

Extended Cubic B-spline Interpolation Method Applied to Linear Two-Point Boundary Value Problems

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Abstract—Linear two-point boundary value problem of order two is solved using extended cubic B-spline interpolation method. There is one free parameters, λ , that control the tension of the solution curve. For some λ , this method produced better results than cubic B-spline interpolation method.

Keywords—two-point boundary value problem, B-spline, extended cubic B-spline.

I. INTRODUCTION

CONSIDER the general form of linear two-point boundary value problem

$$u''(x) + p(x)u'(x) + q(x)u(x) = r(x), \quad (1)$$

$$x \in [a, b], \quad u(a) = \alpha, \quad u(b) = \beta.$$

This problem has a unique solution, $u(x)$, if $p, q, r \in C^1$ and $q(x) < 0$ [1]. Generally, this problem is difficult to solve analytically. Some of the most frequently used numerical methods are shooting, finite difference, finite element and finite volume methods [1], [2]. These methods, although requiring little computational time, evaluate the approximated solutions only at the collocation points, $u(x_i)$ for $i = 0, 1, \dots, n$.

A different approach of solving linear two-point boundary value problem has first been suggested by Bickley in 1968 [3]. He used cubic spline interpolation to model the solution curve and applied the differential equation as well as the boundary conditions to solve for the unknown constants. As a result, a set of equations could be produced approximating the analytical solution. Further work on this approach can be found in [4], [5]. Thirty years later, Caglar et al. proposed the use of cubic B-spline interpolation to solve this problem. The basis function of B-spline is constructed using piecewise polynomial function that satisfies C^2 continuity. The definition and properties of the function as well as their approach can be found in [6] and the references therein. Continuing with this work, we applied the same procedure using extended cubic B-spline interpolation to solve the problem.

Extended B-spline is a generalization of B-spline. One free parameter, λ , is introduced within the basis function that can be used to change the shape of the produced curve. The value

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of λ is varied systematically and the results were analyzed. The value of λ producing the least error is identified. One example is provided at the end.

II. EXTENDED CUBIC B-SPLINE BASIS FUNCTION

For a finite interval $[a, b]$, let $\{x_i\}_{i=0}^n$ be a partition of the interval with uniform step size, h . We can extend the partition using

$$h = \frac{b-a}{n}, \quad x_0 = a, \quad x_i = x_0 + ih, \quad i = \pm 1, \pm 2, \pm 3, \dots$$

Extended cubic B-spline basis function is constructed by linear combination of the cubic B-spline basis function [7]. Here, blending function of degree 4, $EB_{3,i}(x)$, is considered and the resulting function is shown in (2).

$$\frac{1}{24h^4} \begin{cases} b_i(x), & x \in [x_i, x_{i+1}], \\ b_{i+1}(x), & x \in [x_{i+1}, x_{i+2}], \\ b_{i+2}(x), & x \in [x_{i+2}, x_{i+3}], \\ b_{i+3}(x), & x \in [x_{i+3}, x_{i+4}], \end{cases} \quad (2)$$

$$b_i(x) = -4h(\lambda - 1)(x - x_i)^3 + 3\lambda(x - x_i)^4,$$

$$b_{i+1}(x) = (4 - \lambda)h^4 + 12h^3(x - x_{i+1}) + 6h^2(2 + \lambda)(x - x_{i+1})^2 - 12h(x - x_{i+1})^3 - 3\lambda(x - x_{i+1})^4,$$

$$b_{i+2}(x) = (16 + 2\lambda)h^4 - 12h^2(2 + \lambda)(x - x_{i+2})^2 + 12h(1 + \lambda)(x - x_{i+2})^3 - 3\lambda(x - x_{i+2})^4,$$

$$b_{i+3}(x) = -(h + x_{i+3} - x)^3 [h(\lambda - 4) + 3\lambda(x - x_{i+3})].$$

Extended cubic B-spline basis will degenerate into cubic B-spline basis when $\lambda = 0$. For $\lambda \in [-8, 1]$, B-spline and extended B-spline share the same properties: local support, non-negativity, partition of unity and C^2 continuity.

III. EXTENDED CUBIC B-SPLINE INTERPOLATION

Given $\{x_i\}$, the extended cubic B-spline function, $S(x)$ is a linear combination of the extended cubic B-spline basis function,

$$S(x) = \sum_{i=-3}^{n-1} C_i EB_{3,i}(x), \quad x \in [x_0, x_n], \quad (3)$$

where C_i are unknown real coefficients. Since $EB_{3,i}(x_i)$ has a support on $[x_i, x_{i+4}]$, there are three nonzero basis

function evaluated at each $x_i : EB_{3,i-3}(x_i), EB_{3,i-2}(x_i)$ and $EB_{3,i-1}(x_i)$. Thus, from (3), for $i = 0, 1, \dots, n$,

$$\begin{aligned} S(x_i) &= C_{i-3}E_{4,i-3}(x) + C_{i-2}E_{4,i-2}(x) + C_{i-1}E_{4,i-1}(x), \\ &= C_{i-3} \left(\frac{4-\lambda}{24}\right) + C_{i-2} \left(\frac{8+\lambda}{12}\right) + C_{i-1} \left(\frac{4-\lambda}{24}\right), \end{aligned} \quad (4)$$

$$\begin{aligned} S'(x_i) &= C_{i-3}E'_{4,i-3}(x) + C_{i-2}E'_{4,i-2}(x) + C_{i-1}E'_{4,i-1}(x), \\ &= C_{i-3} \left(-\frac{1}{2h}\right) + C_{i-2}(0) + C_{i-1} \left(\frac{1}{2h}\right), \end{aligned} \quad (5)$$

$$\begin{aligned} S''(x_i) &= C_{i-3}E''_{4,i-3}(x) + C_{i-2}E''_{4,i-2}(x) + C_{i-1}E''_{4,i-1}(x), \\ &= C_{i-3} \left(\frac{2+\lambda}{2h^2}\right) + C_{i-2} \left(-\frac{2+\lambda}{h^2}\right) + C_{i-1} \left(\frac{2+\lambda}{2h^2}\right). \end{aligned} \quad (6)$$

Returning to the two-point boundary value problem stated in (1), $S(x)$ is presumed to be the approximation of its solution, $u(x)$. Substituting $S(x)$ into (1), the equation becomes

$$\begin{aligned} u''(x) + p(x)u'(x) + q(x)u(x) &= r(x), \\ x \in [a, b], \quad u(a) &= \alpha, \quad u(b) = \beta. \end{aligned} \quad (7)$$

Substituting (4), (5) and (6) into (7) would result in a system of linear equations of order $(n + 3) \times (n + 3)$. The C_i 's are solved from the system and are substituted in (3). The resulting equation becomes the approximated analytical solution for (1).

IV. VARYING λ

The value of λ is varied systematically in the neighborhood of zero using brute force with suitable step size. At each trial, Max-norm and L^2 -norm for the solution are calculated. The values of λ with the lowest norms are identified. Suppose that the true and approximated solution of (1) are $u(x)$ and $S(x)$, respectively. The norms are calculated using the following equations:

$$\text{Max-norm} = \max_{i=0}^n |S(x_i) - u(x_i)|,$$

$$L^2\text{-norm} = \sum_{i=0}^n [S(x_i) - u(x_i)]^2.$$

V. NUMERICAL EXAMPLE AND CONCLUSION

Problem 5.1 [6]

$$u''(x) - u'(x) = -e^{x-1} - 1, \quad x \in [0, 1], \quad u(0) = u(1) = 0.$$

Exact solution: $u(x) = x(1 - e^{x-1})$.

Problem 5.1 was solved using extended cubic B-spline interpolation method. The numerical results are shown in Table I. The first row is the norms when $\lambda = 0$, that is, for cubic B-spline interpolation method. Using $\lambda = 2.9762 \times 10^{-3}$, the approximated analytical solution is given in (8). The plots of $S(x)$ and $u(x)$ along with the error are presented in Figure 1.

TABLE I
 THE BEST VALUES OF λ FOR EXAMPLE 5.1

λ	Max-Norm	L^2 -Norm
0	2.8996×10^{-4}	6.6089×10^{-4}
2.9762×10^{-3}	3.1415×10^{-6}	7.2625×10^{-6}
2.9776×10^{-3}	3.2452×10^{-6}	7.2555×10^{-6}

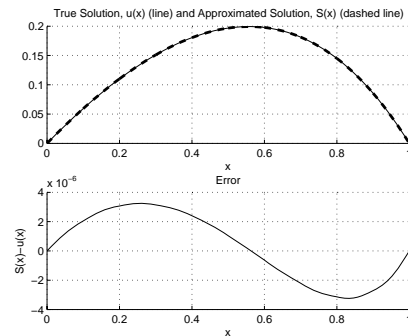


Fig. 1. Comparison between the exact and approximated solutions

$$S(x) =$$

$$\left\{ \begin{aligned} &3.683 \times 10^{-16} + 0.6321x - 0.3679x^2 - 0.1849x^3 - 0.05905x^4, & x \in [0.0, 0.1], \\ &1.380 \times 10^{-6} + 0.6321x - 0.3677x^2 - 0.1835x^3 - 0.06844x^4, & x \in [0.1, 0.2], \\ &2.621 \times 10^{-8} + 0.6322x - 0.3691x^2 - 0.1769x^3 - 0.07915x^4, & x \in [0.2, 0.3], \\ &-5.759 \times 10^{-5} + 0.6331x - 0.3743x^2 - 0.1638x^3 - 0.09136x^4, & x \in [0.3, 0.4], \\ &-3.515 \times 10^{-4} + 0.6362x - 0.3865x^2 - 0.1425x^3 - 0.1053x^4, & x \in [0.4, 0.5], \\ &-0.001306 + 0.6439x - 0.4098x^2 - 0.1112x^3 - 0.1211x^4, & x \in [0.5, 0.6], \\ &-0.003760 + 0.6601x - 0.4497x^2 - 0.06744x^3 - 0.1390x^4, & x \in [0.6, 0.7], \\ &-0.009215 + 0.6905x - 0.5131x^2 - 0.008663x^3 - 0.1595x^4, & x \in [0.7, 0.8], \\ &-0.02017 + 0.7434x - 0.6089x^2 + 0.06834x^3 - 0.1826x^4, & x \in [0.8, 0.9], \\ &-0.04058 + 0.8306x - 0.7484x^2 + 0.1673x^3 - 0.2089x^4, & x \in [0.9, 1.0]. \end{aligned} \right. \quad (8)$$

These results show that extended cubic B-spline has potential to approximate the solution of two-point boundary value problems better than B-spline. Here, we used the exact solution of the problem as a reference to find good values of λ . Therefore, future work will focus on finding the values of λ that produce better approximation from the differential equation in (1) itself without using the exact solution. This study confirmed that for some problems, these values do exist.

REFERENCES

- [1] R. L. Burden and J. D. Faires, *Numerical Analysis*, 8th ed. Brooks/Cole, Cengage Learning, 2005, pp. 641-647, 649-654, 656-660, 662-667, 668-682, 684-685.

- [2] Q. Fang, T. Tsuchiya, and T. Yamamoto, "Finite difference, finite element and finite volume methods applied to two-point boundary value problems," *Journal of Computational and Applied Mathematics*, vol. 139, no. 1, pp. 9 – 19, 2002. [Online]. Available: <http://www.sciencedirect.com/science/article/B6TYH-44FD3DG-2/2/c59d6272888f882a64514091fff104dc>
- [3] W. G. Bickley, "Piecewise cubic interpolation and two-point boundary problems," *The Computer Journal*, vol. 11, no. 2, pp. 206–208, 1968. [Online]. Available: <http://comjnl.oxfordjournals.org/cgi/content/abstract/11/2/206>
- [4] E. L. Albasiny and W. D. Hoskins, "Cubic spline solutions to two-point boundary value problems," *The Computer Journal*, vol. 12, no. 2, pp. 151–153, 1969. [Online]. Available: <http://comjnl.oxfordjournals.org/cgi/content/abstract/12/2/151>
- [5] D. J. Fyfe, "The use of cubic splines in the solution of two-point boundary value problems," *The Computer Journal*, vol. 12, no. 2, pp. 188–192, 1969. [Online]. Available: <http://comjnl.oxfordjournals.org/cgi/content/abstract/12/2/188>
- [6] H. Caglar, N. Caglar, and K. Elfaituri, "B-spline interpolation compared with finite difference, finite element and finite volume methods which applied to two-point boundary value problems," *Applied Mathematics and Computation*, vol. 175, no. 1, pp. 72 – 79, 2006. [Online]. Available: <http://www.sciencedirect.com/science/article/B6TY8-4H74KY7-1/2/60f26c55f79c0a39775e1d76586975da>
- [7] G. Xu and G.-Z. Wang, "Extended cubic uniform b-spline and $[\alpha]$ -b-spline," *Acta Automatica Sinica*, vol. 34, no. 8, pp. 980–984, Aug. 2008. [Online]. Available: <http://www.sciencedirect.com/science/article/B8H0V-4T6VT15-3/2/7c4718af9afe2c6bae6b0d1d4f8e66e3>