Auction in Impartial Games

Ravi Kant Rai

Abstract. Combinatorial game theory (CGT) is a branch of mathematics and theoretical computer science that typically studies sequential games with perfect information. We prove an importance of a tiebreaking marker related with N-position and P-position in the bidding variant of Impartial combinatorial games.

Keywords: Bidding games; Richman; Combinatorial games. 2020 Mathematics Subject Classification: 68R05, 91A05, 91A46, 91B26

I. INTRODUCTION

Let us consider games like Tic-Tac-Toe, Hex, Chess, etc., where instead of playing alternatively, you bid against your opponent to decide who makes a move. For instance, you play the game of Tic-Tac-Toe in which you, along with your opponent, start with 100 coins each. If you bid 15 and your opponent bids 10, simultaneously; then you win the round and make the first move. You give your 15 coins to your opponent and you are left with 85 coins whereas your opponent has 115 coins. The bid for the second move starts and the game goes on till the game ends following the rule of termination. Bidding versions of Tic-Tac-Toe and Hex have been developed online for recreational play. We study such types of games, also known as **Bidding Games [13]**. Bidding games are combinatorial games with an additional layer of bidding.

Combinatorial games are the games of perfect information. Unlike classical games such as *Prisoners Dilemma*, *Rock-Paper-Scissors*, players play alternately in combinatorial games and there is no random elements. These are the games of perfect information. The absence of randomness implies that there are no dice or chance elements.

Combinatorial Game theory (CGT) deals with combinatorial games. CGT is a branch of mathematics and theoretical computer science that involves sequential games with perfect information.

Games which qualify as Combinatorial Games are NIM, CLOBBER, AMAZONS, DOMINEERING and HEX, etc.

Manuscript received on 07 March 2024 | Revised Manuscript received on 13 March 2024 | Manuscript Accepted on 15 March 2024 | Manuscript published on 30 March 2024.
*****Correspondence Author(s)*

Ravi Kant Rai*, Department of Computer Science, University of Liverpool, United Kingdom. E-mail: r.k.rai@liverpool.ac.uk, ORCID ID: 0000-0001-5956-4714

The game of **Nim** [1] is played by two players, where the two players, Left and Right alternately remove objects from distinct piles. On each move, a player must remove at least one object or many, provided they all come from the same pile. Depending on the version we are playing, the objective of the game is either to remove the last object or to avoid taking the last object. This game follows all the properties of Combinatorial Games.

II. BIDDING GAMES

Bidding games are a variant of the Combinatorial games [14]. The examples are Bidding Tic-Tac-Toe, Bidding Hex, Bidding Chess, etc. In these games, the players are involved in an auction to decide who makes the next move. The idea is to play these games under a protocol where players bid for the right to make the next move. It is thus possible that a player will make multiple moves before the other player makes a single move. One needs to carefully balance the bids for making different moves, or else she could not use her early advantage in the game because of losing too much of her budget at the start of the game.

There is a very rich literature on CGT [11, 1, 3][15] and with the introduction of bidding in these games, we get a way to extend CGT to a more economic style of play. Bidding variants of several well-known games like Chess and Hex have been studied in the past [2, 10][12][16]. Richman auctions [8] were designed for any standard combinatorial 2-player game [1], to resolve who is to play next. Instead of alternating play, for each stage of the game, the 2 players, called Left and Right,[2] resolve this crucial moment by a type of auction where the winning player must pay the losing player their bid amount. Poorman auctions are also defined in [7], where the winning player does not pay the losing player their bid amount. Instead, his winning bid is subtracted from his budget. This bidding competition for making a move can be decided by sequential or simultaneous bidding. Each player starts with a (possibly different) fixed budget. The bidding is done alternatively until one player concedes. When a player concedes, the other player gets the right to make a move and the bid amount is paid to the opponent or subtracted from the winner's budget, based on the setting. In the classical *Richman* setting [8,7], the auction is continuous, say with total budget $1, and the players split that budget to, say p and q, with $p + q = 1$. Optimal bids have been resolved for the game tic-tac-toe [4] for the *Richman* setting. All literature on combinatorial games assumes a unique equilibrium, which defines a game value under *optimal play*; see for example [9, 5, 6, 11]. There is also a *Poorman* setting

[1]These names are adapted from standard literature on combinatorial games, Left is 'she' and Right is 'he'.

[4], where the winning bid amount is paid to the (non-existing) auctioneer.

III. SOME TERMINOLOGIES

Now we explain the basic terminology of CGT in a more formal way. Based on the rule of termination, combinatorial games are played under *Normal* and *Misere* play.

Definition 1 (Normal play)**.** In a Normal play of convention, the last player to make a move in a Combinatorial game wins.

Definition 2. (Misere play) In a Misere play of convention, the last player to make a move in a Combinatorial game loses the game.

For example, let us take an example of a combinatorial game, say Subtraction games. We have a heap of 21 coins, and the two players start removing objects alternatively. They can either remove 1,2 or 3 coins at a time. In normal play, the last player to remove the last object wins. However, in misere play, he loses.

There is another way of playing Combinatorial games, known as *Scoring* play.

Definition 3. In a scoring play convention, the winner is not decided by who moves last, but who scored the most cumulative points during the play.

Dots and Boxes and **Go** are examples of scoring play.

. Based on the moves available to the players, they are categorized as *Impartial* and *Partizan* game.

Definition 4. (Impartial games) In an impartial game, the set of allowable moves available from any position does not depend on which player is moving.

Nim, Tic-Tac-Toe, etc. are examples of Impartial games.

Definition 5. (Partizan games) In Partizan games, each player has a different set of allowable moves from a given position.

Chess, Go, Domineering, etc. are example of Partizan games. A game belongs to one of the following four outcome classes.

(1) Left regardless of who moves first.
(2) Right regardless of who moves first.
(3) the Next player whether it is left or right.
(4) the Previous player whether it is left or right.

We call L, R, N and P as the outcome classes. Games in above clasees are said to be in L, R, N and P positions.

For example, let us take an example of a combinatorial game, say Subtraction games. We have a heap of 21 coins, and the two players start removing objects alternatively. They can either remove 1,2 or 3 coins at a time. In normal play, the last player to remove the last object wins. In the example stated above, we thus see that 0, 4, 8 ⋯ 20 are the positions where the Previous player wins. These are called P-positions. 1, 2, 3, 5, 6, 7 ⋯ are positions where the next player to make a move wins.
These are called N-positions.

IV. MODEL

We extend the discrete bidding model by [4] in combinatorial games to all class of Impartial games.

Let G be a game tree with finite rank and finitely many children for each node. There are two players, Left and Right, and the starting position is the root of G. We define two move conventions for deciding who is to play.

The move convention is either normal play [1] or a bidding variation similar to Richman games [11].

If G is played in the bidding convention then we write $G(p,q)$, and there is a total budget partitioned between the players as (p,q), with $p+q = B$. The player to move is determined\ by\ who\ bids\ more\ and\ if\ bids\ are\ equal\ then\ a tie break rule decides. One of the players holds a marker, and this is denoted by (p,q) (Left has the marker) or (p,q) (Right has the bb marker). If bids are equal then the player with the marker wins the bid and the marker shifts to the other player.

Suppose, without loss of generality, that Left has the marker. There are three cases

1. $(\hat{p},q) \rightarrow (\widehat{p-l},q+l)$; Left wins by bidding more, $l > r$.
2. $(\hat{p},q) \rightarrow (\widehat{p+r},q-r)$; Right wins by bidding more, $l < r$.
3. $(\hat{p},q) \rightarrow (p-l,\widehat{q+l})$; Left wins a tie, $l = r$.

Players do not bid at a terminal position.

V. RESULTS

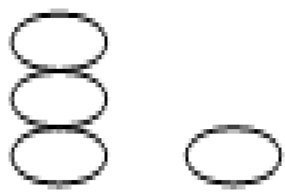
Theorem 6. *Suppose $G \neq \phi$ and let $b \in N_0$. The player with the marker wins bidding $G(b,b)$ if and only if normal play G is an N-position. Let $p,q \in N_0$ and where $p \neq q$. The player with the larger budget wins by bidding $G(p,q)$.*

Proof of Theorem 6: Consider $G(b,b)$ and Left has the marker. Suppose G has a move option to a terminal position. Then G is an N-position in normal play. In the bidding variation, Left bids b and wins the last move.

Suppose next that G is a non-terminal P-position in normal play. The player without the marker (Right) bids 0. If Left bids 0, then Left wins the move and has to move to a normal play N-position, and Right gets the marker. If Left bids $\beta > 0$, then use the second part of the statement, to see that Left will lose.

Suppose next that G is N-position in normal play that does not have a move to a terminal position. Left who has the marker bids 0. If Right bids 0, then he gets the marker, and Left can move to a P-position in normal play. Right tries to bid $\beta > 0$. Then Right will win the bid but Left gets a larger budget, and so wins by the second part (since by assumption each option of G is non-terminal). The result of the theorem 6 is demonstrated in the following example. This is the case when both players have the same budget.

Example 7. Let us see an example of a Nim game with a total budget of 2, partitioned between Left and Right as (1,1). Left has the marker at the start of the game. We have a Nim game with 2 heaps containing 3 and 1 object respectively.



It is already proved in that 0 is a unique optimal bid in the equilibrium.

Retrieval Number: 100.1/ijbsac.H051110080424
DOI: 10.35940/ijbsac.H0511.10070324
Journal Website: www.ijbsac.org

Published By:
Blue Eyes Intelligence Engineering
and Sciences Publication (BEIESP)

- **Case 1**: If Left bids 0 and Right bids 0, Left wins because of the marker and removes two objects from the first heap. Left bids 0 in the next two rounds to force Right to remove the second last object and subsequently remove the last object to win the game.
- **Case 2**: If Left bids 0 and Right bids 1, Right has following four options:

(1) **Right removes one object from the first heap**. Left has the marker and a budget of 2. She bids 0 in the next round and wins the game because of the marker and clears the first heap and then bids 1 (as Right has a budget of 0) to win the round and clears the second heap to win the game.

(2) **Right removes two objects from the first heap**. Left follows the same strategy as above.

(3) **Right clears the first heap**. Left bids 0 to win the next round and remove the object from the second heap to win the game.

(4) **Right clears the second heap**. Left bids 0 to win the next round and remove all three objects from the first heap to win the game.

Now we see the case when one player has a larger budget and we use the result from that 0 is a unique optimal bid in the equilibrium.

Example 8. Let us see an example of a Nim game with a total budget of 1, partitioned between Left and Right as (1,0). Left has the marker at the start of the game. We have the same Nim game with 2 heaps containing 3 and 1 object respectively.

If Left bids 0 and Right bids 0, Left wins by following the same strategy used in **Case 1** of Example 1.

VI. CONCLUSION

We worked on bidding in Impartial games. Here we introduce our tiebreaking rule by defining an available marker with exactly one player at each stage of the game. We proved our result for any impartial game and showed that the marker is, in fact, a property of the N-position and P-position in any Impartial game.

DECLARATION STATEMENT

Funding	No, I did not receive.
Conflicts of Interest	No conflicts of interest to the best of our knowledge.
Ethical Approval and Consent to participate	No, the article does not require ethical approval and consent to participate with evidence.
Availability of data and Material	Not relevant.
Authors contribution	I am only the sole author of the article.

REFERENCES

1. Berlekamp, Elwyn R., John H. Conway, and Richard K. Guy. *Winning ways for your mathematical plays, volume 4*. AK Peters/CRC Press, 2004. https://doi.org/10.1201/9780429487309
2. Bhat J, Payne S. Bidding Chess. The Mathematical Intelligencer. 2009;4(31):37-9. https://doi.org/10.1007/s00283-009-9057-7
3. Conway JH. On numbers and games. AK Peters/CRC Press; 2000 Dec 11. https://doi.org/10.1201/9781439864159
4. Develin M, Payne S. Discrete bidding games. The Electronic Journal of Combinatorics [electronic only]. 2010;17(1): Research-Paper. https://doi.org/10.37236/357
5. Hanner O. Mean play of sums of positional games.
6. Johnson W. The combinatorial game theory of well-tempered scoring games. International Journal of Game Theory. 2014 May;43(2):415-38. https://doi.org/10.1007/s00182-013-0386-6
7. Lazarus AJ, Loeb DE, Propp JG, Stromquist WR, Ullman DH. Combinatorial games under auction play. Games and Economic Behavior. 1999 May 1;27(2):229-64. https://doi.org/10.1006/game.1998.0676
8. Lazarus AJ, Loeb DE, Propp JG, Ullman D. Richman games. Games of no chance. 1996;29:439-49.
9. Milnor J. Sums of positional games. Ann. of Math. Stud. (Contributions to the Theory of Games, HW Kuhn and AW Tucker, eds.), Princeton. 1953 Mar 21;2(28):291-301. https://doi.org/10.1515/9781400881970-017
10. Payne S, Robeva E. Artificial intelligence for Bidding Hex. arXiv e-prints. 2008 Dec:arXiv-0812.
11. Siegel AN. Combinatorial game theory. American Mathematical Soc.; 2013 Aug 1. https://doi.org/10.1090/gsm/146
12. Bashir, S. (2023). Pedagogy of Mathematics. In International Journal of Basic Sciences and Applied Computing (Vol. 10, Issue 2, pp. 1–8). https://doi.org/10.35940/ijbsac.b1159.1010223
13. Kumar, S., & Dwivedi, B. (2019). A Novel Zero-Sum Polymatrix Game Theory Bidding Strategy for Power Supply Market. In International Journal of Recent Technology and Engineering (IJRTE) (Vol. 8, Issue 2, pp. 5669–5675). https://doi.org/10.35940/ijrte.b2889.078219
14. Al-Odhari, A. M. (2023). Algebraizations of Propositional Logic and Monadic Logic. In Indian Journal of Advanced Mathematics (Vol. 3, Issue 1, pp. 12–19). https://doi.org/10.54105/ijam.a1141.043123
15. Narayanan, V., Yegnanarayanan, V., & Srikanth, R. (2019). On Prime Numbers and Related Applications. In International Journal of Innovative Technology and Exploring Engineering (Vol. 8, Issue 12, pp. 4150–4153). https://doi.org/10.35940/ijitee.l3652.1081219
16. Sivaraman, Dr. R., López-Bonilla, Prof. J., & Vidal-Beltrán, S. (2023). On the Polynomial Structure of $rk(n)$. In Indian Journal of Advanced Mathematics (Vol. 3, Issue 2, pp. 4–6). https://doi.org/10.54105/ijam.a1162.103223

AUTHOR PROFILE

Ravi Kant Rai is currently affiliated with the University of Liverpool as a Post Doctoral Research Associate. Previously, he has also held a position at Kyushu University. He was a Postdoc at Kyushu University and was later promoted to Research Assistant Professor. He has his Ph.D. from the Indian Institute of Technology in Mumbai, India. His research interest lies in Game theory, Combinatorial Game theory, Mechanism design, etc. He has served as a Program Committee member of well-esteemed conferences like AAMAS for a couple of years. He is also on a review committee of the journal ``Mathematics and Computer Science".