)} = \frac{\sigma(r+1)}{r+1}, \quad C_0^{(r)} = 1, \qquad (5)$$

$$j\,C_j^{(r)} = \sum_{m=1}^{j}\frac{[m\,(r+1)-j]\,\sigma(m+1)}{m+1}C_{j-m}^{(r)}.$$

From (1) we can to employ determinants to obtain the sum of divisors function in terms of the tau function:

$$\sigma(n) = -\frac{1}{24}\begin{vmatrix} n\tau(n+1) & \tau(2) & \tau(3) & \tau(4) & \cdots & \tau(n) \\ (n-1)\tau(n) & 1 & \tau(2) & \tau(3) & \cdots & \tau(n-1) \\ \vdots & 0 & 1 & \tau(2) & \cdots & \tau(n-2) \\ \vdots & \vdots & \vdots & 1 & \vdots & \vdots \\ 2\tau(3) & 0 & 0 & \vdots & \vdots & \vdots \\ \tau(2) & 0 & 0 & 0 & 0 & 1 \end{vmatrix}, \qquad (6)$$

that is:

$$\sigma(1) = -\frac{1}{24}|\tau(2)|, \quad \sigma(2) = -\frac{1}{24}\begin{vmatrix} 2\tau(3) & \tau(2) \\ \tau(2) & 1 \end{vmatrix}, \quad \sigma(3) = -\frac{1}{24}\begin{vmatrix} 3\tau(4) & \tau(2) & \tau(3) \\ 2\tau(3) & 1 & \tau(2) \\ \tau(2) & 0 & 1 \end{vmatrix}, \ldots \qquad (7)$$

which it is equivalent to:

$$\sigma(n) = \frac{n}{24}\sum_{j=1}^{n}\frac{(-1)^j}{j}A_{n-j}^{(j)}, \quad A_k^{(0)} = \delta_{0k}, \quad A_k^{(1)} = \tau(k+2), \quad A_0^{(k)} = (-24)^k, \qquad (8)$$

$$j\,A_j^{(r)} = -\frac{1}{24}\sum_{m=1}^{j}[m(r+1)-j]\,\tau(m+2)A_{j-m}^{(r)}.$$

III. RAMANUJAN'S FORMULA FOR $\tau(p^n)$

In (3) we can use $m = p^n$, where p is a prime number, thus$|\tau(p^n)| \le (n+1)p^{\frac{11\,n}{2}}$, then it is natural to work with the expression:

$$\frac{\tau(p^n)}{n+1} = Q_n(p)p^{\frac{11\,n}{2}}, \qquad |Q_n(p)| \le 1, \qquad (9)$$

hence $Q_1(p) = \frac{\tau(p)}{2\,p^{\frac{11}{2}}}$ verifying the property (3) proved by Deligne [5][19][20]. We can employ (9) in the recurrence relation (2) to obtain:

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** Correspondence Author (s)*

Dr. R. Sivaraman*, Department of Mathematics, Dwaraka Doss Goverdhan Doss Vaishnav College, Chennai (Tamil Nadu), India. Email: rsivaraman1729@yahoo.co.in, ORCID ID: 0000-0001-5989-4422

Prof. J. López-Bonilla, ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista CP 0778, CDMX, México. Email: jlopezb@ipn.mx, ORCID ID: 0000-0003-3147-7162

S. Vidal Beltrán, ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista CP 0778, CDMX, México. Email: jlopezb@ipn.mx, ORCID ID: 0000-0003-3412-9630

$$(n+3)Q_{n+2} = 2\,(n+2)\,Q_1 Q_{n+1} - (n+1)\,Q_n\,, \qquad (10)$$

whose comparison with the recurrence relation satisfied by the Chebyshev polynomials of the second kind [16]:

$$U_{n+2}(\cos\theta) = 2\cos\theta\, U_{n+1}(\cos\theta) - U_n(\cos\theta), \qquad (11)$$

implies the connections:

$$\cos\theta = Q_1 = \frac{\tau(p)}{2\,p^{\frac{11}{2}}}, \qquad U_n(\cos\theta) = (n+1)Q_n(p) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad (12)$$

verifying the inequality (9) because we know that $|U_n(\cos\theta)| \le (n+1)$. Finally, (9) and (12) generate the following formula published by Ramanujan [1, 2]:

$$\tau(p^n) = \frac{Sin\,(n+1)\theta_p}{Sin\,\theta_p}\, p^{\frac{11\,n}{2}}. \qquad (13)$$

Remark: We note that (2) implies the property:

$$\tau(4n) = 3[-8\,\tau(2n) - 683\,\tau(n)] + \tau(n), \qquad (14)$$

therefore $\tau(4n) \equiv \tau(n) \pmod 3$ [17], and:

$$\tau(4n) = 8\,[-3\,\tau(2n) - 256\,\tau(n)] \quad \therefore \quad \tau(4n) \equiv 0 \pmod k, \; k = 2, 4, 8. \qquad (15)$$

IV. CONCLUSION

Though there are several ways of expressing Ramanujan's Tau function using polynomials, special functions and various tools in mathematics, in this paper, we have expressed the sum of divisors function as a determinant whose entries involves Tau function values as in (6). The first three values are explicitly arrived in (7). A more general form of these expressions are provided in (8). Finally using the Tau conjectures proposed by Ramanujan and using Chebyshev polynomials, we have deduced some interesting congruence related to modulo 2, 4 and 8 as provided in (15). These little observations may provide new insight upon knowing the values of Ramanujan Tau function.

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Authors Contributions	All authors have equal participation in this article.

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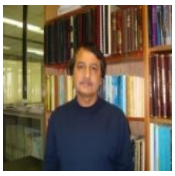
AUTHORS PROFILE

Dr. R. Sivaraman, working as Associate Professor at Dwaraka Doss Goverdhan Doss Vaishnav College, Chennai has 25 years of teaching experience at College level. He has been conferred with National Award for Popularizing mathematics among masses in 2016 by Department of Science and Technology, Government of India. He was conferred with Indian National Science Academy (INSA) Teaching award for the year 2018. He has also received State Government Best Science Book Awards in 2011 and 2012. He has provided more than 400 lectures conveying the beauty, applications of Mathematics. He has published more than 200 research papers and had done his Post Doctoral Research Fellowship and Doctor of Science Degree. He has written 32 books in view of popularizing mathematics among common man. He was a member of the Textbook Writing Committee, Tamil Nadu School Education Department for preparing revised mathematics textbook for eleventh class and chairperson for tenth class.

He has won more than 75 prestigious awards for his distinguished service to mathematics. He has been taking free classes for college students from very poor background for many years. Propagating the beauty and applications of mathematics to everyone was his life mission.



Professor José Luis López-Bonilla is a Teacher and Researcher at National Polytechnic Institute, Mexico city, and his Ph D is in Theoretical Physics: Electrodynamics [motion of classical charged particles], Special and General Relativity [Petrov classification, Lorentz matrix, Lanczos potential, Embedding of Riemannian spaces, Newman-Penrose and 2-spinors formalisms], Quantum Mechanics [matrix elements for several potentials] and Classical Mechanics [constrained Hamiltonian systems]. Besides, he has strong interest in Mathematical Methods Applied to Engineering [for example, importance of the SVD method in 5G technology] and also in certain topics in Number Theory such as: Partitions, Representations of integers as sums of squares, Combinatorics, Bernoulli and Stirling numbers and Recurrence relations for arithmetical functions.

Sergio Vidal – Beltrán is a Professor and Scientific Researcher of the Postgraduate Degree in Telecommunications Engineering Sciences at the National PolytechnicInstitute, Mexico city. He is a specialist in network interconnectivity such as routing protocols and data switching algorithms, performance analysis of data networks in traditional networks and programmable networks, with respect to mobile communications systems, he specializes in 3g and 4g technologies, analyzing the electromagnetic coverage in urban areas, in 5G it specializes in non-orthogonal access techniques and machine learning applications to data networks. He has interest in the decoding of SVD-SCMA signals through matrix transformations, Moore – Penrose pseudoinverse and gradient descent optimization, also about the impact of condition number on total sum rate for NOMA systems.
