Journal of Process Control 138 (2024) 103210

Contents lists available at ScienceDirect

Journal of Process Control

journal homepage: www.elsevier.com/locate/jprocont

Velocity control design of hyperbolic distributed parameter systems using zeroing dynamics and zeroing-gradient dynamics methods

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ARTICLE INFO

Keywords:
Hyperbolic distributed parameter system
Velocity control
Late lumping approach
Zeroing dynamics
Zeroing gradient dynamics
Heat exchanger
Plug flow reactor

ABSTRACT

Velocity control proves to be an effective and a more easily implementable actuation than boundary and distributed actuations for hyperbolic distributed parameter systems. However, the design of velocity control for these systems, following the late lumping approach, i.e., using the partial differential equations model, poses a challenging problem in control engineering. Noticeably, the velocity controller faces a control singularity issue, resulting in a loss of controllability that renders the controller impractical. In this paper, we demonstrate that the zeroing dynamics method is a viable alternative design approach for velocity control of hyperbolic distributed parameter systems following the late lumping approach. Thus, employing the partial differential equations model, a velocity state feedback forcing output tracking is developed based on the zeroing dynamic method. Furthermore, to address the control singularity problem, the zeroing gradient method is combined with the zeroing method to design a state feedback that achieves output tracking even when a singularity occurs. The tracking error convergence is demonstrated for both developed state feedbacks. The effectiveness of these design approaches is clearly demonstrated in the case of a steam-jacketed heat exchanger and a non-isothermal plug flow reactor.

1. Introduction

The dynamical behavior of an important class of distributed parameter systems (DPSs) is described by hyperbolic partial differential equations (PDEs). Examples of such systems include heat exchangers, non-isothermal reactors, packed bed columns, adsorbers, absorbers, and crystallizers [1–5]. Various control actuators can be applied to these systems, such as boundary control [6,7], interior control [8], and velocity control [9]. From a practical standpoint, velocity control, a type of bilinear control, proves to be a natural choice, an effective and a more easily implementable than boundary and distributed actuations [4,9].

It is well known that for control design of hyperbolic distributed parameter system (DPS), the late lumping approach is more effective than the early lumping approach [2,3,10,11]. This can be explained by the fact that the modes of the spatial differential operator of a hyperbolic DPS have the same energy and cannot be captured by a lumped parameter model corresponding to a finite number of modes [8,9,12].

Boundary and distributed control of hyperbolic DPSs, following the late lumping approach, have been extensively studied in the literature [1,5–8,13–15]. Velocity control has been less investigated, and few contributions are reported in the literature. Sira-Ramirez [16] extended the theory of variable structure systems to hyperbolic DPS and developed an infinite dimensional variable structure state feedback. In [4], the Lyapunov direct method is applied to design both linear and nonlinear controllers, while Butkovskiy's maximum principle is employed to design optimal control laws for hyperbolic DPSs. The method of characteristics and sliding mode control theory have been combined by Hanczyc and Palazoglu [17] to design a state feedback controller for a steam heater and non-isothermal plug flow reactor. The input–output linearization approach has been extended by Gundepudi and Friedly [9] to hyperbolic DPS with a single characteristic variable (single flow DPSs) with application to a non-isothermal plug flow reactor. The same approach has been applied by Maidi et al. [6] to the velocity control of a dual flow hyperbolic DPS, i.e., a counter-current heat exchanger. Using the method of characteristics, Shang et al. [12] proposed a control law that enforces the output tracking successfully.

Input–output linearization approach has been successfully applied for control design of DPSs following the late lumping approach [8,9,13,18]. Nevertheless, its application is limited for DPS with boundary or interior actuations. For hyperbolic DPS, with velocity control, even

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https://doi.org/10.1016/j.jprocont.2024.103210
Received 2 February 2024; Received in revised form 21 March 2024; Accepted 28 March 2024
Available online 12 April 2024

the input–output linearization can be applied, but the controller may be impractical due to the control singularity (loss of controllability) that may occur. For instance, the controller fails to kick off with a uniform spatial initial profile [6,9].

Recently, an interesting control design approach based on the Zeroing Dynamics (ZD) method has been developed [19–21]. The design of the controller is carried out using the ZD method and the activation function [22]. Nevertheless, even though the controller design is a straightforward task, the resulting controller, akin to the input–output linearization approach, suffers from control singularity [22,23] when the initial spatial profile is uniform (constant). Therefore, to implement the control, it is necessary to initiate the system in open loop to ensure a non-uniform initial profile before switching to closed loop. However, this solution is economically unattractive. Thus, to address the issue of a uniform initial profile, a Zeroing Gradient Dynamics (ZGD) method, which combines the ZD method and the Gradient Dynamics (GD) method, has been proposed to design a controller without control singularity [19,20,22]. The controller is designed by minimizing an energy function using the GD method [22].

Many successful tracking control applications of the ZD and ZGD methods have been reported in the literature. Li et al. [23] used these methods for tracking control of knee exoskeleton system. Tracking control of a varactor system has been studied by Hu et al. [24]. Nonlinear and robust control have been developed by Zhang et al. [21] for a stirred tank system. Zheng and Zeng [19] investigated the tracking control for robot manipulator both with linear and nonlinear outputs. Both tracking control and disturbance rejection have been investigated in the case of a double-holding water tank by Ding et al. [20]. Applications of ZD and ZGD methods in solving control problems for various systems, including a simple pendulum system, a double-integrator system, an inverted pendulum on a cart, and a Van der Pol oscillator, are discussed in detail in [22,25].

Inspired and motivated by these interesting applications, an extension of these design methods for DPSs is investigated in this work. To the best of our knowledge, the applications of the ZD and ZGD methods are limited only to lumped parameter systems (LPSs), i.e., systems described by Ordinary Differential Equations (ODEs). This paper extends the application of these methods to Distributed Parameter Systems (DPSs), marking the initial exploration in this direction. Thus, in this paper, the ZD and ZGD methods are applied to design a velocity controller for single flow DPSs described by hyperbolic PDEs, and the stability of the resulting closed loop system is investigated. The developed state feedback control strategies are applied to solve the tracking control in the case of a steam-jacketed heat exchanger and a non-isothermal plug flow reactor. The objective consists in enforcing the outlet temperature of the heat exchanger, and the concentration of a species, at the outlet reactor, to track their desired references. Simulation results are given to demonstrate the performances of the designed controller using the ZD and ZGD methods.

The contributions of the paper are summarized by the following points:

- Infinite-dimensional state feedbacks are developed for hyperbolic distributed parameter systems following the late lumping approach based on the ZD and ZGD methods.
- The ZD and ZGD design methods are employed to develop a velocity controller for hyperbolic DPSs, representing the pioneering effort in this field.
- The use of ZGD controller allows us to overcome the challenging control singularity especially in the case of a uniform spatial profile,
- Stability analysis is provided for the ZD and ZGD state feedback control strategies.
- Simulation results are provided to demonstrate the effectiveness of the ZD and ZGD controllers in the case of two processes (steam-jacketed heat exchanger and non-isothermal plug flow reactor).

The paper is outlined as follows: Section 2 is dedicated to the formulation of the velocity control problem for hyperbolic DPSs. Section 3 focuses on the design of the ZD and ZGD state feedbacks, following the late lumping approach, and includes stability analysis for the resulting closed loop system. In Section 4, the practical applicability of the developed controllers is demonstrated through their application to two processes. Finally, Section 5 concludes the paper.

2. Control problem formulation

In this work, let us consider the class of DPSs whose dynamical behavior is described by the following mathematical model

$$\frac{\partial x(t,z)}{\partial t} = A\,\frac{\partial x(t,z)}{\partial z} + f(x(t,z)), \quad z \in \Omega \tag{1}$$

$$x(t,0) = x_0 \tag{2}$$

$$x(0,z) = x^*(z) \tag{3}$$

$$y(t) = \langle c(z), x_i(t,z)\rangle, \quad 1 \le i \le n \tag{4}$$

where $A \in \Re^{n\times n}$ is the following diagonal matrix

$$A = \mathrm{diag}(-u(t), \ldots, -u(t)), \tag{5}$$

$t \in [0,\infty)$ and $z \in \bar{\Omega}$ are independent variables that represent time and space position, respectively. $x(t,z) = \left[x_1(t,z) \ldots x_n(t,z)\right]^T \in \left[\mathcal{L}^2\left(\bar{\Omega}\right)\right]^n$ is the vector of state variables, $u(t) \in \Re^+$ the manipulated variable (flow velocity) and $y(t) \in \Re$ the output variable. $f\ \left(f(x) = \left[f_1(x) \ldots f_n(x)\right]^T\right)$ is a smooth vector function. $c(z)$ is a continuous function defined on the bounded and closed interval $\bar{\Omega}$ that defines the geometric configuration of the sensor. $x^* \in \left[\mathcal{L}^2\left(\bar{\Omega}\right)\right]^n$ is the initial profile and $x_0 \in \Re^n$ is the boundary input. $\Omega = (0, L]$ and $\partial\Omega = \{0\}$ are the interior and boundary of the entire spatial domain $\bar{\Omega}(\bar{\Omega} = \Omega \cup \partial\Omega = [0, L])$, respectively. $\mathcal{L}^2\left(\bar{\Omega}\right)$ denotes the space of real-valued square-integrable functions defined on $\bar{\Omega}$ with the standard scalar product, i.e., for $w_1, w_2 \in \mathcal{L}^2\left(\bar{\Omega}\right)$ defined as

$$\langle w_1(z), w_2(z)\rangle = \int_0^L w_1(z)\,w_2(z)\,dz \tag{6}$$

The aim is to design a state feedback that forces the output variable defined by Eq. (4) to track the desired reference $y^d(t)$.

Remark 1. Christofides and Daoutidis [8, p. 3066–3067, "Review of system-theoretic properties"] investigated the stability of the hyperbolic DPS (1)–(4) based on its linearized model. It is shown that this system is stable since the eigenvalues of the matrix A are negative real, given that the fluid flow velocity $u(t)$ is positive.

3. State feedback design

In this section, the ZD and ZGD methods are used to design, following the late lumping approach, the state feedback that enforces the controlled output defined by Eq. (4) to track a desired trajectory $y^d(t)$.

Assumption 1. The desired reference $y^d(t)$ is bounded and differentiable.

3.1. An overview of ZD and ZGD methods

The ZD and ZGD methods have been developed and successfully applied to LPSs described by ODEs. In this section, the principles of these methods are presented, and further details can be found in [20] and [22].

Consider the lumped parameter system (LPS) described by the following state-space model

$$\dot{x}(t) = f(x(t)) + g(x(t))\,u(t) \tag{7}$$

$$y(t) = h(x(t)) \tag{8}$$

where $x \in \Re^n$, $u \in \Re$, and $y \in \Re$ are the state, control and output, respectively. Here, f and g are smooth vector functions. It is assumed that the relative degree [26], denoted by σ, is finite.

3.1.1. ZD design method

The controller design based on the ZD method involves calculating the following errors

$$e_1(t) = y(t) - y^d(t) \tag{9}$$

$$e_{i+1}(t) = \dot{e}_i(t) + \lambda_i\, e_i(t) \quad \text{for } i = 1,\dots,\sigma-1 \tag{10}$$

where $\lambda_i > 0$ ($i = 1,\dots,\sigma-1$) are positive parameters that determine the dynamics of the errors e_i ($i = 1,\dots,\sigma-1$), respectively, and $y^d(t)$ is the desired reference assumed to be differentiable.

Subsequently, by substituting the different errors e_i into the tracking control design equation given by

$$\dot{e}_\sigma(t) + \lambda_\sigma\, e_\sigma(t) = 0, \quad \lambda_\sigma > 0 \tag{11}$$

the ordinary differential equation (ODE) describing the dynamics of the tracking error e_1 can be expressed in term of Lie derivatives as follows [20]

$$\mathcal{L}_g\mathcal{L}_f^{\sigma-1}h(x)\,u(t) + \sum_{k=0}^{\sigma} c_k\,\mathcal{L}_f^k h(x) - \sum_{k=0}^{\sigma} c_k\, y_d^{(k)}(t) = 0 \tag{12}$$

where c_i ($i = 1,\dots,\sigma-1$) are functions of the parameters λ_i ($i = 1,\dots,\lambda_\sigma$), and $c_\sigma = 1$. The function $\mathcal{L}_f h$ is the Lie derivative of the scalar field h with respect to the vector field f, $\mathcal{L}_f^k h$ is the kth order Lie derivative, and $\mathcal{L}_g\mathcal{L}_f^{\sigma-1}h$ is the mixed Lie derivative [26].

Solving Eq. (12) with respect to the manipulated variable u, yields the following ZD controller

$$u(t) = \frac{\sum_{k=0}^{\sigma} c_k\, y_d^{(k)}(t) - \sum_{k=0}^{\sigma} c_k\,\mathcal{L}_f^k h(x)}{\mathcal{L}_g\mathcal{L}_f^{\sigma-1}h(x)} \tag{13}$$

Note that in the case of control singularity, i.e., $\mathcal{L}_g\mathcal{L}_f^{\sigma-1}h(x) = 0$, the controller (13) becomes impractical. To overcome this problem, a ZGD controller can be used.

3.1.2. ZGD design method

The control u that drives the tracking error e_1 is the solution to Eq. (12). In the case of control singularity, the idea is to minimize, with respect to u, the following energy function

$$\varepsilon(t) = \frac{[J(x(t))]^2}{2} \tag{14}$$

with J is the left-hand side of Eq. (12)

$$J(x(t)) = \mathcal{L}_g\mathcal{L}_f^{\sigma-1}h(x)\,u(t) + \sum_{k=0}^{\sigma} c_k\,\mathcal{L}_f^k h(x) - \sum_{k=0}^{\sigma} c_k\, y_d^{(k)}(t) \tag{15}$$

Hence using the gradient descent technique

$$\dot{u}(t) = -\eta\,\frac{\partial\varepsilon(t)}{\partial u(t)}, \tag{16}$$

yields the following ZGD controller

$$\dot{u}(t) = -\eta\,\mathcal{L}_g\mathcal{L}_f^{\sigma-1}h(x)\,J(x(t)) \tag{17}$$

where $\eta > 0$ is a tuning parameter.

3.2. Application to velocity control design

The characteristic index is a generalization of the concept of relative degree to distributed parameter systems proposed by Christofides and Daoutidis [8]. For the hyperbolic DPS (1)–(4), it is easy to show that the control u appears in the first time derivative of the output (4), meaning that the characteristic index $\sigma = 1$.

3.2.1. State feedback design using the ZD method

To solve the tracking problem of the hyperbolic DPS (1)–(4), given that $\sigma = 1$, the design involves the following tracking error

$$e_1(t) = y^d(t) - y(t) \tag{18}$$

The control design is based on the following first-order ordinary differential equation which forces the error to decrease exponentially

$$\dot{e}_1(t) + \lambda\, e_1(t) = 0 \tag{19}$$

where λ is a strictly positive real number ($\lambda > 0$).

Taking into account Assumption 1 and employing Eq. (19), it follows that

$$\dot{y}^d(t) - \dot{y}(t) + \lambda\, e_1(t) = 0 \tag{20}$$

and using Eq. (4), it gives

$$\dot{y}^d(t) - \left\langle c(z),\, \frac{\partial x_i(t,z)}{\partial t}\right\rangle + \lambda\, e_1(t) = 0 \tag{21}$$

and considering the state Eq. (1), Eq. (21) takes the following form

$$\dot{y}^d(t) - \left\langle c(z),\, -u(t)\,\frac{\partial x_i(t,z)}{\partial z} + f_i(x(t,z))\right\rangle + \lambda\, e_1(t) = 0 \tag{22}$$

or equivalently

$$\left\langle c(z),\, \frac{\partial x_i(t,z)}{\partial z}\right\rangle u(t) + \dot{y}^d(t) - \langle c(z),\, f_i(x(t,z))\rangle + \lambda\, e_1(t) = 0 \tag{23}$$

consequently, the state feedback is derived from Eq. (23) as follows

$$u(t) = \frac{\langle c(z),\, f_i(x(t,z))\rangle - \lambda\, e_1(t) - \dot{y}^d(t)}{\left\langle c(z),\, \frac{\partial x_i(t,z)}{\partial z}\right\rangle} \tag{24}$$

Remark 2. λ is a tuning parameter of the controller (24) that can be used, according to Eq. (19), to adjust the convergence dynamics rate of the tracking error $e_1(t)$.

Proposition 1. *For the hyperbolic DPS (1)–(4), if* $\left\langle c(z),\, \frac{\partial x_i(t,z)}{\partial z}\right\rangle \neq 0$, *the ZD state feedback (24) enforces the output tracking in closed loop, i.e.,* $\lim_{t\to\infty} e_1(t) = 0$.

Proof. The design of the ZD controller (24) is based on Eq. (19). Hence, in closed loop, the tracking error is given by

$$e_1(t) = e_1(0)\, e^{-\lambda t} \tag{25}$$

thus since $\lambda > 0$, at steady-state,

$$\lim_{t\to\infty} e_1(t) = 0, \tag{26}$$

indicating that the output tracking is successfully achieved, that is, $y(t)$ globally exponentially converges to $y^d(t)$.

Note that when $\left\langle c(z),\, \frac{\partial x_i(t,z)}{\partial z}\right\rangle = 0$, the manipulated variable $u(t)$ is infinite, which means that the state feedback (24) is inapplicable. For example, in many DPSs, in the case where the initial profile $x^*(z)$ was chosen uniform (constant), the controller (24) would fail as the denominator of state feedback (24) would be zero. Of course, practically, this issue can be easily avoided for example by choosing a non constant strictly linear initial profile. This control singularity problem is also encountered with the input–output linearization approach [6]. In the following subsection, the ZGD approach is used to design a state feedback that solves the control singularity problem.

3.2.2. State feedback design using the ZGD method

To overcome the control singularity problem of the state feedback (24), the GD method is used to design a valid state feedback. Thus, in agreement with the ZD design method, the function $J(x_i(t,z))$ is defined as the left-hand side of Eq. (23), i.e.,

$$J(x_i(t,z)) = \left\langle c(z), \frac{\partial x_i(t,z)}{\partial z} \right\rangle u(t) + \dot{y}^d(t) - \langle c(z), f_i(x(t,z))\rangle + \lambda e_1(t) \quad (27)$$

Now, by considering the following energy function [22]

$$\varepsilon(t) = \frac{\left[J(x_i(t,z))\right]^2}{2} \quad (28)$$

and using the gradient of energy with respect to the manipulated input similarly to Zhang et al. [22] as a descent technique

$$\dot{u}(t) = -\gamma \frac{\partial \varepsilon(t)}{\partial u(t)} \quad (29)$$

the following state feedback results

$$\dot{u}(t) = -\gamma\, \alpha(t)\left(\alpha(t)\, u(t) + \Phi(x)\right) \quad (30)$$

with

$$\alpha(t) = \left\langle c(z), \frac{\partial x_i(t,z)}{\partial z} \right\rangle, \quad \Phi(x) = \dot{y}^d(t) - \langle c(z), f_i(x(t,z))\rangle + \lambda e_1(t) \quad (31)$$

where γ is a tuning parameter of the controller (30).

Proposition 2. *For a uniformly distributed initial profile $x^*(z)$ and an initial flow velocity control $u(0) \in \Re^+$, the ZGD state feedback (30) ensures a bounded steady state tracking error in closed loop for the hyperbolic DPS (1)–(4).*

Proof. The aim is to demonstrate that the tracking error e_1 in closed loop is bounded. In the same manner as in [19] and [25], two cases are possible, according to the value of α.

- Case of no singularity control $\alpha(t) \neq 0$: In this case, demonstrating that $e_1(t)$ is bounded requires to determine its expression as a function of time variable t. Using the notations of (31), Eq. (23) can be written under the following form

$$\alpha(t)\, u(t) + \Phi(x) = 0 \quad (32)$$

and as Eq. (32) is derived from Eq. (19), hence

$$\dot{e}_1(t) + \lambda e_1(t) = \alpha(t)\, u(t) + \Phi(x) \quad (33)$$

or equivalently

$$\dot{e}_1(t) + \lambda e_1(t) = \alpha(t)\, E(t) \quad (34)$$

where E is the control error defined as

$$E(t) = u(t) - \bar{u}(t) \quad (35)$$

and $\bar{u}(t)$ is the solution of Eq. (32), i.e.,

$$\alpha(t)\, \bar{u}(t) + \Phi(x) = 0, \quad (36)$$

which represents the theoretical flow velocity control that achieves the output tracking.
Now, since

$$-\|\alpha(t)\|\, \|E(t)\| \leq \alpha(t)\, E(t) \leq \|\alpha(t)\|\, \|E(t)\| \quad (37)$$

therefore, as $\alpha(t) \neq 0$, $0 < \alpha^2(t) \leq \beta^2 \leq \infty$ $(\beta > 0)$, hence using Eq. (34), it follows that

$$-\beta\, \|E(t)\| \leq -\|\alpha(t)\|\, \|E(t)\| \leq \dot{e}_1(t) + \lambda e_1(t) \leq \|\alpha(t)\|\, \|E(t)\| \leq \beta\, \|E(t)\| \quad (38)$$

Let us now determine a bound for $\|E(t)\|$. From Eq. (35), it follows that

$$\dot{E}(t) = \dot{u}(t) - \dot{\bar{u}}(t) \quad (39)$$

Substituting $\dot{u}$ according to Eq. (30) gives

$$\dot{E}(t) = -\gamma\, \alpha(t)\, \left(\alpha(t)\, u(t) + \Phi(x)\right) - \dot{\bar{u}}(t) \quad (40)$$

Combining Eq. (40) with Eq. (36) yields

$$\dot{E}(t) = -\gamma\, \alpha(t)\, \left(\alpha(t)\, u(t) - \alpha(t)\, \bar{u}(t)\right) - \dot{\bar{u}}(t) \quad (41)$$

and considering (35), Eq. (41) simplifies as

$$\dot{E}(t) = \mathcal{A}(t)\, E(t) - \dot{\bar{u}}(t) \quad (42)$$

with $\mathcal{A}(t) = -\gamma\, \alpha^2(t)$.
The operator $\mathcal{A}(t)$ generates a two-parameter semigroup $U(t,s)$ given by

$$U(t,s) = e^{I(t)-I(s)}, \quad 0 \leq s \leq t \quad (43)$$

with $I(\tau) = \int_0^\tau \mathcal{A}(\xi)\, d\xi$.
Given that $\gamma > 0$, the semigroup $U(t,s)$ is stable, indicating the existence of two constants $M \geq 1$ and $\omega > 0$ such that Pazy [27]

$$\|U(t,s)\| \leq M\, e^{-\omega(t-s)} \quad (44)$$

The solution of Eq. (42) is given by [27]

$$E(t) = U(t,0)\, E(0) + \int_0^t U(t,s)\, \dot{\bar{u}}(s)\, ds \quad (45)$$

hence, by Gronwall's inequality [28]

$$\|E(t)\| \leq \|U(t,0)\|\, \|E(0)\| + \int_0^t \|U(t,s)\|\, \|\dot{\bar{u}}(s)\|\, ds \quad (46)$$

Since $\gamma > 0$ and $\Phi(x)$ is bounded, it follows from Eq. (30) that $\|\dot{u}(t)\| \leq \eta \leq \infty$. Eqs. (44) and (46) yield

$$\|E(t)\| \leq M\, e^{-\omega t}\, \|E(0)\| + \int_0^t M\, e^{-\omega(t-s)}\, \eta\, ds \quad (47)$$

By evaluating the integral term, Eq. (47) reduces to

$$\|E(t)\| \leq M\, e^{-\omega t} \left(\|E(0)\| - \frac{\eta}{\omega}\right) + \frac{M\,\eta}{\omega} \quad (48)$$

hence, it can be concluded that

$$\lim_{t\to\infty} \|E(t)\| \leq \frac{M\,\eta}{\omega} \quad (49)$$

Considering Eq. (49), Eq. (38) reduces to

$$-\frac{\beta\, M\,\eta}{\omega} \leq \dot{e}_1(t) + \lambda e_1(t) \leq \frac{\beta\, M\,\eta}{\omega} \quad (50)$$

and using Cronwall's inequality [28]

$$-C\, e^{-\lambda t} - \frac{\beta\, M\,\eta}{\lambda\,\omega} \leq e_1(t) \leq C\, e^{-\lambda t} + \frac{\beta\, M\,\eta}{\lambda\,\omega} \quad (51)$$

and since $\frac{\beta\, M\,\eta}{\lambda\,\omega}$ is positive, Eq. (51) simplifies to

$$|e_1(t)| \leq |C|\, e^{-\lambda t} + \frac{\beta\, M\,\eta}{\lambda\,\omega} \quad (52)$$

where C is an integration constant. Consequently,

$$\lim_{t\to\infty} |e_1(t)| \leq \frac{\beta\, M\,\eta}{\lambda\,\omega} \quad (53)$$

Therefore, the tracking error e_1 is bounded.

- Case of singularity control $\alpha(t) = 0$: Let t_0 be the singularity control instant, i.e, $\lim_{t\to t_0} \alpha(t) = 0$. Eq. (30) yields

$$\lim_{t\to t_0} \dot{u}(t) = \lim_{\alpha(t)\to 0} \dot{u}(t) = 0, \quad (54)$$

which means that $u(t)$ is continuous at t_0, i.e., $u(t_0^-) = u(t_0) = u(t_0^+)$. Considering Remark 1, the bounded control sequence $u(t_0^-)$, $u(t_0)$, and $u(t_0^+)$ yields bounded outputs $y(t_0^-)$, $y(t_0)$, and $y(t_0^+)$. As a result, the tracking errors $e_1(t_0^-)$, $e_1(t_0)$, and $e_1(t_0^+)$ remain bounded, given the bounded nature of the desired reference $y^d(t)$ (Assumption 1). Consequently, the tracking error e_1 is bounded.

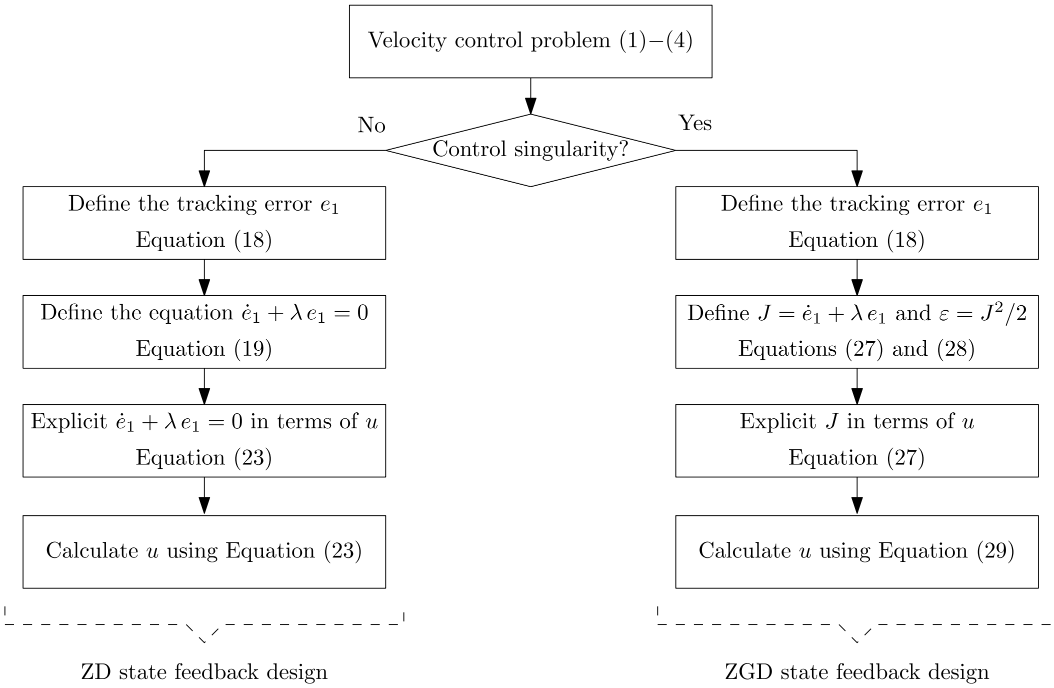

Fig. 1. Velocity control design for hyperbolic DPS (1)–(4).

On account of the above convergence analysis of the tracking error $e(t)$, it can be inferred that the output $y(t)$ remains within finite bounds throughout the tracking operation for both singularity and non-singularity cases.

Remark 3. The ZGD state feedback given by (30) does not suffer from the control singularity problem in comparison to the controller (24) and the controller designed using the input–output linearization approach [6].

Remark 4. For simulation and implementation of the controller (24), an initial value of the velocity, i.e., $u(0)$ must be specified. This can be determined through trial and error or selected as a reasonable practical value.

Remark 5. It is noteworthy that the parameter γ significantly impacts the value of ω, hence in the case of no control singularity ($\alpha(t) \neq 0$), from Eq. (53), it can be seen that the tracking error can be rendered small by increasing the tuning parameters λ and γ.

Remark 6. The state feedbacks (24) and (30) are of infinite dimensionality, thus their implementation requires a knowledge of the entire state of hyperbolic DPS which is achieved by means of an observer that estimates the entire state using the available measurements. Additionally, it is necessary to evaluate the spatial derivative at the outlet $z = L$, which is done numerically. The solution of this issue has been extensively discussed by the authors in the context of geometric control of a counter-current heat exchanger [6]. This solution is still applicable to the ZD and ZGD controllers proposed in this work.

The main velocity control design steps of the hyperbolic DPS (1)–(4) are summarized in Fig. 1

4. Application examples

In this section, the output tracking performances of the controllers (24) and (30) resulting from the ZD and ZDG methods respectively, are evaluated in the case of a steam-jacketed tubular heat exchanger and plug flow reactor. The simulations are carried out using the method of lines based on finite difference schemes [29]. For both processes, the controlled output is defined at outlet point $z = L$, hence the shaping function $c(z)$ is modeled as a Dirac delta function at $z = L$, i.e., $c(z) = \delta_L(z) = \delta(z - L)$. The desired reference $y^d(t)$ is obtained by filtering the set point $y^{\rm sp}(t)$ with a first order filter of time constant τ as

$$\frac{Y^d(s)}{Y^{\rm sp}(s)} = \frac{1}{\tau\, s + 1} \tag{55}$$

where s is the Laplace variable. $Y^d(s)$ and $Y^{\rm sp}$ are the Laplace transforms of $y^d(t)$ and $y^{\rm sp}(s)$, respectively.

4.1. Steam-jacketed tubular heat exchanger

The evolution of the temperature $T(t, z)$ within the tube of a steam-jacketed tubular heat exchanger (Fig. 2), with a length L, is described by the following dimensionless model [12]

$$\frac{\partial T(t, z)}{\partial t} = -v(t)\,\frac{\partial T(t, z)}{\partial z} + H\,\left(T_j(t) - T(t, z)\right) \tag{56}$$

$$T(t, 0) = T_{\rm in} \tag{57}$$

$$T(0, z) = T^*(z) \tag{58}$$

where v is the fluid velocity, $T_{\rm in}$ the inlet fluid temperature, T_j the steam jacket temperature which is assumed to be uniform, $T^*(z)$ the initial temperature profile and H a positive (thermal) constant. Here, all variables are dimensionless. The output to be controlled is the outlet temperature, that is,

$$y(t) = T(t, L) \tag{59}$$

Using Eqs. (24) and (30), the following ZD and ZGD controllers result

$$v(t) = \frac{H\,\langle c(z), \left(T_j(t) - T(t, z)\right)\rangle - \lambda\,\left(y^d(t) - y(t)\right) - \dot{y}^d(t)}{\left.\dfrac{\partial T(t, z)}{\partial z}\right|_{z=L}} \tag{60}$$

Table 1
Steam-jacketed heat exchanger simulation parameters.

Heat exchanger		ZD state feedback		ZGD state feedback	
Parameter	Value	Parameter	Value	Parameter	Value
H	2	λ	1	λ	10
T_j	5			γ	0.1
T_{in}	0			$v(0)$	2
L	1				
v	2				

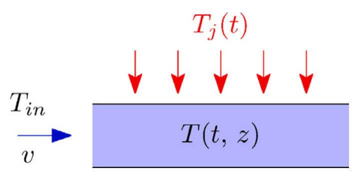

Fig. 2. Steam-jacketed heat exchanger.

$$\dot{v}(t) = -\gamma \left.\frac{\partial T(t,z)}{\partial z}\right|_{z=L} \left[\left.\frac{\partial T(t,z)}{\partial z}\right|_{z=L} v(t) + \lambda \left(y^d(t) - y(t)\right) + \dot{y}^d(t) - H \left\langle c(z), \left(T_j(t) - T(t,z)\right)\right\rangle \right] \tag{61}$$

The simulations are carried out with parameters given in Table 1.

4.1.1. ZD state feedback

In the time interval $[0, 10[$, it is assumed that the heat exchanger is at open loop steady state obtained with parameters given in Table 1. Then, in the time interval $[10, 40]$, the ZD feedback controller is applied. The ZD state feedback performance is evaluated by imposing two set points $y^{\text{sp}}(t) = 3$ and $y^{\text{sp}}(t) = 3.5$ at $t = 10$ and $t = 30$ with the filter time constant $\tau = 2$, respectively. The simulation results are depicted in Figs. 3–4. It can be observed that the ZD state feedback achieves the output tracking (Fig. 3). The flow rate $v(t)$ exhibits physically mild moves (Fig. 4).

4.1.2. ZGD state feedback

In the case of a constant initial temperature profile, the ZGD state feedback should be used. However, a heat exchanger in open loop does not possess a constant temperature profile. Thus, in the first time interval $[0, 10[$, the heat exchanger is operated in open loop with a velocity equal to 2. Then, at $t = 10$, assuming that the actual spatial temperature profile is unknown, a constant initial temperature profile is guessed for the heat exchanger

$$T^*(z) = 3.0783 \qquad \forall\, z \tag{62}$$

equal to the previous outlet temperature in open loop. Thus, a control singularity occurs at $t = 10$. This is due to the derivative of the temperature at the outlet $z = L$ being null. Consequently, the ZD state feedback (60) fails to track the desired reference. Thus, the performance of the ZGD state feedback (61) can be evaluated in the time interval $[10, 40]$ where the same set points as for the ZD control are imposed. The parameters of the ZGD controller are given in Table 1. After a very short transient, the controlled temperature perfectly tracks the reference trajectory (Fig. 5). The flow rate also shows a rapid variation before becoming smoother (Fig. 6). These rapid variations are explained by the fluctuations of the spatial derivative of the outlet temperature (Fig. 7) which is present in the ZGD control law.

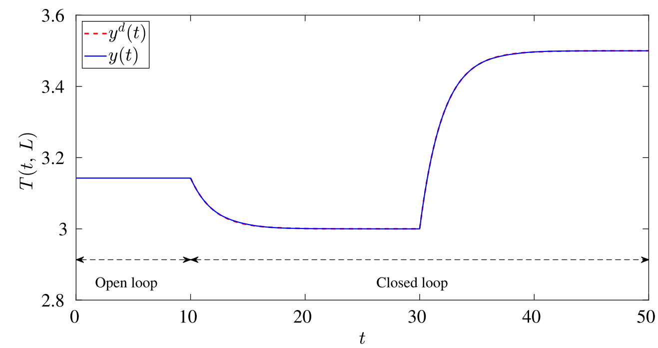

Fig. 3. Steam-jacketed heat exchanger: output evolution with ZD state feedback.

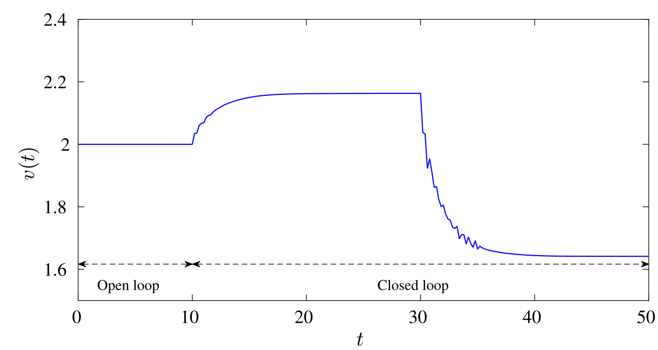

Fig. 4. Steam-jacketed heat exchanger: fluid flow velocity evolution with ZD state feedback.

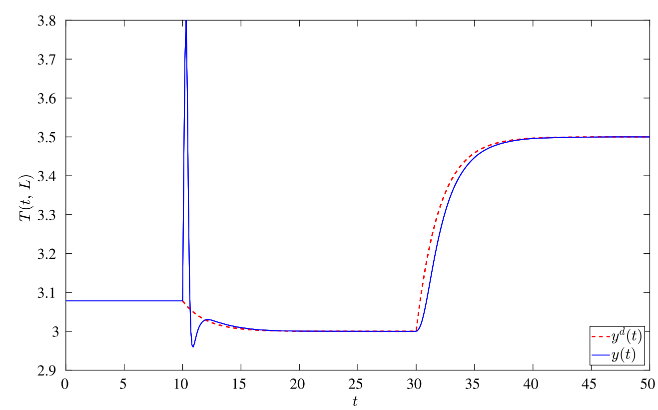

Fig. 5. Steam-jacketed heat exchanger: output evolution with ZGD state feedback.

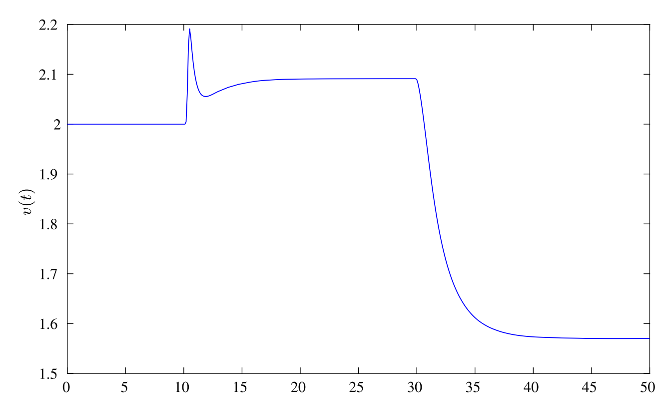

Fig. 6. Steam-jacketed heat exchanger: evolution of the fluid flow velocity $v(t)$ with ZGD state feedback.

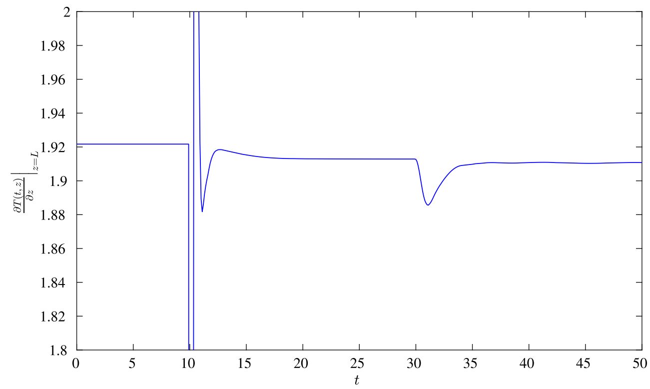

Fig. 7. Steam-jacketed heat exchanger: evolution of the temperature derivative at $z = L$ with ZGD state feedback.

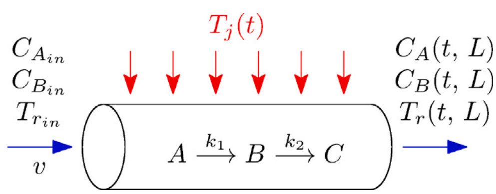

Fig. 8. Non-isothermal plug flow reactor.

4.2. Non-isothermal plug flow reactor

The mathematical model of a plug flow reactor operating under non-isothermal conditions and some reasonable assumptions [8], involving the occurrence of two first order reactions (Fig. 8)

$$A \xrightarrow{k_1} B \xrightarrow{k_2} C \tag{63}$$

is given by the following PDEs

$$\frac{\partial C_A(t,z)}{\partial t} = -\,v(t)\frac{\partial C_A(t,z)}{\partial z} - k_1\, C_A(t,z) \tag{64}$$

$$\frac{\partial C_B(t,z)}{\partial t} = -\,v(t)\frac{\partial C_B(t,z)}{\partial z} + k_1\, C_A(t,z) - k_2\, C_B(t,z) \tag{65}$$

$$\frac{\partial T_r(t,z)}{\partial t} = -\,v(t)\frac{\partial T_r(t,z)}{\partial z} - \frac{\Delta H_1}{\rho_m\, c_{pm}}\, k_1\, C_A(t,z) - \frac{\Delta H_2}{\rho_m\, c_{pm}}\, k_2\, C_B(t,z) + \frac{U_w}{\rho_m\, c_{pm}\, V_r}\left(T_j(t) - T_r(t,z)\right) \tag{66}$$

with the following boundary conditions

$$C_A(t,0) = C_{A,\mathrm{in}}, \quad C_B(t,0) = C_{B,\mathrm{in}}, \quad T_r(t,0) = T_{r,\mathrm{in}} \tag{67}$$

and the initial conditions

$$C_A(0,z) = C_A^*(z), \quad C_B(0,z) = C_B^*(z), \quad T_r(0,z) = T_r^*(z) \tag{68}$$

with the kinetic constants following Arrhenius law

$$k_i = k_{i0}\, e^{-\dfrac{E_i}{R\,T_r(t,z)}} \qquad i = 1, 2 \tag{69}$$

The variables C_A and C_B are the concentrations of species A and B in the reactor, respectively, and T_r the reactor temperature. The various model parameters and their corresponding values are provided in Table 2 [13].

The control objective involves designing a state feedback (flow velocity) to control the concentration of the species B at the reactor outlet $z = L$. Therefore, the output to be controlled is defined as

$$y(t) = C_B(t, L) \tag{70}$$

Table 2
Plug flow reactor parameters [13].

Parameter	Designation	Value	Unit
v	Fluid flow velocity	1	m min^{-1}
L	Reactor length	2	m
V_r	Reactor volume	2	lt
E_1	Activation energy for the first reaction	2×10^4	kcal kmol^{-1}
E_2	Activation energy for the second reaction	2×10^4	kcal kmol^{-1}
k_{10}	Pre-exponential factor	3×10^{12}	min^{-1}
k_{20}	Pre-exponential factor	8×10^{11}	min^{-1}
R	Gas constant	1.987	kcal kmol^{-1} K^{-1}
ΔH_{r_1}	Enthalpy of the first reaction	-4×10^4	kcal kmol^{-1}
ΔH_{r_2}	Enthalpy of the second reaction	-2×10^5	kcal kmol^{-1}
ρ_m	Fluid density	1000	kg lt^{-1}
c_{pm}	Fluid heat capacity	0.231	kcal kg K^{-1}
U_w	Heat transfer coefficient	2×10^4	kcal min^{-1} K^{-1}
T_j	Jacket temperature	350	K
$C_{A,\mathrm{in}}$	Reactant A inlet concentration	1	mol l^{-1}
$C_{B,\mathrm{in}}$	Reactant B inlet concentration	0	mol l^{-1}
$T_{r,\mathrm{in}}$	Inlet stream temperature	350	K

Table 3
Non-isothermal plug flow reactor: controllers and reference parameters.

Designation	ZD state feedback	ZGD state feedback
λ	20	10^2
γ		10^4
$v(0)$		1

By defining $x_1 = C_A$, $x_2 = C_B$, and $x_3 = T_r$, the subscript i in Eq. (4) is equal to 2. Therefore, using Eqs. (24) and (30), the following ZD and ZGD controllers result

$$v(t) = \frac{\langle c(z),\, k_1\, C_A(t,z) - k_2\, C_B(t,z)\rangle - \lambda\left(y^d(t) - y(t)\right) - \dot{y}^d(t)}{\left.\dfrac{\partial C_B(t,z)}{\partial z}\right|_{z=L}} \tag{71}$$

$$\dot{v}(t) = -\gamma\, \left.\frac{\partial C_B(t,z)}{\partial z}\right|_{z=L} \left[\left.\frac{\partial C_B(t,z)}{\partial z}\right|_{z=L} v(t) + \dot{y}^d(t) - \langle c(z),\, k_1\, C_A(t,z) - k_2\, C_B(t,z)\rangle + \lambda\left(y^d(t) - y(t)\right) \right] \tag{72}$$

The used controller parameters are provided in Table 3.

4.2.1. ZD state feedback

The plug flow reactor is operated in open loop at steady state in the time interval [0, 10[min, then in closed loop in [10, 40] min. The steady state profiles are given in Fig. 9, obtained with parameters of Table 2.

To evaluate the tracking performance of the ZD state controller, two set points $y^{sp}(t) = 0.6$ mol l^{-1} and $y^{sp}(t) = 0.5$ mol l^{-1} are imposed at $t = 10$ min and $t = 25$ min with the filter time constant $\tau = 2$ min, respectively. Figs. 10–11 show the obtained results. It can be clearly seen that the ZD state feedback performs well to achieve perfect tracking with smooth fluctuations in the fluid flow velocity (Fig. 11). This is expected because, with the flow velocity as the control variable, and considering the delay between the inlet and the outlet of the reactor, the closed loop system can be understood as a time-varying delay system, leading to oscillations.

4.2.2. ZGD state feedback

To assess the performance tracking of the ZGD state feedback, it is assumed that the fluid crossing the plug flow reactor contains no reactant A in the time interval [0, 10[min so that $C_{A,\mathrm{in}} = 0$, then suddenly the component A is introduced at $t = 10$ min at a concentration $C_{A,\mathrm{in}} = 1$ mol l^{-1}. The reactor is operated in open loop in the time interval [0, 10[min, then in closed loop in [10, 40] min. Due to this step change of concentration, it results that the spatial derivative of the

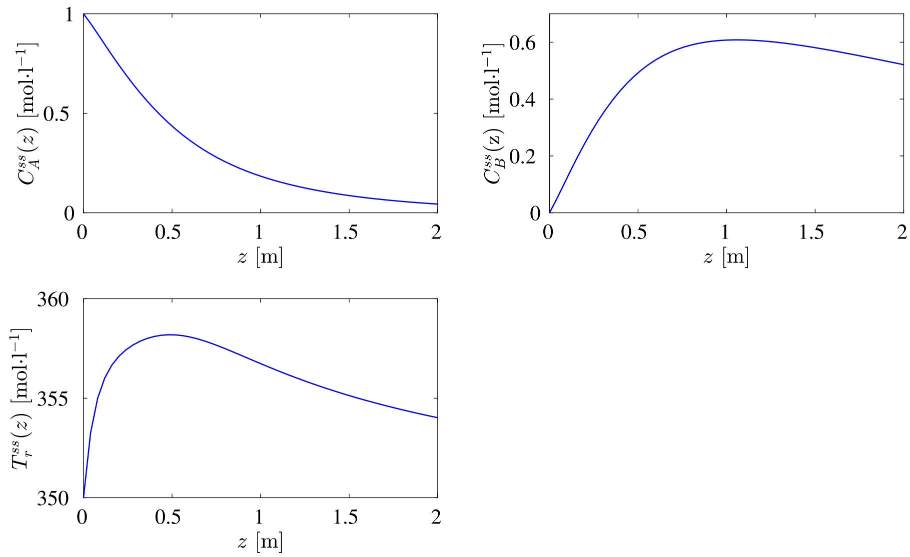

Fig. 9. Reactor steady state profiles.

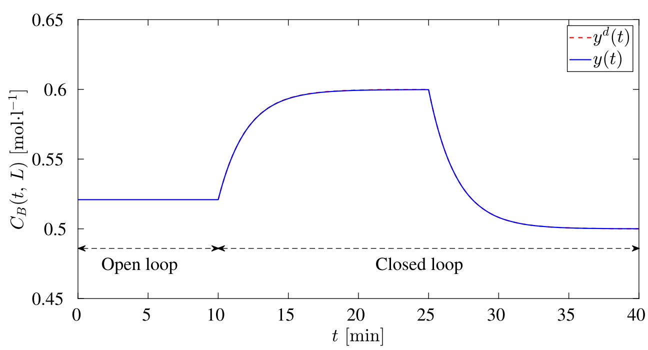

Fig. 10. Non-isothermal plug flow reactor: output evolution with ZD state feedback.

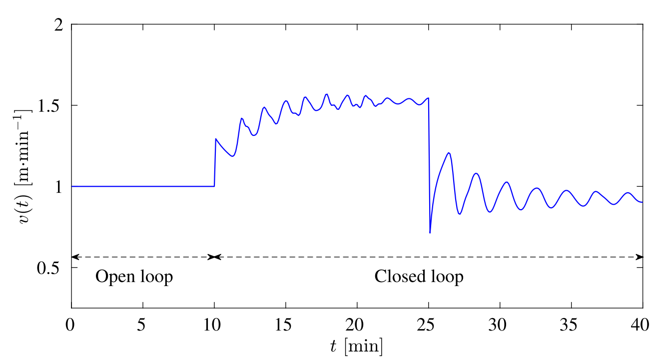

Fig. 11. Non-isothermal plug flow reactor: fluid flow velocity evolution with ZD state feedback.

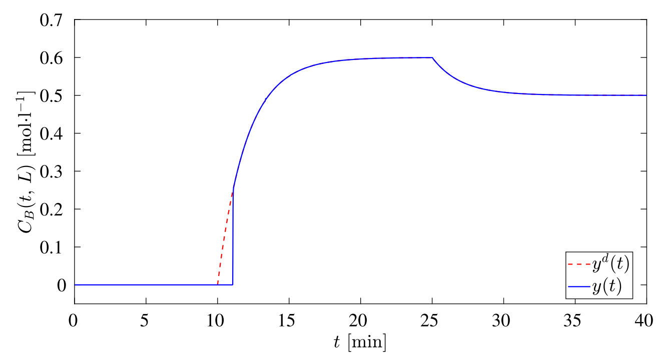

Fig. 12. Non-isothermal plug flow reactor: output evolution with ZGD state feedback.

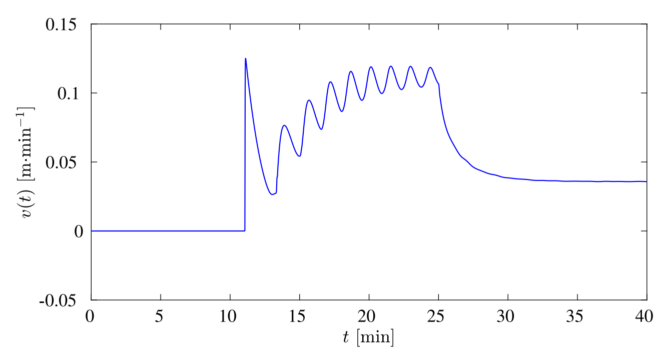

Fig. 13. Non-isothermal plug flow reactor: fluid flow velocity evolution with ZGD state feedback.

outlet temperature is zero at $t = 10$ min creating a control singularity with the ZD controller. Thus, the ZGD control is applied.

The same set points as in the ZD case are imposed. Even, in this situation, the ZGD state feedback enforces the outlet concentration $C_B(L, t)$ to track its desired reference (Fig. 12). It can be clearly seen that the controlled output exhibits a delay and starts with a jump. This is due to the control singularity observed at $t = 10$ min, which leads to a fluid flow velocity (Fig. 13) that exhibits subdued fluctuations in the transient response, particularly at the beginning, but the moves remain physically applicable. It is noteworthy that, after overcoming the singularity issue, the control oscillations are reduced compared to the ZD controller case. This is attributed to the ZGD controller, which can be perceived as applying a filtered control. Fig. 14 displaying the spatial derivative of the outlet concentration shows the effect of the control singularity at $t = 10$ min.

5. Conclusion

In this paper, the ZD and ZGD methods are applied to solve the velocity control problem of hyperbolic DPSs. It is shown that these methods easily allow the design, following the late lumping approach, of a stable infinite-dimensional state feedback that enforces output

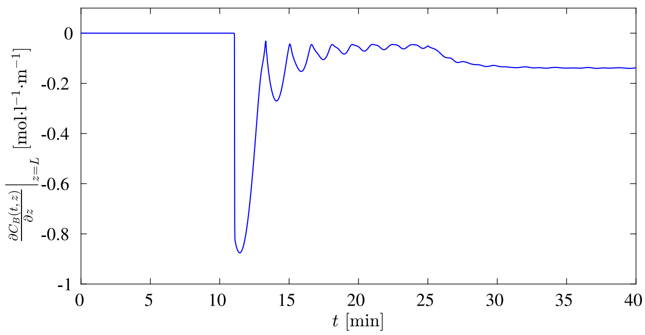

Fig. 14. Non-isothermal plug flow reactor: evolution of the C_B concentration derivative at $z = L$ with ZGD state feedback.

tracking. The design consists solely of evaluating the derivatives of the tracking error.

The ZD method yields a state feedback that is impractical when the initial profile is uniform, leading to a control singularity issue. Therefore, by combining the ZD method with the Gradient Dynamics method, a ZGD state feedback can be designed to address the tracking problem even when the initial profile is uniform. The ZD state feedback involves one tuning parameter that fixes the tracking error convergence rate, whereas ZGD state feedback involves a second tuning parameter that can be used to enhance the transient response.

The tracking capabilities of both ZD and ZGD state feedbacks are demonstrated through simulation in the case of a steam-jacketed tubular heat exchanger and a non-isothermal plug flow reactor. The conducted study demonstrates that the ZGD controller not only successfully addresses the singularity problem but also enhances the performance.

Thus, the present study shows that ZD and ZGD are two effective tools that can be used for the design of infinite-dimensional state feedback for DPSs. The design does not require manipulation of partial differential equations but consists of evaluating derivatives, which is simple to accomplish. Additionally, the tuning of the controllers can be achieved by adjusting the λ and γ parameters. However, for practical implementation of the developed state feedbacks, an observer is essential for estimating the entire state of the hyperbolic DPS.

The developments of the present study can serve as a starting point or catalyst, inspiring the extension of ZD and ZGD methods to other classes of DPS problems and fostering the development of theoretical results. Future work should extend the ZD and ZGD methods to diverse distributed parameter systems, encompassing dual-flow hyperbolic systems and those described by fractional and integro-partial differential equations. Additionally, it should focus on developing state feedback control capable of effectively rejecting disturbances based on the ZD and ZGD methods.

CRediT authorship contribution statement

Ahmed Maidi: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Methodology, Formal analysis, Conceptualization. **Radoslav Paulen:** Writing – review & editing, Visualization, Validation, Supervision, Methodology, Conceptualization. **Jean-Pierre Corriou:** Writing – review & editing, Visualization, Validation, Supervision, Software, Methodology, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Radoslav Paulen reports financial support was provided by Slovak University of Technology in Bratislava. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgments

Radoslav Paulen acknowledges the funding from the Slovak Research and Development Agency under the project APVV-21-0019, the Scientific Grant Agency of the Slovak Republic under the grant 1/0691/21, by the European Commission under the grant no. 101079342 (Fostering Opportunities Towards Slovak Excellence in Advanced Control for Smart Industries).

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