An Accurate Computation of Block Hybrid Method for Solving Stiff Ordinary Differential Equations

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Abstract**—**In this paper, self-starting block hybrid method of order $(5,5,5,5)^T$ is proposed for the solution of the special second order ordinary differential equations with associated initial or boundary conditions. The continuous hybrid formulations enable us to differentiate and evaluate at some grids and off – grid points to obtain four discrete schemes, which were used in block form for parallel or sequential solutions of the problems. The computational burden and computer time wastage involved in the usual reduction of second order problem into system of first order equations are avoided by this approach. Furthermore, a stability analysis and efficiency of the block method are tested on stiff ordinary differential equations, and the results obtained compared favorably with the exact solution.

Keywords**–**Block Method, Hybrid, Linear Multistep Method, Self – starting, Special Second Order.

I. Introduction

LET us consider the numerical solution of the special second order ordinary differential equation of the form

$$y'' = f(x,y), \quad a \le x \le b \tag{1}$$

with associated initial or boundary conditions. The mathematical models of most physical phenomena especially in mechanical systems without dissipation leads to special second order initial value problem of type (1). Solutions to initial value problem of type (1) according to Fatunla [1], [2] are often highly oscillatory in nature and thus, severely restrict the mesh size of the conventional linear multistep method. Such system often occurs in mechanical systems without dissipation, satellite tracking, and celestial mechanics.

Lambert [3] and several authors such as Onumanyi *et al* [4], Awoyemi [5], Yahaya and Adegboye [6], and Fudziah *et al.* [7], have written on conventional linear multistep method

$$\sum_{j=0}^{k} \propto_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j}, k \ge 2 \tag{2}$$

or compactly in the form

$$\rho(E) y_n = h^2 \delta(E) f_n \tag{3}$$

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where E is the shift operator specified by $E^j y_n = y_{n+j}$ while ρ and δ are characteristics polynomials and are given as

$$\rho(\xi) = \sum_{j=0}^{k} \propto_j \xi^j, \ \delta(\xi) = \sum_{j=0}^{k} \beta_j \xi^j \tag{4}$$

y_n is the numerical approximation to the theoretical solution $y(x)$ and $f_n = f(x_n, y_n)$.

In the present consideration, our motivations for the study of this approach is a further advancement in efficiency, i.e obtaining the most accuracy per unit of computational effort, that can be secured with the group of methods proposed in this paper over Taparki and Odekunle [8]; and Adeboye [9].

A. Definition : Consistent, Lambert [3]

The linear multistep method (2) is said to be consistent if it has order $p \ge 1$, that is if

$$\sum_{j=0}^{k} \propto_j = 0 \text{ and } \sum_{j=0}^{k} j \propto_j - \sum_{j=0}^{k} \beta_j = 0 \tag{5}$$

Introducing the first and second characteristics polynomials (4), we have from (5) LMM type (2) is consistent if

$$\rho(1) = 0, \ \rho^1(1) = \delta(1)$$

B. Definition: Zero Stability, Lambert [3]

A linear multistep method type (2) is zero stable provided the roots $\xi_j, j = 0(1)k$ of first characteristics polynomial $\rho(\xi)$ specified as $\rho(\xi) = \det\left|\sum_{j=0}^{k} A(i)\xi^{(k-i)}\right| = 0$ satisfies $|\xi_j| \le 1$ and for those roots with $|\xi_j| = 1$ the multiciplicity must not exceed two. The principal root of $\rho(\xi)$ is denoted by $\xi_1 = \xi_2 = 1$.

C. Definition: Convergence, Lambert [3]

The necessary and sufficient conditions for the linear multistep method type (2) is said to be convergent if it is consistent and zero stable.

D. Definition: Order and Error Constant, Lambert [3]

The linear multistep method type (2) is said to be of order p if $c_0 = c_1 = \cdots c_{P+1} = 0$ but $c_{p+2} \ne 0$ and c_{p+2} is called the error constant, where

$$c_0 = \sum_{j=0}^{k} \propto_j = \propto_0 + \propto_1 + \propto_2 + \ \ldots \ + \propto_k$$

$$c_1 = \sum_{j=0}^{k} j \propto_j = (\propto_1 + 2\propto_2 + 3\propto_3 + \ldots + k\propto_k) - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k)$$

$$c_2 = \sum_{j=0}^{k} \frac{1}{2!} j^2 \propto_j - \sum_{j=0}^{k} \beta_j = \begin{Bmatrix} \frac{1}{2!}(\propto_1 + 2^2\propto_2 + 3^3\propto_3 + \ldots + k^2\propto_k) \\ -(\beta_1 + 2\beta_2 + 3\beta_3 + \ldots + k\beta_k) \end{Bmatrix}$$

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$$c_q = \sum_{j=1}^{k} \{\frac{1}{q!} j^q \propto_j - \frac{1}{(q-2)!} j^{q-2}\beta_j\} = \begin{Bmatrix} \frac{1}{q!}(\propto_1 + 2^q\propto_2 + 3^q\propto_3 + \ldots + k^q\propto_k) \\ -\frac{1}{(q-1)!}(\beta_1 + 2^{(q-1)}\beta_2 + 3^{(q-1)}\beta_3 + \cdots + k^{(q-1)}\beta_k) \end{Bmatrix} \quad (6)$$

E. Theorem: Lambert, [3]

Let $f(x, y)$ be defined and continuous for all points (x, y) in the region D defined by $\{(x, y) : a \le x \le b, -\infty < y < \infty\}$ where *a and b* finite, and let there exist a constant L such that for every x, y, y^* such that (x, y) *and* (x, y^*) are both in D:

$$|f(x, y) - f(x, y^*)| \le L|y - y^*| \quad (7)$$

Then if η is any given number, there exist a unique solution $y(x)$ of the initial value problem (1), where $y(x)$ is continuous and differentiable for all (x, y) *in D*. The inequality (7) is known as a Lipschitz condition and the constant L as a Lipschitz constant.

II. Derivation of the Proposed Method

We proposed an approximate solution to (1) in the form

$$y(x) = \sum_{j=0}^{t+m-1} a_j x^i = y_{n+j}, i = 0(1)m + t - 1 \quad (8)$$

$$y''(x) = \sum_{j=0}^{t+m-1} i(i-1)a_j x^{i-2} = f_{n+j}, \quad (9)$$

$i = 2(3)m + t - 1$ with $m = 5$, $t = 2$ and $p = m+t-1$ where the $a_j, j = 0,1,(m+t-1)$ are the parameters to be determined, t and m are points of interpolation and collocation respectively. Where P, is the degree of the polynomial interpolant of our choice.

Specifically, we collocate (9) at $x = x_{n+j}, j = 0(1)k$ and interpolate (8) at $x = x_{n+j}$, $j = 0(1)k - 2$ using the method described above. Putting in the matrix equation form and then solved to obtain the values of parameters α_j's, j = 0, 1, … which is substituted in (8) yields, after some algebraic manipulation, the new continuous form for the solution.

$$y(x) = \sum_{j=0}^{k-2} \propto_j (x) y_{n+j} + \sum_{j=0}^{k} \beta_j(x) f_{n+j} \quad (10)$$

We set $\xi = (x - x_{n+1})$

If we let k = 3, after some algebraic manipulations we obtain a continuous form of solution

$$y(x) = \{-(\xi)\}y_n + \left\{\left(\frac{h+\xi}{h}\right)\right\}y_{n+1} + \left\{\frac{6(\xi)^6 - 30h(\xi)^5 + 45h^2(\xi)^4 - 20h^3(\xi)^3 + 101h^5(\xi)}{1440h^4}\right\}f_n$$
$$+ \left\{\frac{-6(\xi)^6 + 21h(\xi)^5 + 5h^2(\xi)^4 - 70h^3(\xi)^3 + 60h^4(\xi)^2 + 108h^5(\xi)}{120h^4}\right\}f_{n+1}$$
$$+ \left\{\frac{-6(\xi)^6 + 12h(\xi)^5 + 25h^2(\xi)^4 - 20h^3(\xi)^3 + 27h^5(\xi)}{240h^4}\right\}f_{n+2}$$
$$+ \left\{\frac{54(\xi)^6 - 162h(\xi)^5 - 135h^2(\xi)^4 + 540h^3(\xi)^3 - 459h^5(\xi}{800h^4}\right\}f_{n+\frac{4}{3}}$$
$$+ \left\{\frac{6(\xi)^6 - 3h(\xi)^5 - 15h^2(\xi)^4 + 10h^3(\xi)^3 - 16h^5(\xi)}{1800h^4}\right\}f_{n+3}$$
$$+ \left\{\frac{9(\eta)^5 + 10h(\eta)^4 - 10h^2(\eta)^3 + 11h^4(\eta)}{240h^3}\right\}f_{n+2} \quad (11)$$

Evaluating (11) at $x = x_{n+4/3}$, $x = x_{n+2}$ and $x = x_{n+3}$, yield the following schemes:

(a) $y_{n+\frac{4}{3}} - \frac{4}{3}y_{n+1} + \frac{1}{3}y_n = \frac{h^2}{437400}\{10135f_n + 146580f_{n+1} + 15690f_{n+2} - 73953f_{n+4/3} - 1252f_{n+3}\}$

(b) $y_{n+2} - 2y_{n+1} + y_n = \frac{h^2}{1200}\{85f_n + 1180f_{n+1} + 190f_{n+2} - 243f_{n+4/3} - 12f_{n+3}\}$

(c) $y_{n+3} - 3y_{n+1} + 2y_n = \frac{h^2}{1200}\{155f_n + 2640f_{n+1} + 1470f_{n+2} - 729f_{n+\frac{4}{3}} + 64f_{n+3}\}$

(12)

Taking the first derivative of (11), thereafter, evaluate the resulting continuous polynomial solution at

$x = x_0$ yields

(d) $hz_0 - y_{n+1} + y_n = \frac{h^2}{7200}\{-1625f_n - 6060f_{n+1} - 1110f_{n+2} + 5103f_{n+\frac{4}{3}} + 92f_{n+3}\}$

(13)

(12) and (13) constitute the member of a zero stable block integrators of order $(5,5,5,5)^T$ with $c_7 = \left(\frac{2351}{3936600}, \frac{7}{3600}, \frac{1}{600}, -\frac{143}{50400}\right)$. The application of the block integrators with n = 0 gives the accurate values of unknown as shown in tables I and II of forth section of this paper. To start the IVP integration on the sub interval $[X_0, X_3]$, we combine (12) and (13), when n = 0 i.e the 1-block 4-point method as given in (14). Thus produces simultaneously values for y_1, y_2, y_3 and $y_{\frac{4}{3}}$ without recourse to any predictor like Awoyemi [5] and Tafarki [8] to provide y_1 and y_2 in the main method. Hence, this is an improvement over their cited works. Though, this does not becloud the contribution of these authors.

III. Stability Analysis

Recall, that, it is a desirable property for a numerical integrator to produce solution that behave similar to the theoretical solution to a problem at all times. Thus, several definitions, which call for the method to posses some "adequate" region of absolute stability, can be found in several literatures. See Lambert [3], Fatunla [1, 2] etc.

Following Fatunla [1, 2], the four integrator proposed in this report in (12) and (13) are put in the matrix equation form and for easy analysis the result was normalized to obtain;

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+\frac{4}{3}}\\y_{n+2}\\y_{n+3}\end{bmatrix}=\begin{bmatrix}0&0&-\frac{1}{3}&1\\0&0&-1&3\\0&0&-2&6\\0&0&-1&3\end{bmatrix}\begin{bmatrix}y_{n-2}\\y_{n-\frac{5}{3}}\\y_{n-1}\\y_n\end{bmatrix}$$

$$+h^2\left\{\begin{bmatrix}-\frac{913}{5400}&\frac{523}{14580}&-\frac{313}{109350}&-\frac{14}{81}\\-\frac{81}{400}&\frac{19}{120}&-\frac{1}{100}&-1\\-\frac{243}{400}&\frac{49}{40}&\frac{4}{75}&-4\\\frac{567}{800}&-\frac{37}{240}&\frac{23}{1800}&\frac{1}{6}\end{bmatrix}\begin{bmatrix}f_{n+1}\\f_{n+\frac{4}{3}}\\f_{n+2}\\f_{n+3}\end{bmatrix}+\begin{bmatrix}0&0&0&\frac{2027}{87480}\\0&0&0&\frac{17}{240}\\0&0&0&\frac{31}{240}\\0&0&0&-\frac{65}{288}\end{bmatrix}\begin{bmatrix}f_{n-2}\\f_{n-\frac{5}{3}}\\f_{n-1}\\f_n\end{bmatrix}\right\} \quad (14)$$

with $y_0 = \binom{y_0}{hz_0}$ usually giving along the initial value problem. (14) is the 1- block 4 – point method. The first characteristics polynomial of the proposed 1- block 4 – point method is given

$$\text{by } \rho(\lambda) = \det[\lambda I - A_1^{(1)}] \quad (15)$$

$$\rho(\lambda) = \det\begin{bmatrix}\lambda&0&\frac{1}{3}&-1\\0&\lambda&1&-3\\0&0&\lambda+2&-6\\0&0&1&\lambda-3\end{bmatrix} \quad (16)$$

Solving the determinant of (13), yields
$\rho(\lambda) = \lambda^3(\lambda-1)$, which implies,
$\lambda_1 = \lambda_2 = \lambda_3 = 0$ or $\lambda_4 = 1$

By definition of zero stable and (16), the 1 - block 4 - point method is zero stable and is also consistent as its order $(5,5,5,5)^T > 1$, thus, it is convergent following Henrici [10] and Fatunla [2].

IV. Implementation of the Method

This section deal with numerical experiments by considering the derived discrete schemes in block form for solution of stiff differential equations of second order initial value problems. The idea is to enable us see how the proposed methods performs when compared with exact solutions. The results are summarized in Table I & II.

A. Numerical Experiment

From Taparki and Odekunle [8];
Consider the IVP $y'' = y,\ x \in [0,1], y_0 = 1, y_0' = 1, h = 0.1$, whose exact solution is $y = e^x$

TABLE I
RESULTS FOR THE PROPOSED METHOD

x	Exact Solution	Approximate Value	Error of Proposed Method	Taparki and Odekunle [8]
0.1	1.105170918	1.105170918	0.0000E+00	4.920292×10^{-3}
0.2	1.221402758	1.221402758	0.0000E+00	2.037513×10^{-2}
0.3	1.349858808	1.349858807	5.7600E-10	4.7477416×10^{-2}
0.4	1.491824698	1.491824696	1.6413E-09	8.7788535×10^{-2}
0.5	1.648721271	1.648721269	1.7001E-09	1.4235215×10^{-1}
0.6	1.822118800	1.822118798	2.3905E-09	2.1268728×10^{-1}
0.7	2.013752707	2.013752704	3.4705E-09	3.0047789×10^{-1}
0.8	2.225540928	2.225540924	4.4925E-09	$4.07590071 \times 10^{-1}$
0.9	2.459603111	2.459603107	4.1569E-09	5.3609119×10^{-1}
1.0	2.718281828	2.718281824	4.4590E-09	$6.88271136 \times 10^{-1}$

B. Numerical Experiment

From Adeboye [9]; Consider the BVP $y'' - y = 4x - 5; y(0) = y(1) = 0$, h = 0.1, whose exact solution is

$$y = \frac{7}{4(e^2 - e^{-2})}\left[e^{2x} - e^{-2x}\right] - \frac{3}{4}x$$

TABLE II
RESULTS FOR THE PROPOSED METHOD

x	Exact Solution	Approximate Value	Error of Proposed Method	Adeboye [9]
0.0	0.00000000000	0.0000000000	0.000000000E+00	0.000000000E+00
0.1	0.14735784232	0.1473578284	1.390000000E-08	6.598600000E-06
0.2	0.25015214537	0.2501521164	2.890000000E-08	9.454000000E-06
0.3	0.31341504348	0.3134150000	4.340000000E-08	1.156300000E-05
0.4	0.34178302747	0.3417825591	4.680000000E-07	1.204180000E-05
0.5	0.33954334810	0.3395424500	8.981000000E-07	8.902600000E-06
0.6	0.31067692433	0.3106755871	1.337200000E-06	1.922800000E-06
0.7	0.25889818576	0.2588965200	1.665700000E-06	2.803580000E-05
0.8	0.18769224781	0.1876902363	2.011500000E-06	8.259870000E-05
0.9	0.10034979197	0.1003474152	2.376700000E-06	1.870490000E-04
1.0	0.00000000000	-0.0000023895	2.389500000E-06	3.379500000E-06

V. CONCLUSION

In this paper, a new block method with uniform integrators of order $(5,5,5,5)^T$ was developed. The resultant numerical integrators posses the following desirable properties:

i. Zero stability i.e. stability at the origin
ii. Convergent schemes
iii. An addition of equation from the use of first derivative
iv. Being self – starting as such it eliminate the use of predictor – corrector method
v. Facility to generate solutions at 4 points simultaneously
vi. Produce solution over sub intervals that do not overlaps
vii. Apply uniformly to both IVP_s and BVP_s with adjustment to the boundary conditions

In addition, the new schemes compares favourably with the theoretical solution and the results are more accurate than Taparki and Odekunle [8], and Adeboye [9], see table I and II respectively. Hence, this work is an improvement over other cited works.

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