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Graphs with Metric Dimension Two- A Characterization

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Abstract**—**In this paper, we define distance partition of vertex set of a graph G with reference to a vertex in it and with the help of the same, a graph with metric dimension two (i.e. $\beta(G)=2$) is characterized. In the process, we develop a polynomial time algorithm that verifies if the metric dimension of a given graph G is two. The same algorithm explores all metric bases of graph G whenever $\beta(G)=2$. We also find a bound for cardinality of any distance partite set with reference to a given vertex, when ever $\beta(G)=2$. Also, in a graph G with $\beta(G)=2$, a bound for cardinality of any distance partite set as well as a bound for number of vertices in any sub graph H of G is obtained in terms of diam H .

Keywords**—**Metric basis, Distance partition, Metric dimension.

I. INTRODUCTION

EVERY network can be viewed as a graph in which the vertices represent the processors and an edge between any two vertices indicate the connection between the processors corresponding to the vertices. In Samir Khuller et al [8], navigations are studied in a graph-structured framework in which the navigating agent (the robot) moves from vertex to vertex of a graph space. The robot can locate itself by the presence of the distinct codes assigned for the vertices of the graph. There are several methods to associate a code for a vertex. For example, in [7], Paul F. Tsuchiya assigned the codes by decomposing the network into sub networks. The method of approach in [7] is random and the code associated depends only on the number of sub divisions. However, a mathematical approach for the assignment of distinct codes given by F. Harary et al [4] and further studied by various other authors [2, 6, 8-13], purely depends on the other invariants associated with the network namely, diameter, distance between two vertices etc. The codes generated by the methods given in the above references can easily be implemented to locate any vertex in the graph network. In this paper, we define distance partition of vertex set of a graph G with reference to a vertex in it and with the help of the same, we characterize graphs with metric dimension two (i.e. $\beta(G)=2$). In the process, we develop a polynomial time algorithm that verifies if the metric dimension of a given graph G is two.The same algorithm explores all metric bases of graph G whenever $\beta(G)=2$.

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We also find a bound for cardinality of any distance partite set with reference to a given vertex, whenever $\beta(G)=2$.

Throughout this paper, we write $G(V,E)$ or simply G, to denote a graph on a finite non empty set V of vertices and E of edges. All the graphs considered in this paper are simple, finite, undirected and connected. For all the other basic notations we refer to [1, 3, and 5]. For any two vertices u and v, the distance between u and v denoted by $d(u, v)$, is the length of the shortest path between them. For a given graph G, there are number of properties related to distance between two vertices and are widely studied by various authors.

II. DEFINITIONS

Definition 2.1: Let $G=(V,E)$ be a connected, undirected graph and u,v,w be vertices in V . A vertex v is said to resolve the vertices u and w if the distance of u from v is different from distance of w from v . A set $S\subseteq V$ is a resolving set of $V(G)$ if for each pair of distinct vertices $u,w\in V$ there exists at least one $v\in S$ such that v resolves u and w .

Remark 2.2: Note that V itself is a resolving set of V .

Definition 2.3: A resolving set T with minimum cardinality amongst all resolving sets is known as a metric basis of a graph G . Further, the cardinality of any metric basis is the metric dimension of G and is denoted by $\beta(G)$.

Remark 2.4: Clearly, metric basis exists for every given graph (by definition) and it need not be unique. For example, given a path graph G with pendant vertices u and v , $\{u\}$ and $\{v\}$ are metric bases for G .

Remark 2.5: A metric basis of a graph G is minimal (set theoretic sense) among all resolving sets, but converse need not be true. For example, in case of u and v adjacent vertices in a path graph G and neither among u and v are pendant vertices, then $\{u,v\}$ is a minimal resolving set but not a metric basis.

It is learnt further that a path graph has metric dimension one, a cyclic graph has metric dimension two and a complete graph on n vertices has metric dimension $(n-1)$.

Definition 2.6: Let G be a graph with vertex set $V(G)$ and v be a vertex in it. Then $\{V_0, V_1, V_2 .... V_k\}$ is called a

distance partition of $V(G)$ with reference to the vertex v if $V_0 = \{v\}$ and V_i contains those vertices which are at distance i from v for $0 < i \le k$, where k is the eccentricity of v in G. The sets $V_0, V_1, V_2 .... V_k$ are called distance partite sets.

The result given in the following proposition was observed by Samir Khuller et al [8], and is an important tool in deriving several interesting results of the present paper.

Proposition 2.7 (Samir khuller et al)
In a graph $G(V,E)$, consider any three vertices u, v and w such that $(u,v) \in E$. If $d = d(u,w)$ then $d(v,w)$ is one of $d-1, d$ and $d+1$.

In the following corollaries we consider a graph G with $\beta(G) = 2$, metric basis $\{v_1, v_2\}$ and distance partite sets $V_0, V_1 ... V_k$ with reference to v_1. Proof is immediate from the above proposition and the definition of $\beta(G) = 2$.

Corollary 2.8: *Given any vertex $v \in V_i$ there exist at most three vertices in V_{i+1} adjacent to v, where $0 \le i \le k-1$. Similarly there exist at most three vertices in V_{i-1} adjacent to v when $1 \le i \le k$.*

Corollary 2.9: *Every pair of vertices w_1 and w_2 from different distance partite sets are resolved by at least v_1 and when w_1 and w_2 are from same distance partite set then v_2 resolves them.*

III. PROPERTIES OF DISTANCE PARTITION

In this section we shall discuss some characteristics of the graph due to the properties of distance partition, proofs are straight forward.

Let v be a vertex in $V(G)$ and $\{V_0, V_1, V_2 .... V_k\}$ be the distance partition of $V(G)$ with reference to the vertex v.

Theorem 3.1: *Every vertex in V_j is adjacent to atleast one vertex in V_{j-1} for every j with $2 \le j \le k$ and every vertex in V_1 is adjacent to v.* ♦

Theorem 3.2: *Let G be a graph and $|G| = n$. Then the following are equivalent:*

i) There exists a $v \in V(G)$, such that $|V_i| = 1$ for each of distance partite set V_i of G with reference to the vertex v.

ii) G is a path graph and v is a pendant vertex in it.

iii) There exists $v \in V(G)$ such that $e(v) = n-1$.

In fact, in the above there exist exactly two vertices in $V(G)$ such that with reference to each of them the number of distinct distance partite sets of G is equal to n ♦

Theorem 3.3: *Let G be a graph and k_v be the number of distance partite sets with reference to a vertex $v \in V(G)$. Then $k_v = 2$ for every $v \in V(G)$ if and only if G is a complete graph.* ♦

The following corollary is a result given by Samir Khuller et al [8] and is immediate from Theorem 3.2.

Corollary 3.4: *Metric dimension of a graph G is one if and only if G is a path graph.*

IV. RESULTS

In this section, we establish some results pertaining to structure of a graph G with $\beta(G) = 2$.

Let G be a graph with $\beta(G) = 2$ and $\{v_1, v_2\}$ be a metric basis of G. Further, let $\{V_0, V_1, V_2 .... V_k\}$ be the distance partition of G with reference to the vertex v_1.

The results of the theorem 4.3, 4.4 and 4.6 are due to Samir Khuller et al [8] and we give a simple alternative proof using the concept of distance partition.

Theorem 4.1: *For any vertex $v \in V_j$ there exists a shortest path of length j between v_1 and v. In fact, a shortest path from v_1 to v contains exactly one vertex $w_i \in V_i$ for $1 \le i \le j$. , and the distance $d(w_i, v) = j - i$.*

Theorem 4.2: *If G is a graph with $\beta(G) = 2$ and metric basis $\{v_1, v_2\}$,then there exists a unique shortest path between v_1 and v_2.*

Theorem 4.3: *Let $\{v_1, v_2\}$ be a metric basis of G with $\beta(G) = 2$ then degree of both v_1 and v_2 are less than or equal to three.*

Theorem 4.4: *Let $\{v_1, v_2\}$ be a metric basis of G where $\beta(G) = 2$ .For any vertex v on the unique shortest path between v_1 and v_2, there exists at most one vertex adjacent to it in the distance partite set with respect to v_1 to which it belongs to. Further, v has exactly one vertex adjacent to it in the preceding distance partite set.*

Theorem 4.5: *Let $\{v_1, v_2\}$ be a metric basis of G, where $\beta(G) = 2$.* The *Maximum degree of any vertex v on the unique shortest path between v_1 and v_2 is five. In the case of $\deg(v) = 5$, the adjacency of v is as shown in the following structure (Figure 1).*

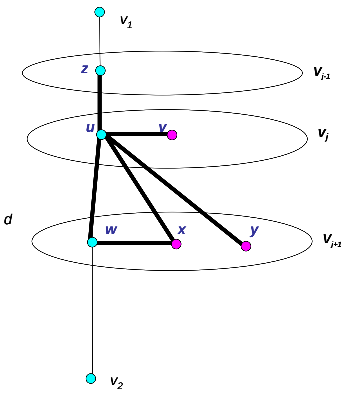

Fig. 1 Vertex u on the unique shortest path between v_1 and v_2 having degree five

Proof: Proof is immediate from Theorem 4.4 and corollary 2.8. ♦

Theorem 4.6: *Let $\{v_1, v_2\}$ be a metric basis of G, where $\beta(G)=2$. Consider distance partite sets $V_0, V_1,...V_k$ with reference to v_1 . Any connected component of the graph induced by a distance partite set is a path and in fact, degree of any vertex in the graph induced by the distance partite set is at most two.*

The corollary 4.7 given below is due to Samir Khuller et al [8] and corollary 4.8 is due to Sooryanarayan B [13] and corollary 4.9 is due to Sooryanarayan B, Murali, K.S.Harinath [14] are immediate consequences of the Theorem 4.6.

Corollary 4.7 *A graph G with $\beta(G)=2$ cannot have K_5.*

Corollary 4.8: *Let $\{v_1, v_2\}$ be a metric basis of G with $\beta(G)=2$. Then for a triangle T in G , if any, all the vertices of T cannot be at the same distance from v_1 or v_2.*

Proof: Proof is immediate from Theorem 4.6. ♦

Corollary 4.9: *For any graph G with $\beta(G)=2$, the metric basis of G cannot have a vertex v of a sub graph K_4 of G.*

Proof: Let $\{v_1, v_2\}$ be a metric basis of G and $v_1 \in V(K_4)$. Consider the distance partition of $V(G)$ with reference to v_1. Then the distance partite set V_1 has the other three vertices of K_4 which induce a cycle, a contradiction to Theorem 4.6.

♦

Definition : The shortest path from vertex v_1 to a vertex u_j of V_j is said to be downward extendable if there exists vertex u_{j+1} in V_{j+1} such that u_{j+1} is adjacent to u_j.

Note that the path $v_1 \to .... \to u_j \to u_{j+1}$ is a shortest path from v_1 to u_{j+1} . Any path $v_1 \to .... \to u_j \to u_{j+1} \to .... \to u_{j+t}$ is said to be a downward extension of a path $v_1 \to .... \to u_j$. In the following theorem, we shall observe that a maximal downward extension of the unique shortest path $v_1 \to .... \to v_2$ is unique.

Theorem 4.10: *Let $\{v_1, v_2\}$ be a metric basis of G where $\beta(G)=2$. The maximal downward extension of the unique shortest path $v_1 \to v_2$ ($v_2 \in V_j$) is unique and has at most one vertex u_{j+t} from V_{j+t} where $0 \le t \le k-j$.*

Theorem 4.11: *The maximum degree of any vertex in a graph G with $\beta(G)=2$ is eight and it is realizable.*

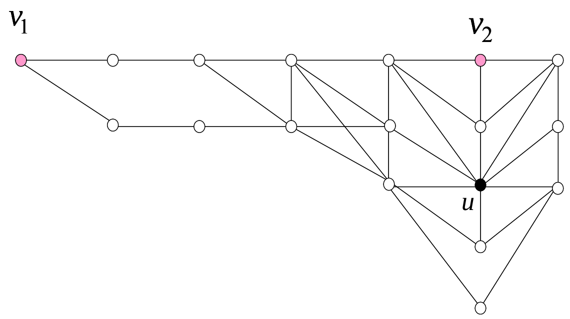

Fig. 2 vertex u with maximum degree eight

Remark 4.12: The above theorem gives an upper bound for degree of any vertex in a graph G with $\beta(G)=2$.

Theorem 4.13: *Let $\{v_1, v_2\}$ be a metric basis of G, where $\beta(G)=2$. Then G cannot have $K_5 - e$ as a sub graph.*

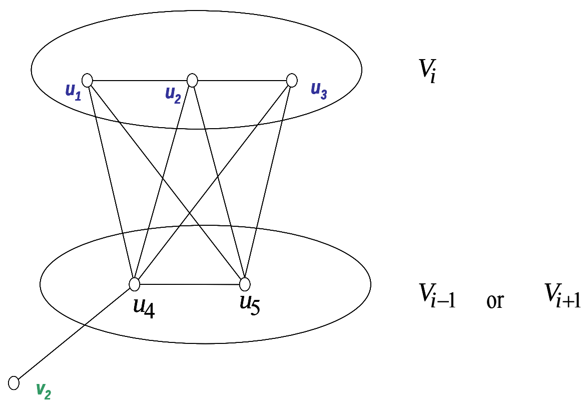

Fig. 3. A Graph G cannot have $K_5 - e$ as a sub *graph.*

Remark 4.14: From theorem 4.13 it is clear that neither K_5 nor $K_5 - \{e\}$ can be a sub graph of a graph with metric dimension two. so it is of natural curiosity that how further smaller sub graph of K_5 can be excluded from being a sub graphs of a graph from the class of graphs with metric dimension two in the following we realize that $K_5 - 2e$ could be a sub graph of some graph G with $\beta(G) = 2$.

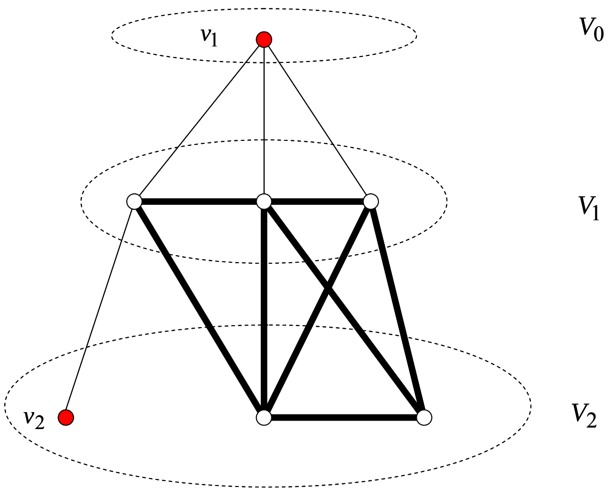

Fig. 4. A Graph G can have $K_5 - 2e$ as a sub graph.

Similar to the above, we can prove the following theorem.

Theorem 4.15: *If G is a graph with $\beta(G) = 2$ then G cannot have $K_{3,3}$ as a sub graph .* ♦

Theorem 4.16: *Let $\{v_1, v_2\}$ be a metric basis of G, where $\beta(G) = 2$. Let $e(v_1) = k$ and $|V(G)| = n$. Then eccentricity of the second resolving vertex v_2 is greater than or equal to $\left[\frac{n-4}{k-1}\right]$, $k \neq 1$, where $[x]$ is the integer part of the number x.*

Remark 4.17*:* In a graph G with $\beta(G) = 2$ and metric basis $\{v_1, v_2\}$, if $e(v_2) = s$ then any distance partite set V with respect to the vertex v_1 other than the one with v_2 in it contains at most S vertices (to be precise, not more than $s + 1 - d(V, v_2)$, where $d(V, v_2)$ is shortest of distances between any vertex v of V and v_2, distance partite set that contains v_2 may have $s+1$ vertices.

Proof: Proof is immediate from Corollary 2.9 and Theorem 4.6. ♦

Though Theorem 4.21 and corollary 4.22 provide much stronger result, for general understanding of structure of G with $\beta(G) = 2$, we give following results.

Theorem 4.18: *Let G be a graph with $\beta(G) = 2$ and $\{v_1, v_2\}$ be a metric basis of G .Let P be the Petersen graph. Then neither of v_1 and v_2 are in $V(P)$. Further, if eccentricity of any of v_1 and v_2 is not more than three, then P cannot be a sub graph of G.*

Proof: Consider distance partite sets $\{V_0, V_1, V_2, ...V_k\}$ with reference to v_1. If $v_1 \in V(P)$, then V_2 consists of at least six vertices of $V(P)$ which induces a cycle in V_2 so a contradiction. Hence $v_1 \notin V(P)$. Similarly $v_2 \notin V(P)$.

Suppose that P is a sub graph of G and $e(v_2) = 3$. Now consider distance partite sets with reference to v_1. From the remark 4.17 at most one V_j which contains v_2 may have four vertices and the remaining V_i's have no more than three vertices. As $v_1 \notin V(P)$ and diameter of $P = 2$, $V(P)$ is distributed among three V_j 's such that one having four vertices of $V(P)$ and other two having three each . This implies $v_2 \notin V(P)$, a contradiction. ♦

Theorem 4.19: *Let G be a graph with $\beta(G) = 2$ then there is no connected sub graph H of G such that diameter of $H < \sqrt{m-1}$, where m is cardinality of $V(H)$.*

Proof: Consider a metric basis $\{v_1, v_2\}$ of G , where $\beta(G) = 2$, and distance partition $\{V_0, V_1, ...V_k\}$ of $V(G)$ with reference to one among the basis elements, say v_1. Let H be any connected sub graph of G with diameter of H equal to D. Any pairs of vertices, among vertices of H and in the same partite set, (say V_j), are resolved by v_2 . Since distance between any pair of vertices from $\{v_h | v_h \in V(H) \cap V_j\}$ is not more than diameter D of H , $d(v_2, v_h)$ takes distinct values among $d, d+1, ...d+D$, where $d = \min_{v \in H \cap V_j} \{d(v, v_2)\}$. So, the cardinality of $H \cap V_j$ is at most $D+1$. Further , as diameter $H = D$, the vertices of H could be distributed among at most $D+1$ consecutive V_i's. Hence the cardinality of H is at most $(D+1)(D+1)$. That is $m \leq (D+1)^2$, where m is cardinality of $V(H)$. Therefore $\sqrt{m-1} \leq D$. This proves the result. ♦

Corollary 4.20: *The complete graph K_5 or the Petersen graph P cannot be a sub graph of a graph G with $\beta(G) = 2$.*

Proof: Proof is immediate from the Theorem 4.19 and K_5 is of diam 1 with order 5, and Peterson graph is of diam 2 with order 10. ♦

Lemma 4.21: *Let G be a graph with $\beta(G) = 2$ and $\{v_1, v_2\}$ be a metric basis of G. Further, let $\{V_0, V_1 ....V_k\}$ be the* *distance partition of $V(G)$ with reference to the vertex v_1. Then every distance partite set can have at most two vertices more than the maximum possible cardinality of preceding distance partite set.*

Theorem 4.22: *Let G be a graph with $\beta(G)=2$ and $\{v_1,v_2\}$ be a metric basis of G. Further, let $\{V_0,V_1,......V_k\}$ be the distance partition of $V(G)$ with reference to one of the vertices in the metric basis. Then maximum number of vertices in any distance partite set, say V_i, for $0\le i\le k$ is $(2i+1)$.*

V. BOUND FOR NUMBER OF VERTICES IN A GRAPH G WITH $\beta(G)=2$

In the following a sharper bound for number of vertices in a graph G with $\beta(G)=2$ is given .

Let G be a graph with $\beta(G)=2$ and $\{v_1,v_2\}$ be a metric basis of G. Further, let $\{V_0,V_1,......V_k\}$ be the distance partition of $V(G)$ with reference to v_1 and $v_2\in V_i$. Let $e(v_2)=s$. V_i can have atmost $(s+1)$ vertices only if $(s+1)\le 2i+1$. Thus to find maximum number of vertices that G with $\beta(G)=2$ can have, we consider the following two cases.

Case (i): $(s+1)\le 2i+1$, i.e. $s\le 2i$.

Then V_i can have $(s+1)$ vertices. Let t be the smallest integer such that $s-t+1\le 2(i-t)+1$ and $s-t>2(i-t-1)+1$ i.e. $s\le 2i-t$ and $s>2i-t-1$.

Then the distance partite sets $V_i,V_{i-1},....V_{i-t}$ can have atmost $s+1,s,.......s-t+1$ vertices respectively. The distance partite sets $\{V_0,V_1,.......V_{i-t-1}\}$, can have atmost $1,3,5,\ldots.. 2(i-t-1)+1=2(i-t)-1$. vertices respectively and finally the distance partite sets $V_{i+1},V_{i+2},......V_k$ can have atmost $s,s-1,......s-(k-i-1)$ vertices respectively.

Thus the maximum number of vertices in a graph G with $\beta(G)=2$ is given by

$$\sum_{m=1}^{i-t}(2m-1)+\sum_{n=-1}^{t-1}(s-n)+\sum_{r=0}^{k-i-1}(s-r)$$

$$=\left[(i-t)^2\right]+\left[(t+1)s-\frac{(t-1)t}{2}+1\right]+(k-i)s-\frac{(k-i-1)(k-i)}{2}$$

Case(ii): $(s+1)>2i+1$, **i.e.,** $s>2i$.

Then obviously, $s>2i-1,2i-3,........ 3,1$

Hence the distance partite sets $V_0,V_1,V_2,.........V_i$ can have at most $1,3,.......2i+1$ vertices respectively.

Let t be the minimum positive integer such that $s>2i+3t$ and $s\le 2i+3t+3$. Then distance partite sets $V_{i+1},V_{i+2},..........V_{i+t}$ can have atmost $2(i+1)+1,2(i+2)+1,........2(i+t)+1$ vertices respectively and the distance partite sets $V_{i+t+1},............V_k$ can have at most $s-t,s-t-1,............s-(k-i-1)$ vertices respectively. Thus the maximum number of vertices in G with $\beta(G)=2$ is

$$\sum_{m=0}^{i+t}(2m+1)+\sum_{n=t}^{k-i-1}(s-m)$$

$$=(i+t+1)^2+(k-i-t)s-\left(\frac{(k-1)k}{2}\right)+\frac{t(t-1)}{2}.$$

VI. CHARACTERIZATION OF GRAPHS WITH METRIC DIMENSION TWO

The following is the characterization of graphs with metric dimension two.

Theorem 6.1: *Let G be a graph which is not a path with $V(G)=\{v_1,v_2..........v_n\}$ and $\{V_{i0},V_{i1}..........V_{ik_i}\}$ be the distance partition of $V(G)$ with reference to the vertex v_i where k_i is the eccentricity of v_i, $1\le i\le n$. The metric dimension of G is 2 if and only if there exist vertices v_i and v_j such that $\left|V_{i_k}\cap V_{j_l}\right|\le 1$ for every k and l with $1\le k\le e(v_i)$ and $1\le l\le e(v_j)$.*

Proof: Given v_p and v_r, $\left|V_{p_q}\cap V_{r_s}\right|>1$ for some p_q and r_s implies that there exists atleast two vertices, say u_1 and u_2 in $V_{p_q}\cap V_{r_s}$ such that $d(v_p,u_1)=d(v_p,u_2)=q$ and $d(v_q,u_1)=d(v_q,u_2)=s$ and hence u_1 and u_2 are not resolved by both v_p and v_r so , $\left|V_{p_q}\cap V_{r_s}\right|>1$ for all p_q and r_s implies no pair of vertices v_p and v_r resolves $V(G)$, in other words $\beta(G)\ne 2$.

Conversely if there exist v_p and v_r such that $\left|V_{p_q}\cap V_{r_s}\right|\le 1$ for all p_q and r_s, then given any pair of vertices w_1 and w_2 from $V(G)$ we have $w_1\in V_{p_{q_1}}\cap V_{r_{s_1}}$ and $w_2\in V_{p_{q_2}}\cap V_{r_{s_2}}$ where at least p_{q_1} is different from p_{q_2} or r_{s_1} is different from r_{s_2}. This implies that w_1 and w_2 are resolved by at least one of v_p and v_r. So $\beta(G)\le 2$ and in fact, $\beta(G)=2$ as G is not a path. ♦

VII. ALGORITHM TO CHECK WHETHER THE METRIC DIMENSION OF A GIVEN GRAPH G IS TWO

The following algorithm follows from the Theorem 6.1.

Step 1: Input: distance matrix

Input is the distance matrix ordered according to vertices of a graph G which is not a path.

Step 2 : Check if $\beta(G)\ne 2$ from number of elements in $V(G)$

If number of vertices in G i.e., $|V(G)| > (D-1)^2 + 8$ where D is the diameter of the graph G then $\beta(G) \neq 2$ (Samir khuller et al) .

Step 3: Selection of vertices for finding metric basis

Select only those vertices in G with degree less than or equal to three (Samir khuller et al) .

Step 4: Formation of distance partitions

Form distance partition $\{V_{i_0}, V_{i_1}, ...V_{ik_i}\}$ of $V(G)$ with reference to every vertex v_i having degree less than or equal to three $1 \le i \le n$.

Step 5: Identify the pair of vertices for finding metric basis

i) Given a pair (u_1, u_2) if eccentricity of a vertex u_2 is less than number of vertices at distance $d(u_1, u_2)$ and vice versa then $\{u_1, u_2\}$ cannot be a metric basis for G . Consider only a remaining pairs.

ii) Among the pairs (u_i, u_j) remaining, consider only the pairs with unique shortest path between them.

Step 6: Find intersection

If there exists vertices v_i and v_j $(i \neq j)$ with $\left|V_{i_k} \cap V_{j_l}\right| \le 1$ for every k and l with $1 \le k \le e(v_i)$ *and* $1 \le l \le e(v_j)$, then $\{v_i, v_j\}$ is a metric basis for the graph G . Otherwise the metric dimension of G is not equal to two.

VIII.COMPLEXITY

Let G be a graph with diameter D on n vertices. Every set in distance partition of $V(G)$ with reference to a vertex v is to be compared with at most $(n-1)D$ sets. Therefore totally there are $((n-1)D)D$ comparisons for v . For the next vertex the number of comparisons needed is $((n-2)D^2)$. Similarly for the last vertex the number of comparisons needed is $(n-(n-1))D^2$. Therefore the total number of set comparisons required is

$$((n-1)D^2 + (n-2)D^2 + ...+1.D^2) = D^2\left[\frac{n(n-1)}{2}\right].$$

In every comparison of two sets there can be at most D^2 comparisons of elements. Hence total number of element comparisons is $D^2\left[\frac{n(n-1)}{2}\right]D^2$.

Thus the complexity of the algorithm is $=\left[\frac{n(n-1)}{2}\right]D^4$.

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