Stability analysis in a fractional order delayed predator-prey model

Changjin Xu, Peiluan Li

Abstract—In this paper, we study the stability of a fractional order delayed predator-prey model. By using the Laplace transform, we introduce a characteristic equation for the above system. It is shown that if all roots of the characteristic equation have negative parts, then the equilibrium of the above fractional order predator-prey system is Lyapunov globally asymptotical stable. An example is given to show the effectiveness of the approach presented in this paper.

Keywords—Fractional predator-prey model, laplace transform, characteristic equation.

I. Introduction

IN recent years, the interaction between the predator and prey has attracted a lot of attention and many good results have already been reported. For example, May [1] discussed briefly the stability of the following delayed predator-prey system

$$\begin{cases} \dot{x}(t) = x(t)[r_1 - a_{11}x(t-\tau) - a_{12}y(t)], \\ \dot{y}(t) = y(t)[-r_2 + a_{21}x(t) - a_{22}y(t)], \end{cases} \quad (1)$$

where $x(t)$ and $y(t)$ can be interpreted as the population densities of prey and predator at time t, respectively; $\tau \geq 0$ is the feedback time delay of the prey to the growth of the species itself; $r_1 > 0$ denotes intrinsic growth rate of the prey and $r_2 > 0$ denotes the death rate of the predator; the parameters $a_{ij}(i, j = 1, 2)$ are all positive constants. Song and Wei [2] further made a discussion on the dynamical behavior of system (1). In 2006, Yan and Li [3] incorporated the same delay τ into the population density of the predator in the second equation of system (1) and obtained the following system

$$\begin{cases} \dot{x}(t) = x(t)[r_1 - a_{11}x(t-\tau) - a_{12}y(t)], \\ \dot{y}(t) = y(t)[-r_2 + a_{21}x(t) - a_{22}y(t-\tau)]. \end{cases} \quad (2)$$

Regarding the delay τ as the bifurcation parameter, they investigated the stability of system (2) and studied the properties of Hopf bifurcation for system (2) by using the normal form theory and the center manifold theorem which is different from that used in Song and Wei [2].

In 2001, Faria[4] focused on the stability and Hopf bifurcation of the following system with two different delays:

$$\begin{cases} \dot{x}(t) = x(t)[r_1 - a_{11}x(t) - a_{12}y(t-\tau_2)], \\ \dot{y}(t) = y(t)[-r_2 + a_{21}x(t-\tau_1) - a_{22}y(t)]. \end{cases} \quad (3)$$

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Manuscript received May 12, 2012; revised August 11, 2012.

According the view point of Kuang [5], Yan and Zhang [6] considered the stability and Hopf bifurcation of the following delayed system

$$\begin{cases} \dot{x}(t) = x(t)[r_1 - a_{11}x(t-\tau) - a_{12}y(t-\tau)], \\ \dot{y}(t) = y(t)[-r_2 + a_{21}x(t-\tau) - a_{22}y(t-\tau)]. \end{cases} \quad (4)$$

Xu ea al. [7] argued that in real situations, the feedback time delay of the prey to the growth of the species itself and the feedback time delay of the predator to the growth of the species itself are different, then they explored the dynamics of the following more generalized delayed system

$$\begin{cases} \dot{x}(t) = x(t)[r_1 - a_{11}x(t-\tau_1) - a_{12}y(t-\tau_2)], \\ \dot{y}(t) = y(t)[-r_2 + a_{21}x(t-\tau_1) - a_{22}y(t-\tau_2)]. \end{cases} \quad (5)$$

We all know that fractional calculus and it applications to physics, biology and engineering is a recent focus of interest to many researchers [8]. Although the fractional(-order) calculus equations have almost the same history as those of typical differential equations, they did not attract much attention till recent decades [9-11]. It was lately found that a lot of systems can be modelled via using fractional derivatives. These systems display fractional-order dynamics, such as heart transfer, viscoelasticity, electrical circuit, electro-chemistry, dynamics, economics, polymer physics and control [12-13]. Analysis of stability is fundamental to any control system. Recently, considerable attention has given to the stability problems arising from neutral systems. Various analysis techniques have been applied to derive stability criteria for the systems [14-22]. However, the work on the topic of stability for fractional order delayed predator-prey system is rare. This motivated our research. In this paper, we will focus on the stability of the following fractional order predator-prey with time delays

$$\begin{cases} {}^C_0 D_t^{\alpha_1} x(t) = x(t)[r_1 - a_{11}x(t-\tau_1) \\ \qquad\qquad - a_{12}y(t-\tau_2)], \\ {}^C_0 D_t^{\alpha_2} y(t) = y(t)[-r_2 + a_{21}x(t-\tau_1) \\ \qquad\qquad - a_{22}y(t-\tau_2)], \end{cases} \quad (6)$$

where $\alpha_i(i = 1, 2)$ are real and lies in (0, 1). The initial values $x(t) = \varphi_1(t)$ and $y(t) = \varphi_2(t)$ are given for $t \in [-\tau, 0]$.

Following the mehtod of [23], we make use of the Laplace transform, a characteristic equation for the above system is introduced. We find that if all roots of the characteristic equation have negative parts, then the equilibrium of the above fractional order predator-prey system is Lyapunov globally asymptotical stable if the equilibrium exists which is almost the same as that of classical differential equations. Finally, one numerical example is given to illustrate the effectiveness of the obtained results.

World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:7, No:5, 2013

II. PRELIMINARIES

This section start with recalling the essentials of the fractional calculus. The idea of fractional calculus has been known since the development of the regular calculus, with the first reference probably being associated with Leibniz and LHospital in 1695 where half-order derivative was mentioned. The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer-order differentiation and n-fold integration. There are three main uesed definitions of fractional integration and differentiation, such as Grunwald- Letnikovs definition, Riemann-Liouvilles definition, Caputos fractional derivative. The former two definitions are often used by pure mathematicians, while the last one is adopted by applied scientists, since it is more convenient in engineering applications. Here we only discuss Caputo derivative:

$$
{}_0^C D_t^q x(t) = \frac{d^q x(t)}{dt} = J^{m-q} x^{(m)}(t), \alpha > 0, \qquad (7)
$$

where $m = [q]$, i.e., m denotes the first integer that is not less than q, $x^{(m)}$ denotes a conventional m-th order derivative, J is the β-th order Riemann-Liouville integral operator, which is expressed in the form

$$
J^{\beta} x(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} x(s) ds, \beta > 0. \qquad (8)
$$

In engineering, the fractional order q often lies in (0,1), so we always suppose that the 'order' q is a positive number but less than 1 in this paper.

III. STABILITY OF SYSTEM (6)

It is easy to see that the delayed predator-prey model (6) has a unique positive equilibrium point $E_0(x^*, y^*)$, where

$$
x^* = \frac{r_1 a_{22} + r_2 a_{12}}{a_{11} a_{22} + a_{12} a_{21}}, y^* = \frac{r_1 a_{21} - r_2 a_{11}}{a_{11} a_{22} + a_{12} a_{21}},
$$

if the following condition

$$
(H1) \quad r_1 a_{21} - r_2 a_{11} > 0
$$

holds.

Let $\bar{x}(t) = x(t) - x^*, \bar{y}(t) = y(t) - y^*$ and still denote $\bar{x}(t), \bar{y}(t)$ by $x(t), y(t)$, respectively, then (6) becomes

$$
\begin{cases}
{}_0^C D_t^{\alpha_1} x(t) = m_1 x(t) + m_2 x(t-\tau_1) + m_3 y(t-\tau_2) \\
\qquad + m_4 x(t) x(t-\tau_1) + m_5 x(t) y(t-\tau_2), \\
{}_0^C D_t^{\alpha_2} y(t) = n_1 y(t) + n_2 x(t-\tau_1) + n_3 y(t-\tau_2) \\
\qquad + n_4 x(t-\tau_1) y(t) + n_5 y(t) y(t-\tau_2),
\end{cases} \qquad (9)
$$

where

$$
\begin{aligned}
m_1 &= r_1 - a_{11} x^* - a_{12} y^*, m_2 = -a_{11} x^*, \\
m_3 &= -a_{12} x^*, m_4 = -a_{11}, m_5 = -a_{12}, \\
n_1 &= -r_2 + a_{21} x^* - a_{22} y^*, n_2 = a_{21} y^*, \\
n_3 &= -a_{22} y^*, n_4 = a_{21}, n_5 = -a_{22}.
\end{aligned}
$$

The linearization of Eq. (9) at $(0,0)$ is

$$
\begin{cases}
{}_0^C D_t^{\alpha_1} x(t) = m_1 x(t) + m_2 x(t-\tau_1) + m_3 y(t-\tau_2), \\
{}_0^C D_t^{\alpha_2} y(t) = n_1 y(t) + n_2 x(t-\tau_1) + n_3 y(t-\tau_2).
\end{cases} \qquad (10)
$$

Next, we study the stability of system (6). Taking Laplace transform [24] on both sides of (10) gives

$$
\begin{cases}
s^{\alpha_1} X_1(s) = m_1 X_1(s) + s^{\alpha_1-1} \varphi_1(0) \\
\qquad + m_2 e^{-s\tau_1} (X_1(s) + \int_{-\tau_1}^0 e^{-st} \varphi_1(t) dt) \\
\qquad + m_3 e^{-s\tau_2} (X_2(s) + \int_{-\tau_2}^0 e^{-st} \varphi_2(t) dt), \\
s^{\alpha_2} X_2(s) = n_1 X_2(s) + s^{\alpha_2-1} \varphi_2(0) \\
\qquad + n_2 e^{-s\tau_1} (X_1(s) + \int_{-\tau_1}^0 e^{-st} \varphi_1(t) dt) \\
\qquad + n_3 e^{-s\tau_2} (X_2(s) + \int_{-\tau_2}^0 e^{-st} \varphi_2(t) dt),
\end{cases} \qquad (11)
$$

where $X_1(s)$ and $X_2(s)$ are the Laplace transform of $x(t)$ and $y(t)$ with $X_1(s) = L(x(t))$ and $X_2(s) = L(y(t))$.

System (11) can be rewrite as follows

$$
\Delta(s). \begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = \begin{pmatrix} k_1(s) \\ k_2(s) \end{pmatrix}, \qquad (12)
$$

in which

$$
\begin{cases}
k_1(s) = s^{\alpha_1-1} \varphi_1(0) + m_2 e^{-s\tau_1} \int_{-\tau_1}^0 e^{-st} \varphi_1(t) dt \\
\qquad + m_3 e^{-s\tau_2} \int_{-\tau_2}^0 e^{-st} \varphi_2(t) dt, \\
k_2(s) = s^{\alpha_2-1} \varphi_2(0) + n_2 e^{-s\tau_1} \int_{-\tau_1}^0 e^{-st} \varphi_1(t) dt \\
\qquad + n_3 e^{-s\tau_2} \int_{-\tau_2}^0 e^{-st} \varphi_2(t) dt
\end{cases} \qquad (13)
$$

and

$$
\begin{aligned}
\Delta(s) &= \begin{pmatrix} s^{\alpha_1} - m_1 - m_2 e^{-s\tau_1} & m_3 e^{-s\tau_2} \\ -n_2 e^{-s\tau_1} & s^{\alpha_2} - n_1 - n_3 e^{-s\tau_2} \end{pmatrix} \\
&= s^{\alpha_1+\alpha_2} - n_1 s^{\alpha_1} - m_1 s^{\alpha_2} - n_3 s^{\alpha_1} e^{-s\tau_2} \\
&\quad - m_2 s^{\alpha_2} e^{-s\tau_2} + m_1 n_3 e^{-s\tau_2} + m_2 n_1 e^{-s\tau_1} \\
&\quad + (m_2 n_3 + m_3 n_2) e^{-s(\tau_1+\tau_2)}.
\end{aligned} \qquad (14)
$$

We call $\Delta(s)$ a characteristic matrix of system (6) for simplicity and $\det \Delta(s)$ a characteristic polynomial of (6). The distribution of $\det \Delta(s)$ s eigenvalues totally determines the stability of system (6). This can be seen from the following discussion.

Remark 3.1. If $\tau_1 = \tau_2 = \tau$ and $\alpha_1 = \alpha_2 = 1$, then the characteristic matrix and characteristic equation of (6) are reduced to

$$
s^2 - (m_1 + n_1)s - [(m_2 + n_3)s - m_1 n_3 - m_2 n_1]e^{-s\tau} + (m_2 n_3 + m_3 n_2)e^{-2s\tau}, \qquad (15)
$$

and

$$
s^2 - (m_1 + n_1)s - [(m_2 + n_3)s - m_1 n_3 - m_2 n_1]e^{-s\tau} + (m_2 n_3 + m_3 n_2)e^{-2s\tau} = 0, \qquad (16)
$$

respectively. They coincide with the usual definitions of the characteristic matrix and characteristic equation of delayed equations.

Theorem 3.1. Under the conditions (H1), if all the roots of the characteristic equation $\det \Delta(s)$=0 have negative real parts, then the positive equilibrium points $E_0(x^*, y^*)$ of system (6) is Lyapunov globally asymptotically stable.

Proof: Multiplying s on both sides of (12) gives

$$
\Delta(s). \begin{pmatrix} s X_1(s) \\ s X_2(s) \end{pmatrix} = \begin{pmatrix} s k_1(s) \\ s k_2(s) \end{pmatrix}, \qquad (17)
$$

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Vol:7, No:5, 2013

If all roots of the transcendental equation $\det(\Delta(s)) = 0$ lie in open left half complex plane, i.e., $\text{Re}(s) < 0$, then we consider (17) in $\text{Re}(s) \geq 0$. In this restricted area, (17) has a unique solution $sX_11(s)sX_2(s)$. Thus have $\lim_{s\to 0, \text{Re}(s)\geq 0} sX_i(s) = 0, i = 1, 2$. From the assumption of all roots of the characteristic equation $\det(\Delta(s)) = 0$ and the final-value theorem of Laplace transform [24], we get

$$\lim_{t\to+\infty} x_1(t) = \lim_{s\to 0, \text{Re}(s)\geq 0} sX_1(s) = 0$$

and

$$\lim_{t\to+\infty} x_2(t) = \lim_{s\to 0, \text{Re}(s)\geq 0} sX_2(s) = 0.$$

It implies that the fractional order predator-prey system is Lyapunov globally asymptotically stable which completes the proof.

IV. NUMERICAL EXAMPLES

This section will give one example to show the effectiveness of our new criteria for asymptotic stability of fractional order predator-prey system. We consider the following special case of system (6)

$$\begin{cases} {}^{C}_{0}D^{0.5}_{t}x(t) = x(t)[0.4 - 0.4x(t - \tau_1) \\ \qquad - y(t - \tau_2)], \\ {}^{C}_{0}D^{0.5}_{t}y(t) = y(t)[-0.4 + x(t - \tau_1) \\ \qquad - y(t - \tau_2)], \end{cases} \qquad (18)$$

It is easy to obtain that system (18) has a unique positive equilibrium point $E(0.5714, 0.1714)$. Let $\tau_1 = 0.3$ and $\tau_2 = 2.5$, then the characteristic equation of this systems (18) is

$$s - 0.3428s^{0.5} + 0.4s^{0.5}e^{-2.5s}$$
$$-0.0588e^{-2.5s} + 0.481e^{-2.8s} = 0 \qquad (19)$$

With a simple calculation in the Matlab toolbox, all the roots of the characteristic equation have negative real parts. According to Theorem 3.1, the system is asymptotically stable which is illustrated by the computer simulations (see Figs.1-4).

Fig.1

Fig.2

Fig.3

Fig.4
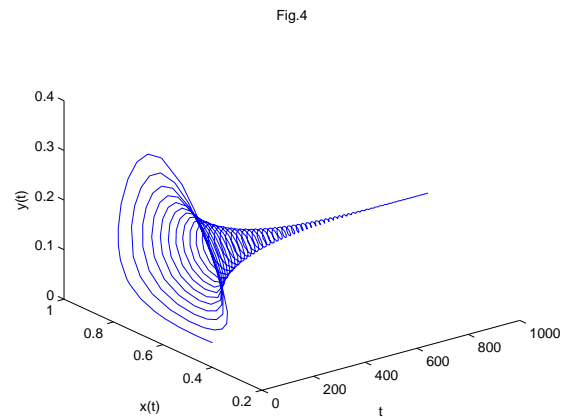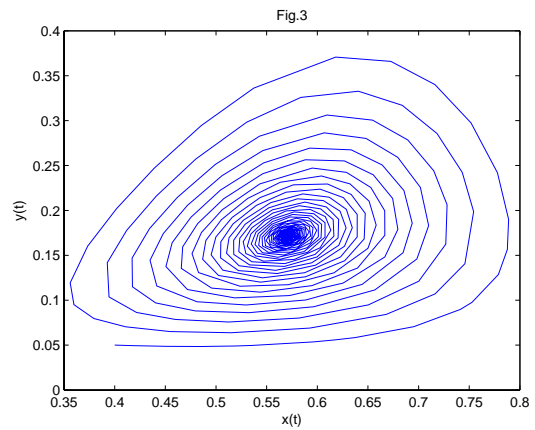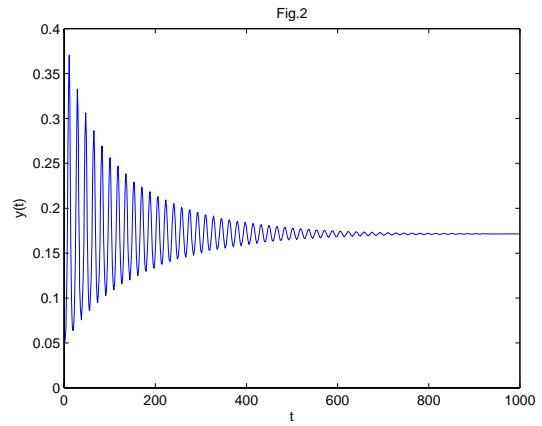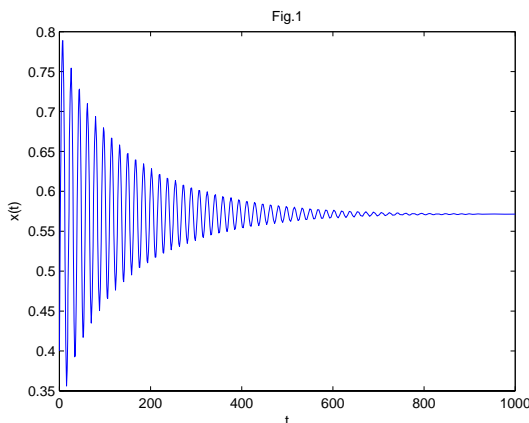
Figs.1-4 Dynamic behavior of system (18): times series of x. A Matlab simulation of the asymptotically stable zero equilibrium to system (18) with $\tau_1 = 0.3$ and $\tau_2 = 2.5$. The initial value is (0.4,0.05).

V. CONCLUSIONS

In this paper, We have investigated the stability of a fractional order delayed predator-prey model. The characteristic equation is introduced for the fractional order delayed predator-prey system by using the Laplace's transformation. A stability conditions for the fractional order delayed predator-prey system are obtained. An illustrative example is in line with the theoretical analysis.

REFERENCES

[1] R.M. May, Time delay versus stability in population models with two and three trophic levels. Ecology 54 (2) (1973) 315-325.

[2] Y.L. Song and J. Wei, Local Hopf bifurcation and global periodic solutions in a delayed predator-prey system. J. Math. Anal. Appl. 301 (1) (2005) 1-21.

[3] X.P. Yan and W.T. Li, Hopf bifurcation and global periodic solutions in a delayed predator-prey system. Appl. Math. Comput. 177 (1) (2006) 427-445.

[4] T. Faria, Stability and bifurcation for a delayed predator-prey model and the effect of diffusion. J. Math. Anal. Appl. 254 (2) (2001) 433-463.

[5] Y. Kuang, Delay Differential Equations With Applications in Population Dynamics. Academic Press, INC, 1993.

[6] Yan, X. P. and C.H. Zhang, Hopf bifurcation in a delayed Lokta-Volterra predator-prey system. Nonlinear Anal.: Real World Appl. 9 (1) (2008) 114-127.

[7] C.J. Xu, X.H. Tang, M.X. Liao and X.F. He, Bifurcation analysis in a delayed Lokta-Volterra predator-prey model with two delays. Nonlinear Dyn. 66 (1-2) (2011) 169-183.

[8] Eva Kaslik and Seenith Sivasundaram, Nonlinear dynamics and chaos in fractional-order neural networks. Neural netw. 32 (2012) 245C256.

[9] K.S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley Interscience, New York, 1993.

[10] I. Podlubny, Fractional Differential Equations. Academic Press, New York, 1999.

[11] P.L. Butzer and U. Westphal, An Introduction to the Fractional Calculus. World Scientific, Singapore, 2000.

[12] O.P. Agrawal, J.A. Tenreiro Machado and J. Sabatier, Introduction, Nonlinear Dyn. 38 (1-2) (2204) 1-2.

[13] S. Kempfle, I. Sch$a$fer and H. Beyer, Fractional calculus: theory and applications. Nonlinear Dyn. 29(1-4) (2002) 99-127.

[14] M.W. Spong, A theorem on neutral delay systems. Systems & Control Letters, 6 (4) (1985)291-294.

[15] Y. He, M. Wu, J.H. She, G.P. Liu, Delay-dependent robust stability criteria for uncertain neutral systems with mixed delays. Systems & Control Letters 51 (1) (2004) 57-65.

[16] C.H. Lien, K.W. Yu, Y.-J. Chung, Y.F. Lin, L.Y. Chung and J.D. Chen, Exponential stability analysis for uncertain switched neutral systems with interval time-varying state delay. Nonlinear Anal.: Hybrid Syst. 3 (3) (2009) 334-342.

[17] L.L. Xiong, S.M. Zhong, M. Ye and S.L. Wu, New stability and stabilization for switched neutral control systems. Chaos, Solitons & Fractals 42 (3) (2009) 1800-1811.

[18] X.G. Liu, M. Wu, Ralph Martin and M.L. Tang, Stability analysis for neutral systems with mixed delays. J. Comput. Appl. Math. 202 (2) (2007) 478-497.

[19] L.L. Xiong, S.M. Zhong and J.K. Tian, Novel robust stability criteria of uncertain neutral systems with discrete and distributed delays. Chaos, Solitons & Fractals 40 (2) (2009) 771-777.

[20] W.J. Xiong and J.L. Liang. Novel stability criteria for neutral systems with multiple time delays. Chaos, solitions & Fractals 32 (5) (2007) 1735-1741.

[21] H. Li, H.B. Li, S.M. Zhong. Some new simple stability criteria of linear neutral systems with a single delay. J. Comput. Appl. Math. 200 (1) (2007) 441-447.

[22] L.L. Xiong, Y. Zhao and T. Jiang, Stability analsysi of linear fractional order neutral system with multiple delays by algebraic approach. World Academy of Science, Engineering and Technology 76 (2011) 983-986.

[23] W.H. Deng, C.P. Li and J.H. L$\ddot{u}$, Stbility analysis of linear fractional differential system with multiple time delays. Nonlinear Dyn. 48 (4) (2007) 409-416.

[24] E.J. Muth, Transform Methods with Applications to Engineering and Operations Research. Prentice-Hall, New Jersey, 1977.

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