

LMI Approach to Regularization and Stabilization of Linear Singular Systems: The Discrete-time Case

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Abstract—Sufficient linear matrix inequalities (LMI) conditions for regularization of discrete-time singular systems are given. Then a new class of regularizing stabilizing controllers is discussed. The proposed controllers are the sum of predictive and memoryless state feedbacks. The predictive controller aims to regularizing the singular system while the memoryless state feedback is designed to stabilize the resulting regularized system. A systematic procedure is given to calculate the controller gains through linear matrix inequalities.

Keywords—Singular systems; Discrete-time systems, Regularization; LMIs.

I. INTRODUCTION

Singular systems arise in many engineering disciplines including electrical networks, power systems, and aerospace engineering. Since the late of 1970s singular systems have attracted the attention of many researchers and the stabilization of such systems has been the subject of numerous research papers. Singular systems are those the dynamics of which are governed by a mixture of differential equations and algebraic equations. The inherently complex nature of singular systems causes many difficulties in the control of such systems. In that sense the question of their uniqueness and existence of solution, solvability, question of consistent initial conditions and stability deserves great attention. Several books and survey papers dealing with these systems have addressed the issues of solvability, controllability, pole assignment and elimination of impulse behavior and so on [1], [2], [3], [4].

As we have mentioned before, singular systems are defined as dynamical systems subject to algebraic constraints. Therefore, the process of elimination of these constraints is called a *regularization problem*. Actually, elimination of these algebraic constraints needs a special feedback that does not always exist. The majority of works that have dealt with this problem were in the continuous-time case, see [5], [6], [7], [8], [9] and the references therein. Furthermore, the main contributions were focused only upon the conditions of existence of regularizing controllers without showing how to find, in efficient computationally way, the gains of such regularizing feedbacks.

In this note sufficient LMI conditions for regularization of discrete-time singular systems are given. Subsequently, a novel form of regularizing stabilizing controllers that involve the action of predictive and memoryless state feedbacks is introduced. Our goal is two folds. First, we regularize the singular systems by applying a predictive controller that necessitates estimation of the system states at the forward

iteration. Second, we feedback the resulting system by a classical memoryless controller that achieves the stability of the regularized system. The control action appears as the sum of two independent controllers computed separately through two linear matrix inequalities. The class of the proposed controller seems interesting, in the sense, that the closed-loop system behaves as a nonsingular system and consequently, all the properties and behavioral phenomena of singular system disappear under the action of such feedback.

The paper is organized as follows. In section II, the problem of regularization by both full state and output predictive controllers is addressed. The conditions of existence of such controllers are formulated in linear matrix inequalities. In section III, the stabilization procedure of the regularized system is given and the main result of this paper is stated. A numerical example is also provided to highlight the efficiency of the proposed technique. Finally, some concluding remarks are given. Throughout this paper we note by \mathbb{R} the set of real number. The notation $X > 0$ (resp. $X < 0$) means that the matrix X is positive definite (resp. negative definite). A' stands for the transpose of A . We note by I the identity matrix of appropriate dimension and by $\mathbf{0}$ the null matrix of appropriate dimension.

II. REGULARIZATION OF DISCRETE-TIME SINGULAR SYSTEMS

Consider the singular discrete-time system:

$$\begin{aligned} E x_{k+1} &= A x_k + B u_k, \\ y_k &= C x_k, \end{aligned} \quad (1)$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the control input, and $y_k \in \mathbb{R}^p$ is the system output. The matrix $E \in \mathbb{R}^{n \times n}$ is supposed to be a singular square matrix. We suppose that system (1) is completely observable and (E, A) is a regular pair, i.e., $\det(sE - A) \neq 0, \forall s$. The imposed regularity property guarantees the existence and the uniqueness of solutions.

In this section, we will investigate the conditions of existence of a controller gain $L \in \mathbb{R}^{m \times n}$ such that system (1) under the feedback

$$u_k = -L x_{k+1} + v_k, \quad (2)$$

is equivalent to a discrete-time regular system of the form:

$$\begin{aligned} E_r x_{k+1} &= A x_k + B v_k, \\ y_k &= C x_k, \end{aligned} \quad (3)$$

where $E_r = E + BL$ is invertible or a full rank square matrix. We call the feedback $-L x_{k+1}$ a regularizing controller and $v_k \in \mathbb{R}^m$ is the new control input to be designed

later. The computation of L is fulfilled through the solution of a linear matrix inequality. Subsequently, we give similar LMI condition that guarantee the existence of regularizing controller of the form:

$$u_k = -Hy_{k+1} + v_k, \quad (4)$$

where H is a constant matrix of dimension $m \times p$. For both cases, we assume that x_{k+1} is available for feedback by an appropriate observer.

A. Regularization by a predictive static feedback

A predictive static feedback is defined as in Eq. (2). Computing the full state of the singular system at the stage $k + 1$ necessitates a full order observer. We refer the reader to the references [10], [11] for more details on observer design for discrete-time singular systems. A necessary condition for the existence of the gain L is that [2]

$$\text{rank}[E \ B] = n. \quad (5)$$

In this subsection we give a sufficient LMI condition to regularize system (1) by a predictive static feedback of the form (2). We summarize the result of this subsection in the following statement.

Theorem 1: If there exist a positive and definite matrix $X \in \mathbb{R}^{n \times n}$ and a matrix $Y \in \mathbb{R}^{m \times n}$ such that the following LMI holds

$$\begin{bmatrix} XE' + EX + Y'B' + BY & X \\ X & X \end{bmatrix} > 0. \quad (6)$$

Then there exists $L = YX^{-1}$ such that the matrix $E_r = E + BL$ has a full rank and consequently, system (1) is regularized by the predictive static feedback

$$u_k = -YX^{-1}x_{k+1} + v_k. \quad (7)$$

Proof: The matrix $E_r = E + BYX^{-1}$ has a full rank if and only if $E_r'E_r > 0$. This means that $(E_r'E_r)^{-1}$ exists and consequently, E_r^{-1} exists. If $E_r'E_r > 0$, then

$$E_r'X^{-1}XX^{-1}E_r > 0.$$

Replacing E_r by $E + BYX^{-1}$, this gives

$$(E'X^{-1} + X^{-1}Y'B'X^{-1})X(X^{-1}E + X^{-1}BYX^{-1}) > 0$$

Since for given symmetric matrices Z , and W , we have

$$Z'W + W'Z \leq Z'XZ + W'X^{-1}W, \quad (8)$$

then if we put $Z' = E'X^{-1} + X^{-1}Y'B'X^{-1}$ and $W = I$, we obtain

$$(E'X^{-1} + X^{-1}Y'B'X^{-1})X(X^{-1}E + X^{-1}BYX^{-1}) \geq E'X^{-1} + X^{-1}Y'B'X^{-1} + X^{-1}E + X^{-1}BYX^{-1} - X^{-1} \quad (9)$$

If the following inequality holds

$$E'X^{-1} + X^{-1}Y'B'X^{-1} + X^{-1}E + X^{-1}BYX^{-1} - X^{-1} > 0, \quad (10)$$

then by pre- and post multiplying the last inequality by X , we have

$$XE' + EX + BY + Y'B' - X > 0 \quad (11)$$

and then $E_r'X^{-1}E_r > 0$ is verified which translates that E_r is invertible. By the Schur complement lemma, inequality (11) is equivalent to (6). This ends the proof.

B. Regularization by a predictive static output feedback

Regularizing the singular system (1) by a predictive static output feedback of the form (4) is a special case of the regularization by full state predictive static feedback discussed in the last subsection. The information y_{k+1} can be obtained by designing an observer for the singular discrete-time system (1) or by extrapolating the discrete outputs y_k . The output prediction y_{k+1} can also be obtained from an estimate of the first derivative of y_k . We refer the interested reader to the reference [12] to see some recent works on estimation of the output time-derivatives in the discrete-time case. The sufficient condition for the existence of a predictive static output feedback that regularizes the singular system (1) is given by the following theorem.

Theorem 2: If there exist a matrix $H \in \mathbb{R}^{m \times p}$ and $0 < \epsilon \leq 1$ such that the the following LMI holds

$$\begin{bmatrix} E'E + E'BHC + C'H'B'E + C'H'B' + BHC & I \\ I & \epsilon I \end{bmatrix} > 0 \quad (12)$$

then, the controller

$$u_k = -Hy_{k+1} + v_k, \quad (13)$$

regularizes system (1) and the matrix $E + BHC$ has a full rank.

Proof: As we have shown in the last subsection, the invertibility condition of the matrix $E + BHC$ is equivalent to the following inequality

$$(E + BHC)'(E + BHC) > 0. \quad (14)$$

The last inequality can be rewritten as

$$E'E + E'BHC + C'H'B'E + C'H'B'BHC > 0. \quad (15)$$

For any $0 < \epsilon \leq 1$, we have

$$C'H'B'BHC \geq \epsilon C'H'B'BHC. \quad (16)$$

Furthermore, by the use of (8), we can write

$$\epsilon C'H'B'BHC + \frac{1}{\epsilon}I \geq C'H'B' + BHC \quad (17)$$

This implies that if

$$E'E + E'BHC + C'H'B'E + C'H'B' + BHC - \frac{1}{\epsilon}I > 0 \quad (18)$$

then (15) is verified. Using the Schur complement lemma, then (18) is equivalent to (12).

III. STABILIZATION OF THE REGULARIZED SYSTEM

A. Preliminaries

Before giving the main result of this paper we would rather begin by exposing some preliminary results.

Lemma 1: The following statements are equivalent

- i) there exist symmetric and positive definite matrices P and Q such that

$$A'PA - P + Q < 0; \quad (19)$$

- ii) there exist $P = P' > 0$ and $Q = Q' > 0$ such that

$$\begin{bmatrix} -P^{-1} & P^{-1}A' & P^{-1} \\ AP^{-1} & -P^{-1} & \mathbf{0} \\ P^{-1} & \mathbf{0} & -Q^{-1} \end{bmatrix} < 0. \quad (20)$$

Proof: First we prove i) implies ii). If (19) is satisfied then, we have

$$P^{-1}(A'PA - P + Q)P^{-1} < 0, \quad (21)$$

or

$$-P^{-1} + P^{-1}A'PAP^{-1} + P^{-1}QP^{-1} < 0 \quad (22)$$

By the Schur complement lemma, we conclude that

$$-P^{-1} + P^{-1}A'PAP^{-1} < 0 \quad (23)$$

and

$$\begin{bmatrix} -P^{-1} + P^{-1}A'PAP^{-1} & P^{-1} \\ P^{-1} & -Q^{-1} \end{bmatrix} < 0, \quad (24)$$

which is equivalent to (20).

Now the equivalence ii) implies i) can be demonstrated as follows. Starting from inequality (20), then we have for any $P = P' > 0$ and $Q = Q' > 0$

$$\begin{bmatrix} P & A'P & Q \end{bmatrix} \begin{bmatrix} -P^{-1} & P^{-1}A' & P^{-1} \\ AP^{-1} & -P^{-1} & \mathbf{0} \\ P^{-1} & \mathbf{0} & -Q^{-1} \end{bmatrix} \begin{bmatrix} P \\ PA \\ Q \end{bmatrix} < 0. \quad (25)$$

By expanding the last inequality we get exactly inequality (19).

The result of lemma 1 is interesting, in the sense, that the stabilizability problem of discrete-time systems can be easily solved by the use of this lemma. For this purpose, we introduce the following corollary.

Corollary 1: Consider the discrete-time system

$$x_{k+1} = Ax_k + Bu_k \quad (26)$$

where $x_k \in \mathbb{R}^n$ is the state vector and $u_k \in \mathbb{R}^m$ is the system input. Then if there exist a symmetric and positive definite matrices $P \in \mathbb{R}^{n \times n}$, and $Z \in \mathbb{R}^{n \times n}$ and a matrix $W \in \mathbb{R}^{m \times n}$ such that

$$\begin{bmatrix} -P & PA' + W'B' & P \\ BW + AP & -P & \mathbf{0} \\ P & \mathbf{0} & -Z \end{bmatrix} < 0 \quad (27)$$

then the controller $u_k = WP^{-1}x_k$ stabilizes system (26) in the origin.

Proof: Using result of lemma 1, then if we replace P by P^{-1} , Q^{-1} by Z , and A by $A + BWP^{-1}$ in (20) the condition of the stabilizability of system (26) by the controller $u_k = WP^{-1}x_k$ is reduced to the solvability of the LMI (27).

B. Main result

In this subsection the design of the controller v_k such that the feedback (2) asymptotically stabilizes the singular system (1) is given. Since the feedback gain L that guarantees the invertibility of the matrix $E + BL$ can be found without any care of the stability of system (1), then the design of v_k is seen as a problem of the stabilizability of a regular discrete-time system of the form $x_{k+1} = A_r x_k + B_r v_k$ where A_r and B_r stand for the new resulting nominal matrices. The main result of this paper that gathers the solvability of the regularization problem along with the stabilization of system (1) is given by the following theorem.

Theorem 3: Consider the discrete-time singular system (1). If there exist a set of symmetric and positive matrices $X \in \mathbb{R}^{n \times n}$, $P \in \mathbb{R}^{n \times n}$, and $Z \in \mathbb{R}^{n \times n}$, and two matrices $Y \in \mathbb{R}^{m \times n}$ and $W \in \mathbb{R}^{m \times n}$ such that the LMI (6) holds and

$$\begin{bmatrix} -P & PA'_r + W'B'_r & P \\ B_r W + A_r P & -P & \mathbf{0} \\ P & \mathbf{0} & -Z \end{bmatrix} < 0 \quad (28)$$

where $A_r = (E + BYX^{-1})^{-1}A$ and $B_r = (E + BYX^{-1})^{-1}B$. Then the controller

$$u_k = -YX^{-1}x_{k+1} + WP^{-1}x_k \quad (29)$$

stabilizes system (1) in the origin.

Proof: The proof of this theorem is already proved by the use of results of theorem 1 and corollary 1.

Remark 1: The LMI (28) and (6) must be satisfied simultaneously. LMI (6) must be solved first with respect to X and Y before solving the LMI (28). Evidently, the regularized matrices A_r and B_r are function of the solution of (6).

With the same analysis, if there exist H and $0 < \epsilon \leq 1$ satisfying the LMI (12), a matrix W of appropriate dimension, and two matrices $P = P' > 0$ and $Z = Z' > 0$ of dimensions $n \times n$ such that the following LMI holds

$$\begin{bmatrix} -P & P\tilde{A}'_r + W'\tilde{B}'_r & P \\ \tilde{B}_r W + \tilde{A}_r P & -P & \mathbf{0} \\ P & \mathbf{0} & -Z \end{bmatrix} < 0 \quad (30)$$

where $\tilde{A}_r = (E + BHC)^{-1}A$ and $\tilde{B}_r = (E + BHC)^{-1}B$. Then the controller

$$u_k = -Hy_{k+1} + WP^{-1}x_k \quad (31)$$

stabilizes system (1).

C. Example

Consider the singular discrete-time system (1) with

$$\begin{aligned}
 E &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \\
 A &= \begin{bmatrix} 0.1 & -0.3 & 2 \\ 0.5 & -3 & -1 \\ 0.2 & 0.4 & -0.1 \end{bmatrix}, \\
 B &= \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, \\
 C &= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.
 \end{aligned} \tag{32}$$

The solution of the LMI (6) with respect to X and Y gives

$$\begin{aligned}
 X &= \begin{bmatrix} 1.4194 & -0.1741 & -0.1209 \\ -0.1741 & 1.4194 & -0.1209 \\ -0.1209 & -0.1209 & 1.6123 \end{bmatrix}, \\
 Y &= \begin{bmatrix} -1.3481 & -0.0149 & 0 \\ -0.3693 & -0.9936 & 0.3544 \end{bmatrix}.
 \end{aligned} \tag{33}$$

The solution of (12) with respect to H and ϵ gives

$$\begin{bmatrix} -0.5100 & 0.0997 \\ 0.0997 & -1.1001 \end{bmatrix}, \quad \epsilon = 0.8684.$$

This means that the system can be regularized by both full state and output predictive controllers. Using the full state predictive controller, we obtain the new matrices

$$\begin{aligned}
 A_r &= \begin{bmatrix} -0.1534 & 0.5710 & 1.1679 \\ 0.2497 & -1.8175 & -1.1113 \\ 0.2037 & -0.4034 & 0.4434 \end{bmatrix}, \\
 B_r &= \begin{bmatrix} -1.0050 & 0.4064 \\ 0.7175 & -1.0204 \\ 0.2875 & 0.6140 \end{bmatrix}.
 \end{aligned} \tag{34}$$

By solving the LMI (28), we get

$$\begin{aligned}
 P &= \begin{bmatrix} 0.7187 & 0.0808 & 0.0325 \\ 0.0808 & 0.4429 & 0.1136 \\ 0.0325 & 0.1136 & 0.7059 \end{bmatrix}, \\
 Z &= \begin{bmatrix} 1.3087 & 0.0422 & 0.0272 \\ 0.0422 & 1.2103 & 0.0537 \\ 0.0272 & 0.0537 & 1.3045 \end{bmatrix}, \\
 W &= \begin{bmatrix} -0.1024 & 0.4365 & 0.5713 \\ -0.1032 & -0.4029 & -0.6107 \end{bmatrix}.
 \end{aligned} \tag{35}$$

IV. CONCLUSION

The problem of regularization of discrete-time singular systems by predictive controllers is addressed. Sufficient LMI conditions for the existence of such regularizing feedbacks are given. The problem of the stabilizability of singular discrete-time systems by means of sum of predictive and memoryless feedbacks is also treated in an LMI framework. The results of this paper can be seen as an extension of existing results on the stabilizability of singular systems by memoryless feedbacks.

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