An Alternative Proof for the Topological Entropy of the Motzkin Shift

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Abstract—A Motzkin shift is a mathematical model for constraints on genetic sequences. In terms of the theory of symbolic dynamics, the Motzkin shift is nonsofic, and therefore, we cannot use the Perron-Frobenius theory to calculate its topological entropy. The Motzkin shift M(M,N) which comes from language theory, is defined to be the shift system over an alphabet A that consists of N negative symbols, N positive symbols and M neutral symbols. For an x in the full shift, x will be in the Motzkin subshift M(M,N) if and only if every finite block appearing in x has a non-zero reduced form. Therefore, the constraint for x cannot be bounded in length. K. Inoue has shown that the entropy of the Motzkin shift M(M,N) is log(M + N + 1). In this paper, a new direct method of calculating the topological entropy of the Motzkin shift is given without any measure theoretical discussion.

Keywords—Motzkin shift, topological entropy.

I. INTRODUCTION

A DYNAMICAL system is an abstract mathematical model describing the time dependence of point's position in its space. This is conventionally modeled by a map whose iterates denote the passage of time.

Broadly, a dynamical system is a set with a map $T : X \longrightarrow X$. This is discrete time. Continuous time considers a flow $\varphi_t : X \longrightarrow X$.

Symbolic dynamics is a powerful tool used in the study of dynamical systems. Its advantage lies in the fact that this technique reduces a complicated system into a set of sequences, the latter of which being much easier to analyze! We will see various instances of this simplification.

Let $\mathcal{A} = \{1, 2, \ldots, n - 1\}$ be a finite set called alphabet. The dynamical system $\sigma : \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathcal{A}^{\mathbb{Z}}$ is defined to be the set of all functions $f : \mathbb{Z} \longrightarrow \mathcal{A}$, and is called a shift dynamical system. The field of symbolic dynamics is referred to the study of the shift dynamical systems. In symbolic dynamics, subshifts of sequences are defined as sets of bi-infinite sequences of symbols over a finite alphabet avoiding a given set of finite factors (or blocks) called forbidden factors. Well known classes of shifts of sequences are shifts of finite type which avoid a finite set of forbidden factors and sofic shifts which avoid a regular set of forbidden factors. Sofic shifts may also be defined as labels of bi-infinite paths of a finite-state labelled graph where there are no constraints of initial or infinitely repeated states. We refer to [1] and [3] for an introduction to this theory.

Let A be an $n \times n$ adjacency matrix with entries in $\{0, 1\}$. Using these elements we construct a directed graph $G = (V, E)$ with V the set of vertices, the set of edges E defined with A. Let Y be the set of all infinite admissible

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sequences of edges, where by admissible it is meant that the sequence is a walk of the graph. Let σ be the shift operator on such sequences; it plays the role of the time-evolution operator of the dynamical system. A subshift of finite type is then defined as a pair (Y, σ) obtained in this way. If the sequence extends to infinity in only one direction, it is called a one-sided subshift of finite type, and if it is bilateral, it is called a two-sided subshift of finite type.

Formally, one may define the sequence of edges as

$$\Sigma_A^+ = \left\{ (x_0, x_1, \ldots) : x_j \in V, A_{x_j x_{j+1}} = 1, j \in \mathbb{N} \right\}.$$

This is the space of all sequences of symbols such that the symbol p can be followed by the symbol q only if the $(p, q)^{th}$ entry of the matrix A is 1. The space of all bi-infinite sequences is defined analogously:

$$\Sigma_A = \left\{ (x_0, x_1, \ldots) : x_j \in V, A_{x_j x_{j+1}} = 1, j \in \mathbb{Z} \right\}.$$

The shift operator σ maps a sequence in the one- or two-sided shift to another by shifting all symbols to the left, i.e.

$$(\sigma(x))_j = x_{j+1}.$$

Clearly this map is only invertible in the case of the two-sided shift.

A Motzkin shift (first suggested by [2]) is a mathematical model for constraints on genetic sequences. In terms of the theory of symbolic dynamics, the Motzkin shift is nonsofic, and therefore, we cannot use the Perron-Frobenius theory to calculate its topological entropy. The Motzkin shift $\mathfrak{M}_{M,N}$ which comes from language theory, is defined to be the shift system over an alphabet $\mathcal{A}$ that consists of N negative symbols, N positive symbols and M neutral symbols. For an x in the full shift $\mathcal{A}^{\mathbb{Z}}$, x is in $\mathfrak{M}_{M,N}$ if and only if every finite block appearing in x has a non-zero reduced form. Therefore, the constraint for x cannot be bounded in length. A beautiful way to describe the Dyck shift is in terms of its syntactic monoid.

Let $\mathcal{A} = \{\ell_1, \ell_2, \cdots, \ell_N, r_1, r_2, \cdots, r_N, 1_1, 1_2, \cdots, 1_M\}$, $M, N \in \mathbb{Z}^+$. Let **M** be a monoid (with zero) with generators $\ell_i, r_i, 1 \le i \le N, 1_j, 1 \le j \le M$ and **1**. The defining relations on the monoid are:

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$$\ell_i \circ r_j = \mathbf{1} \quad \text{if} \quad i = j \quad 1 \le i, j \le N,$$
$$1_i \circ 1_j = \mathbf{1} \quad \text{if} \quad 1 \le i, j \le M,$$
$$\ell_i \circ r_j = 0 \quad \text{if} \quad i \ne j \quad 1 \le i, j \le N,$$
$$\alpha \circ \mathbf{1} = \mathbf{1} \circ \alpha = \alpha \quad \alpha \in \mathcal{A} \cup \{\mathbf{1}\},$$
$$\alpha \circ 1_i = 1_i \circ \alpha = \alpha \quad 1 \le i \le K, \quad \alpha \in \mathcal{A} \cup \{\mathbf{1}\},$$
$$\alpha \circ 0 = 0 \circ \alpha = 0 \quad \alpha \in \mathcal{A} \cup \{\mathbf{1}\},$$
$$0 \circ 0 = 0.$$

We use a mapping $red() : \mathcal{A}^* \longrightarrow \mathbf{M}$ such that for

$$\omega = \omega_1 \omega_2 \cdots \omega_n \in \mathcal{A}^* (n \ge 1),$$
$$red(\omega) = \omega_1 \circ \omega_2 \circ \cdots \circ \omega_n, \text{ and } red(\varepsilon) = \mathbf{1}.$$

where $\mathcal{A}^*$ denotes the set of all finite sequences with letters taken from $\mathcal{A}$.

Definition 1: The Motzkin shift $\mathfrak{M}_{M,N}$ [2] is defined by

$$\mathfrak{M}_{M,N} = \left\{ x \in \mathcal{A}^{\mathbb{Z}} : \text{ if } i \le j, \text{ then } red(x_{[i,j)}) \ne 0 \right\},$$

where $x_{[i,j)} = x_i x_{i+1} \cdots x_{j-1}$.

Therefore, the Motzkin shift can be regarded as a shift defined by a simple directed graph G which has one vertex and $(2N + M)$-loops named by the elements of the set $\mathcal{A}$, and the loop named $\ell_i, r_i, 1 \le i \le N$ carry the labels ℓ, r respectively, the loop named $1_j, (1 \le j \le M)$ carries the label 1, that is, $\mathfrak{M}_N$ is the Dyck type inverse monoid. And such a presentation of subshifts is called S-presentation, where S is an inverse semigroup of Dyck type. Note that if $M = 0$, the monoid is the Dyck monoid $\mathfrak{D}_N$ and the subshift $\mathfrak{M}_{N,0}$ is the Dyck shift $\mathfrak{D}_N$.

Example 1: The following point is an element of the Motzkin shift over the alphabet $\mathcal{A} = \{\ell_1, \ell_2, r_1, r_2, 1_1, 1_2, 1_3\}$.

$$x = \cdots \ell_2 \ell_1 1_3 r_1 \ell_1 \ell_2 \cdots$$

Example 2: Let $\mathcal{A} = \{(, [,],), 1_1\}$. Then the following words are allowed

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while these are forbidden

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The Motzkin constraint: the symbol (is matched with), the symbol [is matched with a]

II. THE TOPOLOGICAL ENTROPY

The topological entropy of a dynamical system (X, T), denoted by $h(T)$, is a nonnegative real number that measures the complexity of the orbits in the system. For a system given by an iterated function, the topological entropy represents the exponential growth rate of the number of distinguishable orbits of the iterates. For shifts, it is defined as the asymptotic growth rate of the number of occurring blocks of large sequences. That is, the number $|\mathcal{B}_n(X)|$ of n-blocks appearing in points of a shift space X gives some idea of the complexity of X. Instead of using the individual numbers $|\mathcal{B}_n(X)|$ for $n = 1, 2, \ldots$, we can summarize their behavior by computing their growth rate in the following definition.

Definition 2: Let X be a shift space. The entropy of X is defined by

$$h(X) = \lim_{n \longrightarrow \infty} \frac{1}{n} \log |\mathcal{B}_n(X)|.$$

We will always use the base 2 for the logarithms, so that log means $\log_2$.

Recall the Perron-Frobenius Theorem which asserts every non-negative irreducible matrix A has a positive eigenvalue λ_A such that $\lambda_A = |\mu|$ for any other eigenvalue μ and also that $Av = \lambda_A v$ for some vector v all of whose entries are positive, and that no other eigenvalue has an eigenvector with all positive entries.

As is well known the topological entropy of a one-dimensional subshift is completely determined by an adjacency matrix of the shift if the shift is of finite type or sofic by using the Perron-Frobenius theory. Unfortunately, this method is no longer available to calculate the entropy of the Motzkin shift $\mathfrak{M}_{M,N}$ as it is nonsofic. K. Inoue in [2] has shown that the entropy of the Motzkin shift is $\log(M + N + 1)$. In the following we have found a method of calculating the topological entropy of $\mathfrak{M}_{M,N}$ by counting possible paths without any theoretical discussion.

A. Main Result

The topological entropy of the Motzkin shift $\mathfrak{M}_{M,N}$, computed in [2] is as follows:

Proposition 1: [2] The entropy of the Motzkin shift $(\mathfrak{M}_{M,N}, \sigma)$ is

$$h_{top}(\sigma) = \log(M + N + 1).$$

Proof: The proof in details can be found in [2]. ∎

In the following, we will provide an alternative proof of the topological entropy for the Motzkin shift. This proof is similar to the proof of Niteckis' topological entropy proof in [4]. In order to give the proof, we need first to know what is meant by the notion of **Motzkin balanced** words.

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Recall the following fact from [4]. In the Dyck shift [5], a word $\omega = \omega_0\omega_1 \ldots \omega_n$ of even length is called **balanced** if its entries can be paired subject to

- a pair of entries consists of a left delimiter to the left of a matching right delimiter: if ω_α is paired with ω_β, where $0 \leq \alpha < \beta \leq n$, then $\omega_\alpha = \ell_i$ for some index i and $\omega_\beta = r_i$ for the same index;
- distinct pairs are nested or disjoint: given $\alpha < \beta$ as above, every intermediate $\omega_\iota, (\alpha < \iota < \beta)$ is paired with some other intermediate $\omega_\kappa, (\alpha < \kappa < \beta)$.

Definition 3: A finite sequence $\alpha = \alpha_0\alpha_1 \ldots \alpha_n$, is called a **Motzkin balanced** if α satisfies one of the following:

- $\alpha = \omega$, where ω is a Dyck balanced word;
- $\alpha = a$, where $a \in \mathcal{A}_1^* = \{1_1, 1_2, \cdots, 1_M\}^*$;
- $\alpha = a_0\omega_0 a_1\omega_1 \ldots a_n\omega_n a_{n+1}$, whereby the word $\omega_0\omega_1 \ldots \omega_n$ is a Dyck balanced word and (possibly empty) $a_i \in \mathcal{A}_1^*, 0 \leq i \leq n+1$.

We regard the empty word ε as balanced. Now, we specify the (infinite) list of disallowed words.

$$\mathcal{F} = \{\ell_i b r_j : b \text{ is a Motzkin balanced word and } i \neq j\}.$$

The subshift on the set of sequences $\mathfrak{M}_{M,N} \subset \mathcal{A}^{\mathbb{Z}}$ in which no element of $\mathcal{F}$ appears is the (two-sided) Motzkin shift.

The alternative proof of the Motzkin topological entropy is given as follows.

Theorem 1: The Motzkin shift $(\mathfrak{M}_{M,N}, \sigma)$ has

$$h_{top}(\sigma) = \log(M + N + 1).$$

Proof:
An admissible word has the general form

$$\omega = b_0 r_{i_1} b_1 r_{i_2} \ldots b_{k-1} r_{i_k} b_k \ell_{j_1} b_{k+1} \ldots b_{k+m-1} \ell_{j_m} b_{k+m}$$

where each $b_\alpha, \alpha = 0, \ldots, k+m$, is a (possibly empty) Motzkin balanced sub-word, and the $k \geq 0$ *right* delimiters which are not matched in ω all occur to the left of the $m \geq 0$ unmatched *left* delimiters in ω. This leads to a natural decomposition of any admissible word as a concatenation of three (possibly empty) sub-words

$$\omega = ABC$$

where $B = b_k$ is balanced, while $A = b_0 \ldots r_{i_k}$ (*resp.* $C = \ell_{j_1} \ldots b_{m+k}$) ends (*resp.* start) with an unmatched right (*resp.* left) delimiter.

To calculate the topological entropy, note first that every admissible word ω is the initial sub-word of at least $M + N + 1$ admissible words of length $|\omega| + 1$: the N and M words $\omega\ell_i, i = 1, \ldots, N, \omega 1_j, j = 1, \ldots, M$ are always admissible, and ωr_{j_m} is admissible if $m \geq 0$ while all words ωr_i are admissible if $m = 0$. Thus

$$|\mathcal{B}_{n+1}(X)| \geq (M + N + 1)|\mathcal{B}_n(X)|, \text{ for all } n,$$

where, $\mathcal{B}_n(X)$ equals the set of admissible words $\omega \in \mathcal{A}^n$. So

$$h_{top}(f) = \limsup_n \frac{1}{n} \log |\mathcal{B}_n(X)| \geq \log(M + N + 1).$$

To handle the opposite inequality, we first estimate the cardinality of the sets A_n, B_n, C_n of admissible words of length n whose decomposition has only one nonempty factor, of the type indicated by the letter.

We begin with Motzkin balanced words: It is well known that from [4] the set of all length n Dyck balanced words has cardinality

$$|\mathsf{B}_n| \leq (N + 1)^n.$$

Thus, the details given about the Motzkin balanced words assures that the cardinality of the set of all Motzkin balanced words of length n is

$$|\mathsf{B}_n| \leq (M + N + 1)^n.$$

We now consider the set C_n of words beginning with an unmatched left delimiter, noting that the initial length k sub-word of any $\omega \in \mathsf{C}_n$ itself belongs to C_k. Given $\omega \in \mathsf{C}_n$, we immediately have $\omega\ell_i, \omega 1_j \in \mathsf{C}_{n+1}$ for $i = 1, \ldots, N$, $j = 1, \ldots, M$ and $\omega r_i \in \mathsf{C}_{n+1}$ provided that ω has at least two unmatched left delimiters, the last of which is ℓ_i. This gives us

$$|\mathsf{C}_{n+1}| \leq (M + N + 1)|\mathsf{C}_n|,$$

and since

$$|\mathsf{C}_1| = N,$$

this implies

$$|\mathsf{C}_n| \leq (M + N + 1)^n.$$

A similar estimate can be obtained for $|\mathsf{A}_n|$, either by repeating the argument or by noting the bijection between $|\mathsf{A}_n|$ and $|\mathsf{C}_n|$ obtained by reversing letter order and interchanging ℓ with r (keeping indices).

Finally, to estimate $|\mathcal{B}_n(X)|$ we consider, for each ordered triple (i, j, k) of nonnegative integers summing to n, the set of words of the form $\omega = ABC$ with $|A| = i$, $|B| = j$ and $|C| = k$. Since an arbitrary factoring is possible, the number of such words is

$$|\mathsf{A}_i| \cdot |\mathsf{B}_j| \cdot |\mathsf{C}_k| \leq (M + N + 1)^{i+j+k} = (M + N + 1)^n.$$

But the number of possible triples (i, j, k) summing to n is less than $(n + 1)^3$, so

$$|\mathcal{B}_n(X)| \leq (n + 1)^3(M + N + 1)^n.$$

The asymptotic growth rate of the right-hand quantity is $\log(M + N + 1)$, so

$$h_{top}(\sigma) = \log(M + N + 1).$$

■

III. Conclusion

In this paper, we have found an alternative proof of the topological entropy of the Motzkin shift.

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