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Regular Generalized Star Star closed sets in Bitopological Spaces

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Abstract—The aim of this paper is to introduce the concepts of $\boldsymbol{\tau_1\tau_2}$-regular generalized star star closed sets , $\boldsymbol{\tau_1\tau_2}$-regular generalized star star open sets and study their basic properties in bitopological spaces.

Keywords—$\tau_1\tau_2$-regular closed sets; $\tau_1\tau_2$-regular open sets; $\tau_1\tau_2$-regular generalized closed sets; $\tau_1\tau_2$-regular generalized star closed sets; $\tau_1\tau_2$-regular generalized star star closed sets.

I. INTRODUCTION

IN 1963, J.C. Kelly [11] initiated the study of bitopoloical spaces as a natural structure by studying quasi metrics and its conjugate. This structure ia a richer structure than that of a topological space. Considerable effort had been expended in obtaining appropriate generalizations of standard topological properties to bitopological category by various authors. Most of them deal with the theory itself but very few with applications.

K. Chandrasekhara Rao and N. Palaniappan [6] introduced the concepts of regular generalized star closed sets and regular generalized star open sets in a topological space and they are extended to bitopological settings by K. Chandrasekhara Rao and K. Kannan [1]. On the other hand Chandrasekhara Rao and N. Palaniappan [7] introduced the concept of regular generalized star star closed sets and regular generalized star star open sets in topological spaces and study their properties. In this sequel, we introduce the concepts of $\tau_1\tau_2$-regular generalized star star closed sets ($\tau_1\tau_2$-rg^* closed sets) and $\tau_1\tau_2$-regular generalized star star open sets ($\tau_1\tau_2$-rg^* open sets) and study their basic properties in bitopological spaces.

Throughout this paper, (X, τ_1, τ_2) or simply X denote a bitopological space. The intersection (resp. union) of all τ_i-closed sets containing A (resp. τ_i-open sets contained in A) is called the τ_i-closure (resp. τ_i-interior) of A, denoted by τ_i-cl(A) {resp. τ_i-int(A)}. The closure and interior of B relative to A with respect to the topology τ_i are written as τ_i-cl$_A(B)$ and τ_i-int$_A(B)$ respectively.

For any subset $A \subseteq X$, τ_i-rint(A) and τ_i-rcl(A) denote the regular interior and regular closure of a set A with respect to the topology τ_i respectively. The regular closure and regular interior of B relative to A with respect to the topology τ_i are written as τ_i-rcl$_A(B)$ and τ_i-rint$_A(B)$ respectively. The set of all τ_2-regular closed sets in X is denoted by τ_2-R.C(X, τ_1, τ_2). The set of all $\tau_1\tau_2$ -regular open sets in X is denoted by $\tau_1\tau_2$-R.O(X, τ_1, τ_2). A^C denotes the complement of A in X unless explicitly stated.

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We shall require the following known definitions and results:

Definition 1.1: [1] A set A of a bitopological space (X, τ_1, τ_2) is called

(a) $\tau_1\tau_2$-*regular closed* if τ_1-cl[τ_2-int(A)] $=A$.
(b) $\tau_1\tau_2$-*regular open* if τ_1-int[τ_2-cl(A)] $=A$.
(c) $\tau_1\tau_2$-*regular generalized closed* ($\tau_1\tau_2$-rg closed) in X if τ_2-cl(A) $\subseteq U$ whenever $A \subseteq U$ and U is $\tau_1\tau_2$-regular open in X.
(d) $\tau_1\tau_2$-*regular generalized open* ($\tau_1\tau_2$-rg open) in X if $F \subseteq \tau_2$-int(A) whenever $F \subseteq A$ and F is $\tau_1\tau_2$-regular closed in X.
(e) $\tau_1\tau_2$-*regular generalized star closed* ($\tau_1\tau_2$-rg^* closed) in X if and only if τ_2-rcl(A) $\subseteq U$ whenever $A \subseteq U$ and U is $\tau_1\tau_2$-regular open in X.
(f) $\tau_1\tau_2$-*regular generalized star open* ($\tau_1\tau_2$-rg^* open) in X if and only if its complement is $\tau_1\tau_2$-*regular generalized star closed* ($\tau_1\tau_2$-rg^* closed) in X.

Lemma 1.2: [1] Let A be a τ_1-open set in (X, τ_1, τ_2) and let U be $\tau_1\tau_2$-regular open in A. Then $U = A \cap W$ for some $\tau_1\tau_2$-regular open set W in X.

Lemma 1.3: [1] If A is $\tau_1\tau_2$-open and U is $\tau_1\tau_2$-regular open in X then $U \cap A$ is $\tau_1\tau_2$-regular open in A.

II. $\tau_1\tau_2$-REGULAR GENERALIZED STAR STAR CLOSED SETS

Definition 2.1: A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-regular generalized star star closed ($\tau_1\tau_2$-rg^{**} closed) in X if and only if τ_2-$cl[\tau_1$-$int(A)] \subseteq U$ whenever $A \subseteq U$ and U is $\tau_1\tau_2$-regular open in X.

Example 2.2: Let $X = \{a, b, c\}, \tau_1 = \{\phi, X, \{a\}\}$, $\tau_2 = \{\phi, X, \{a\}, \{b, c\}\}$. Then the set of all subsets $P(X)$ of X are $\tau_1\tau_2$-rg^{**} closed sets in (X, τ_1, τ_2).

Theorem 2.3: Let A be a subset of a bitopological space (X, τ_1, τ_2). If A is $\tau_1\tau_2$-rg^{**} closed then τ_2-$cl[\tau_1$-$int(A)] - A$ does not contain non empty $\tau_1\tau_2$-regular closed sets.

Proof: Suppose that A is $\tau_1\tau_2$-rg^{**} closed . Let F be a $\tau_1\tau_2$-regular closed set such that $F \subseteq \tau_2$-$cl[\tau_1$-$int(A)] - A$. Then $F \subseteq \tau_2$-$cl[\tau_1$-$int(A)] \cap A^C$. Since $F \subseteq A^C$, we have $A \subseteq F^C$. Since F is $\tau_1\tau_2$-regular closed set, we have F^C is $\tau_1\tau_2$-regular open . Since A is $\tau_1\tau_2$-rg^{**} closed, we have τ_2-$cl[\tau_1$-$int(A)] \subseteq F^C$. Therefore, $F \subseteq [\tau_2$-$cl[\tau_1$-$int(A)]]^C$. Also $F \subseteq \tau_2$-$cl[\tau_1$-$int(A)]$. Hence $F \subseteq \phi$. Therefore, $F = \phi$. ■

Theorem 2.4: If A is $\tau_1\tau_2$-rg^{**} closed and B is $\tau_1\tau_2$-g closed, then $A \cup B$ is $\tau_1\tau_2$-rg^{**} closed.

Proof: Let $A \cup B \subseteq U$ and U is $\tau_1\tau_2$-regular open in X. Since $A \subseteq U$ and A is $\tau_1\tau_2$-rg^{**} closed, we have

τ_2-$cl[\tau_1$-$int(A)] \subseteq U$. Since $B \subseteq U$ and B is $\tau_1\tau_2$-g closed, we have τ_2-$cl(B) \subseteq U$. Now, τ_2-$cl[\tau_1$-$int(A \cup B)] \subseteq \tau_2$-$cl[\tau_1$-$int[A \cup \tau_2$-$cl(B)]] \subseteq U$. Therefore, $A \cup B$ is $\tau_1\tau_2$-rg^{**} closed. ■

Theorem 2.5: If a subset A is $\tau_1\tau_2$-rg closed then A is $\tau_1\tau_2$-rg^{**} closed.

Proof: Let $A \subseteq U$ and U is $\tau_1\tau_2$-regular open. Since A is $\tau_1\tau_2$-rg closed, we have τ_2-$cl(A) \subseteq U$. Hence τ_2-$cl[\tau_1$-$int(A)] \subseteq U$. Therefore, A is $\tau_1\tau_2$-rg^{**} closed. ■

Definition 2.6: Let $B \subseteq Y \subseteq X$. A subset B of Y is said to be $\tau_1\tau_2$-rg^{**} closed relative to Y if B is $\tau_1\tau_2$-rg^{**} closed in the subspace Y.

Theorem 2.7: Let $(Y, \tau_{1/Y}, \tau_{2/Y})$ be a subspace of (X, τ_1, τ_2). Suppose that a subset B of Y is $\tau_1\tau_2$-rg^{**} closed relative to $(Y, \tau_{1/Y}, \tau_{2/Y})$ and Y is $\tau_1\tau_2$-open and $\tau_1\tau_2$-g closed in (X, τ_1, τ_2) then B is $\tau_1\tau_2$-rg^{**} closed in (X, τ_1, τ_2).

Proof: Let $B \subseteq U$ and U is $\tau_1\tau_2$-regular open in X. Since Y is $\tau_1\tau_2$-open in X, we have $U \cap Y$ is $\tau_1\tau_2$-regular open in Y {by Lemma 1.3} and $B \subseteq U \cap Y$. Since B is $\tau_1\tau_2$-rg^{**} closed relative to $(Y, \tau_{1/Y}, \tau_{2/Y})$, we have τ_2-$cl_Y[\tau_1$-$int_Y(B)] \subseteq U \cap Y$. Hence τ_2-$cl[\tau_1$-$int(B)] \cap Y \subseteq U \cap Y$. Let $G = U \cup \{X - \tau_2$-$cl[\tau_1$-$int(B)]\ \}$. Then G is τ_1-open and $Y \subseteq G$. Since Y is $\tau_1\tau_2$-g closed, we have τ_2-$cl(Y) \subseteq G$. Now, τ_2-$cl[\tau_1$-$int(B] = \tau_2$-$cl(Y) \cap \tau_2$-$cl[\tau_1$-$int(B)] \subseteq G \cap \tau_2$-$cl[\tau_1$-$int(B)] = U \cap \tau_2$-$cl[\tau_1$-$int(B)] \cup \phi \subseteq U$. Therefore, B is $\tau_1\tau_2$-rg^{**} closed in (X, τ_1, τ_2). ■

Theorem 2.8: Suppose that a subset B of Y is $\tau_1\tau_2$-rg^{**} closed in (X, τ_1, τ_2) and Y is $\tau_1\tau_2$-open in (X, τ_1, τ_2) then B is $\tau_1\tau_2$-rg^{**} closed relative to $(Y, \tau_{1/Y}, \tau_{2/Y})$.

Proof: Let $B \subseteq U$ and U is $\tau_1\tau_2$-regular open in Y. Since Y is τ_1-open, we have $U = Y \cap W$ for some $\tau_1\tau_2$-regular open set W in (X, τ_1, τ_2) {by Lemma 1.2}. Since $B \subseteq Y \cap W \subseteq W$ and B is $\tau_1\tau_2$-rg^{**} closed in (X, τ_1, τ_2), we have τ_2-$cl[\tau_1$-$int(B)] \subseteq W$. Therefore, τ_2-$cl_Y[\tau_1$-$int_Y(B)] = \tau_2$-$cl[\tau_1$-$int(B)] \cap Y \subseteq W \cap Y = U$. Hence B is $\tau_1\tau_2$-rg^{**} closed relative to Y. ■

Theorem 2.9: Let A and B are subsets such that $A \subseteq B \subseteq \tau_2$-$cl[\tau_1$-$int(A)]$. If A is $\tau_1\tau_2$-rg^{**} closed, then B is $\tau_1\tau_2$-rg^{**} closed

Proof: Let $B \subseteq U$ and U is $\tau_1\tau_2$-regular open in X . Since $A \subseteq B$, we have $A \subseteq U$. Since A is $\tau_1\tau_2$-rg^{**} closed, we have τ_2-$cl[\tau_1$-$int(A)] \subseteq U$. Since $B \subseteq \tau_2$-$cl[\tau_1$-$int(A)]$, we have τ_2-$cl[\tau_1$-$int(B)] \subseteq \tau_2$-$cl(B) \subseteq \tau_2$-$cl[\tau_1$-$int(A)] \subseteq U$. Therefore, B is $\tau_1\tau_2$-rg^{**} closed. ■

Theorem 2.10: Suppose that $\tau_1\tau_2$-$R.O(X, \tau_1, \tau_2) \subseteq \tau_2$-$C(X, \tau_1, \tau_2)$. Then every subset of X is $\tau_1\tau_2$-rg^{**} closed.

Proof: Let A be a subset of X. Let $A \subseteq U$ and U is $\tau_1\tau_2$-regular open in X. Since $\tau_1\tau_2$-$R.O(X, \tau_1, \tau_2) \subseteq \tau_2$-$C(X, \tau_1, \tau_2)$, we have U is τ_2-closed in X. Since $A \subseteq U$, we have τ_2-$cl(A) \subseteq \tau_2$-$cl(U) = U$. Therefore, τ_2-$cl[\tau_1$-$int(A)] \subseteq \tau_2$-$cl[A] \subseteq U$. Hence A is $\tau_1\tau_2$-rg^{**} closed . ■

III. $\tau_1\tau_2$-Regular generalized star star open sets

Definition 3.1: A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-regular generalized star star open ($\tau_1\tau_2$-rg^{**} open) in X if and only if its complement is $\tau_1\tau_2$-regular generalized star star closed ($\tau_1\tau_2$-rg^{**} closed) in X .

Example 3.2: Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a\}\}$, $\tau_2 = \{\phi, X, \{a\}, \{b, c\}\}$. Then the set of all subsets $P(X)$ are $\tau_1\tau_2$-rg^{**} open sets in (X, τ_1, τ_2).

A necessary and sufficient condition for a set A to be a $\tau_1\tau_2$-rg^{**} open set is obtained in the next theorem.

Theorem 3.3: A subset A of a bitopological space (X, τ_1, τ_2) is $\tau_1\tau_2$-rg^{**} open if and only if $F \subseteq \tau_2$-$int[\tau_1$-$cl(A)]$ whenever $F \subseteq A$ and F is $\tau_1\tau_2$-regular closed in X.

Proof: **Necessity:** Let $F \subseteq A$ and F is $\tau_1\tau_2$-regular closed in X . Then $A^C \subseteq F^C$ and F^C is $\tau_1\tau_2$-regular open in X . Since A is $\tau_1\tau_2$-rg^{**} open, we have A^C is $\tau_1\tau_2$-rg^{**} closed. Hence, τ_2-$cl[\tau_1$-$int(A^C)] \subseteq F^C$. Consequently, $[\tau_2$-$int[\tau_1$-$cl(A)]]^C \subseteq F^C$. Therefore, $F \subseteq \tau_2$-$int[\tau_1$-$cl(A)]$.

Sufficiency: Let $A^C \subseteq U$ and U is $\tau_1\tau_2$-regular open in X . Then $U^C \subseteq A$ and U^C is $\tau_1\tau_2$-regular closed in X. By our assumption, we have $U^C \subseteq \tau_2$-$int[\tau_1$-$cl(A)]$. Hence $[\tau_2$-$int[\tau_1$-$cl(A)]]^C \subseteq U$. Therefore, τ_2-$cl[\tau_1$-$int(A^C)] \subseteq U$. Consequently A^C is $\tau_1\tau_2$-rg^{**} closed. Hence A is $\tau_1\tau_2$-rg^{**} open. ■

Theorem 3.4: Let A and B be subsets such that τ_2-$int[\tau_1$-$cl(A)] \subseteq B \subseteq A$. If A is $\tau_1\tau_2$-rg^{**} open, then B is $\tau_1\tau_2$-rg^{**} open.

Proof: Let $F \subseteq B$ and F is $\tau_1\tau_2$-regular closed in X. Since $B \subseteq A$, we have $F \subseteq A$. Since A is $\tau_1\tau_2$-rg^{**} open, we have, $F \subseteq \tau_2$-$int[\tau_1$-$cl(A)]$ {by Theorem 3.3}. Since τ_2-$int[\tau_1$-$cl(A)] \subseteq B$, we have τ_2-$int[\tau_2$-$int[\tau_1$-$cl(A)]] \subseteq \tau_2$-$int(B) \subseteq \tau_2$-$int[\tau_1$-$cl(B)]$. Hence $F \subseteq \tau_2$-$int[\tau_1$-$cl(A)] \subseteq \tau_2$-$int[\tau_1$-$cl(B)]$. Therefore, B is $\tau_1\tau_2$-rg^{**} open. ■

Theorem 3.5: If a subset A is $\tau_1\tau_2$-rg^{**} closed, then τ_2-$cl[\tau_1$-$int(A)] - A$ is $\tau_1\tau_2$-rg^{**} open .

Proof: Let $F \subseteq \tau_2$-$cl[\tau_1$-$int(A)] - A$ and F is $\tau_1\tau_2$-regular closed. Since A is $\tau_1\tau_2$-rg^{**} closed, we have τ_2-$cl[\tau_1$-$int(A)] - A$ does not contain nonempty $\tau_1\tau_2$-regular closed {by Theorem 2.3}. Therefore, $F = \phi$. Hence τ_2-$cl[\tau_1$-$int(A)] - A$ is $\tau_1\tau_2$-rg^{**} open. ■

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