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Modeling and Numerical Simulation of Sound Radiation by the Boundary Element Method

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***Abstract*—**The modeling of sound radiation is of fundamental importance for understanding the propagation of acoustic waves and, consequently, develop mechanisms for reducing acoustic noise. The propagation of acoustic waves, are involved in various phenomena such as radiation, absorption, transmission and reflection. The radiation is studied through the linear equation of the acoustic wave that is obtained through the equation for the Conservation of Momentum, equation of State and Continuity. From these equations, is the Helmholtz differential equation that describes the problem of acoustic radiation. In this paper we obtained the solution of the Helmholtz differential equation for an infinite cylinder in a pulsating through free and homogeneous. The analytical solution is implemented and the results are compared with the literature. A numerical formulation for this problem is obtained using the Boundary Element Method (BEM). This method has great power for solving certain acoustical problems in open field, compared to differential methods. BEM reduces the size of the problem, thereby simplifying the input data to be worked and reducing the computational time used.

***Keywords*—**Acoustic radiation, boundary element

I. Introduction

ENGINEERING problems are often described by physical laws, which are commonly expressed by partial differential equations. Among these problems, we highlight the noise sound that is present in various situations of everyday life.

A very common mechanism for generating sounds consists of a vibrating structure. Structures move cyclically vibrating air molecules around it, generating local concentration and rarefaction of these, which causes pressure variations.

The sound propagation outdoors is usually studied in terms of three components: the sound source, the transmission path and receiver. First, the source emits a certain power level, generating a noise level that can be measured near the source. From there, the sound level is attenuated as the sound travels between the source and receiver along a particular path.

The modeling of acoustic radiation is of fundamental importance for understanding the propagation of acoustic waves and, consequently, develops mechanisms for attenuation of acoustic noise. To estimate sound pressure levels, it is necessary to know the sound power levels of the sources in question.

The distribution of pressure in fluid subject to a source of vibration is given by the Helmholtz equation. The derivation of this equation begins with the equations governing the fluid, with some restrictions.

In many cases, a mathematical representation of alternative equivalent problem is found in terms of boundary integral equations.

The most general and effective numerical technique for solving boundary integral equations is the method of boundary element.

This paper presents the formulation and analytical solution of the wave equation for an infinite cylinder which is vibrating (expanding and contracting) uniformly in the radial direction with constant amplitude. This solution is then compared with the literature. A numerical formulation for this problem is also presented, through the formulation of the direct method of boundary element. Currently, the Method of Boundary Element is one of the most advanced and used especially when it comes to problems considering ways infinite and semi-infinite, since it allows reducing the size of the problem by reducing the number of equations used, allowing the solution only contour, without the need to analyze your entire domain.

The numerical solution to the problem described, using the method of boundary element has been implemented and compared with results obtained by [6].

II. Wave Equation

The equation that governs the phenomenon of acoustic radiation is found from the equations of state, conservation of mass and conservation of momentum.

For fluid media, the equation of state relates physical quantities that describe the thermodynamic behavior of the fluid and is given for

$$P - P_0 = \beta \frac{(\rho - \rho_0)}{\rho_0} \tag{1}$$

where P is the instantaneous pressure at one point, P_0 is the pressure of the fluid balance, β is the adiabatic module (coefficient of thermal expansion of the fluid), ρ is the instantaneous density at one point and ρ_0 is the density of the fluid balance .

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You can set the condensation s at one point as the density variation of equilibrium $s = \frac{\rho - \rho_0}{\rho_0}$ and express (1) in terms of sound pressure p and of the condensation s

$$p \approx \beta s \tag{2}$$

where $p = P - P_0$ is the acoustic pressure.

This approach is limited to amplitude waves whose relatively small change in density of the medium is small compared with its equilibrium value, ie, condensation s must be very small, $|s| << 1$, [4]. This supposition is necessary to arrive at a simple theory for sound in fluids, which aptly describes the most common phenomena in acoustics.

To relate the movement of fluid with its compression or expansion, you need a function that relates the speed $\vec{u}$ particle fluid with its instantaneous density ρ .

It is an infinitesimal element of fluid volume, fixed in space. The continuity equation relates the rate of growth in mass volume element, with the mass flow through the closed surface that surrounds it. Since the mass flow must be equal to the rate of growth, we obtain the continuity equation.

$$\frac{\partial s}{\partial t} + \nabla \vec{u} = 0 \tag{3}$$

The equation of motion relates the sound pressure p at the speed $\vec{u}$ instantaneous particle to an adiabatic fluid and not sticky, that is, the effects of fluid viscosity are neglected. Thus is the Euler equation (force equation) to acoustic phenomena of small amplitude.

$$\rho_0 \frac{\partial \vec{u}}{\partial t} = -\nabla p \tag{4}$$

Combining (1), (3) and (4) it is obtained the linearized wave equation for the propagation of sound in fluids, expressed in terms of sound pressure

$$\nabla^2 p = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} \tag{5}$$

where ∇^2 is the Laplacian operator and $c = \sqrt{\frac{\beta}{\rho}}$ is the speed of propagation of acoustic wave in the middle [7].

For non-viscous fluid, the particle velocity is irrotational, $\nabla \times \vec{u} = 0$. This means that the speed can be written as the gradient of a scalar function ϕ called velocity potential, $\vec{u} = \nabla \phi$. Thus, we obtain the linear wave equation, expressed in terms of velocity potential of acoustic wave

$$\nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} . \tag{6}$$

For the solution of (6), it is assumed that the velocity potential with harmonic dependence on time. From (6), is the Helmholtz equation independent of time for a medium without losses, expressed in terms of velocity potential of acoustic wave

$$\nabla^2 \phi_f(\vec{r}) + k^2 \phi_f(\vec{r}) = 0 . \tag{7}$$

onde ϕ_f representa a parte espacial do potencial de velocidade, ω representa a freqüência angular de vibração e k é o número de onda.

III. ANALYTICAL SOLUTION

Using the method of separation of variables, we obtain the spatial part of the solution of the Helmholtz equation, given by (6)

$$\phi(r, \psi, z) = -\frac{V_0}{k} \frac{H_0^{(2)}(kr)}{H_1^{(2)}(ka)}, \quad r \geq a \tag{8}$$

where $H_0^{(2)}(u)$ and $H_1^{(2)}(u)$ are the Hankel functions of 2nd kind and order 0 and 1, respectively, k is the wave number, a is the radius of the cylinder, r is the radial direction of the velocity potential.

The solution, given by (8) is obtained for the special case of uniform radiation (monopole mode), ie, the surface of the cylinder is vibrating uniformly in the radial direction with an amplitude of V_0 meters per second at a frequency of f hertz.

IV. RESULTS AND COMPARISONS

The analytical solution of the Helmholtz differential equation was implemented in Matlab and results were compared with those obtained by [6], shown in Table 1 and exhibited in Figure 1. In analyzing the data, we calculated the absolute error e_{abs}, and relative error of the result obtained in the literature e_{rel}, the way defined below

$$e_{abs} = val1 - val \tag{9}$$

and

$$e_{rel} = \frac{e}{val1} \tag{10}$$

where $val1$ is the value obtained in the literature, val is the value obtained in this work.

TABLE I
COMPARISON OF RESULTS: ANALYTIC [6] AND ANALYTIC (MATLAB)

Freq. (Hz)	Analytic [6]	Analytic (MatLab)	Abs. Error	Rel. Error
62,5	0,7709467	0,7709467	-7,00000E-09	-9,07974E-09
125	0,4193960	0,4193960	7,00000E-09	1,66906E-08
250	0,2159032	0,2159032	1,50000E-08	6,94755E-08
500	0,1088600	0,1088600	1,00000E-09	9,18611E-09

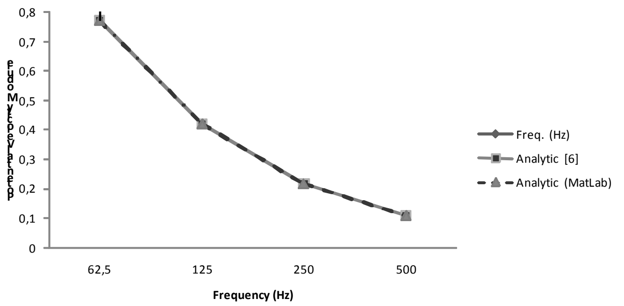

Fig. 1 Comparison of analytical results implemented

It appears from Fig. 1 that the module of the velocity potential decreases as the frequency increases. This can be seen in (8), where it is shown that the potential speed is inversely proportional to the wave number.

The results of this work, presented in Table 1 and shown in Fig. 1, are quite satisfactory when compared to results obtained by [6], by calculating the absolute and relative errors.

V. NUMERIC METHOD

A. Advantages of the Boundary Element Method

One of the most widely used numerical techniques for solving boundary integral equations is the boundary element method. A peculiarity of the boundary element method is that it provides a continuous model of the domain, since no discretization of the same is required, making it thus an effective method for solving problems of infinite domain.The solutions in the internal points are calculated after the unknown boundary was calculated, similar to a post-processing.

B. Boundary Integral Equation s

The classical differential equation, Helmholtz equation that describes the problem of acoustic radiation from an infinite cylinder pulse was determined in previous chapters. To find the integral equation on the boundary, from the Helmholtz equation, it is a two-dimensional body B immersed in an infinite domain Ω, shown in Fig. 2 below.

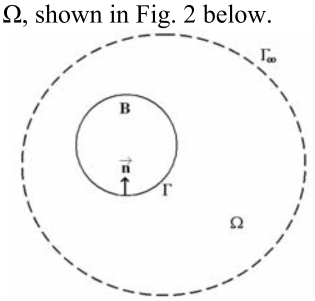

Fig. 2 Representation of the field and the boundary

The solution of (7) that describes the problem considered subject to boundary condition

$$\frac{\partial \phi}{\partial n} = 1 \text{ (Neumann Condition)} \tag{11}$$

It is found by solving boundary integral equations. Equation (7), which is a differential equation is valid at all points of the domain Ω. To turn it into an integral equation, it is assumed that it is she can not be zero throughout the domain, thus generating a residual r. With this you can write it as follows

$$\nabla^2 \phi(\vec{r}) + k^2 \phi(\vec{r}) = r, \forall \vec{r} \in \Omega . \tag{12}$$

The residue of (12) is evaluated at each point, using the method of weighted residues, which gets the sum of the residues in the field. For this, part of a weight function u^* for this sum is zero. Thus, we obtain

$$\int_\Omega r\, u^* \, d\Omega = 0 \ . \tag{13}$$

Substituting (12) in (13) we obtain

$$\int_\Omega \left(\nabla^2 \phi + k^2 \phi \right) u^* \, d\Omega = 0 \ . \tag{14}$$

For the solution of the equation above, one must make use of some vector identities, to know

$$u^* \nabla^2 \phi = \nabla \cdot \left(u^* \nabla \phi \right) - \nabla u^* \cdot \nabla \phi \tag{15}$$

and

$$\nabla \phi \cdot \nabla u^* = \nabla \cdot \left(\phi \nabla \mathrm{u}^* \right) - \phi \nabla^2 u^* . \tag{16}$$

Applying the distributive property and (15) in (14) we obtain

$$\int_\Omega \nabla \cdot \left(u^* \nabla \phi \right) d\Omega + \int_\Omega \left(u^* k^2 \phi - \nabla u^* \cdot \nabla \phi \right) d\Omega = 0. \tag{17}$$

Applying the divergence theorem in (17) we obtain

$$\int_\Gamma u^* \nabla \phi \, d\Gamma + \int_\Omega \left(u^* k^2 \phi - \nabla u^* \cdot \nabla \phi \right) d\Omega = 0 \tag{18}$$

It is known that

$$\nabla \phi \cdot d\Gamma = \nabla \phi \cdot \hat{n} \, d\Gamma = \frac{\partial \phi}{\partial n} \tag{19}$$

Thus, (18) becomes

$$\int_{\Gamma} u^* \frac{\partial \phi}{\partial n}\, d\Gamma + \int_{\Omega} \left(u^* k^2 \phi - \nabla u^* \cdot \nabla \phi \right) d\Omega = 0 \tag{20}$$

Substituting (16) in (20) we obtain

$$\int_{\Gamma} u^* \frac{\partial \phi}{\partial n}\, d\Gamma + \int_{\Omega} \left(u^* k^2 \phi + \phi \nabla^2 u^* \right) d\Omega - \int_{\Omega} \nabla \cdot \left(\phi \nabla u^* \right) d\Omega = 0. \tag{21}$$

Applying the divergence theorem in (21) is

$$\int_{\Gamma} u^* \frac{\partial \phi}{\partial n}\, d\Gamma - \int_{\Gamma} \phi \frac{\partial u^*}{\partial n}\, d\Gamma + \int_{\Omega} \phi \left(\nabla^2 u^* + k^2 u^* \right) d\Omega = 0. \tag{22}$$

Due to the property of Dirac delta function, we have

$$\int_{\Omega} \left(\nabla^2 u^* + k^2 u^* \right) \phi\, d\Omega = -\phi(\vec{r}). \tag{23}$$

Substituting (23) in (22) we obtain

$$\int_{\Gamma} u^* \frac{\partial \phi}{\partial n}\, d\Gamma - \int_{\Gamma} \phi \frac{\partial u^*}{\partial n}\, d\Gamma - \phi(\vec{r}) = 0 \cdot \tag{24}$$

The above equation was obtained for placement points $\vec{r}$ belonging to the domain, where u^* it is the fundamental solution, represented by the Green's function. This solution is presented for the two-dimensional body [1], as

$$u^* = \frac{i}{4} H_0^1(kR) \tag{25}$$

and its derivative is given by

$$q^* = \frac{\partial u^*}{\partial \vec{n}} = -\frac{ik}{4} H_1^1(kR) \frac{\partial R}{\partial \vec{n}} \tag{26}$$

Where R it is the distance between the point $\vec{r}$ and the application point $\vec{r}\,'$ in the domain Ω.

In the boundary element method, this equation is applied in the boundary. When $\vec{r} = \vec{r}\,'$, the value of R it will be zero, causing a problem of singularity in (25) and (26). One way to avoid this problem is to consider a point $\vec{r}\,'$ in the boundary, but the area around it being increased by a semi-circle radius ε, and to examine the solution in the limit when the radius ε→0. As this process limit depends only on the order of the singularity of the velocity potential ϕ,which is the same for operators of Laplace and Helmholtz second [5], we conducted a study of boundary integrals of (24) using the fundamental solution for Laplace equation in Ω. Thus, one arrives at the Boundary Integral Equation

$$\int_{\Gamma} u^* \frac{\partial \phi}{\partial n}\, d\Gamma - \int_{\Gamma} \phi \frac{\partial u^*}{\partial n}\, d\Gamma = \frac{1}{2} \phi(Y) \tag{27}$$

A more general way of representing it, in which $\vec{r}\,'$ may be located in the area, the boundary or outside the field, can be formulated using a free term $c(\vec{r}\,')$ related to to the position of $\vec{r}\,'$.

$$\int_{\Gamma} u^* \frac{\partial \phi}{\partial n}\, d\Gamma - \int_{\Gamma} \phi \frac{\partial u^*}{\partial n}\, d\Gamma = c(Y)\phi(Y). \tag{28}$$

If the point $\vec{r}\,'$ belongs to the field outside the body studied, its value is 0 if the point belongs to the boundary then its value will be $\frac{1}{2}$ and 1 if the point belongs to the interior of the body [1].

C. Discretization of Variables

For the discretization of the physical and geometrical variables of the problem, the boundary is discretized into N elements. It assumes a constant distribution of variables and u^* and $\frac{\partial u^*}{\partial n}$ over the elements on which the contour was discretized. Thus, from (23) and (24) can be written

$$\frac{1}{2}\phi_j(\vec{r}\,') + \sum_{j=1}^{N} \phi_j(\vec{r}\,') \int_{\Gamma_j} \frac{\partial u^*(\vec{r},\vec{r}\,')}{\partial n}\, d\Gamma(\vec{r}) = \sum_{j=1}^{N} \frac{\partial \phi_j}{\partial n} \int_{\Gamma_j} u^*(\vec{r},\vec{r}\,')\, d\Gamma(\vec{r}) \tag{29}$$

The integrates $\int_{\Gamma_j} \frac{\partial u^*}{\partial n} d\Gamma$ and $\int_{\Gamma_j} u^* d\Gamma$ in (29) are called coefficients of influence, it relates the influence of the solution at point P, when the fundamental solution is integrated over the element Q. Renaming it the integrals above, we obtain

$$G_{ij} = \int_{\Gamma_j} u^* d\Gamma \tag{30}$$

and

$$\hat{H}_{ij} = \int_{\Gamma_j} \frac{\partial u^*}{\partial n} d\Gamma \tag{31}$$

where *i* represents the point of release and *j* the element into account to be integrated. Thus (29) can be written as follows

$$\frac{1}{2}\phi_j(\vec{r}\,') + \sum_{j=1}^{N} \phi_j \hat{H}_{ij} = \sum_{j=1}^{N} \frac{\partial \phi_j}{\partial n} G_{\mathrm{ij}} \tag{32}$$

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$$H_{ij} = \begin{cases} \widehat{H}_{ij} & para\ i \neq j \\ \widehat{H}_{ij} + \dfrac{1}{2} & para\ i = j \end{cases} \tag{33}$$

Because $\phi(\vec{r}\,')$ it is zero for the elements that do not contain the singularity and one for which it contains. Thus, one arrives at the following equation

$$\sum_{j=1}^{N} \phi_j H_{ij} = \sum_{j=1}^{N} G_{ij} q_j \tag{34}$$

where $q = \dfrac{\partial \phi}{\partial n} = 1$ (Neumann condition) as defined above. In the more general case, the variables u^* and $\dfrac{\partial u^*}{\partial n}$ are approximated by interpolation functions of the form

$$\phi(Q) = \sum_{m=1}^{E} \phi_m N_m(Q) \tag{35}$$

where E is the degree of the interpolating function. It is assumed that the position of node i also varies from 1 to N. Thus, the fundamental solution is applied on each node, which enables you to check the influence of all other elements in the node and the uniqueness of it on himself, resulting in a system of equations expressed in matrix form, for each point of contour, as following

$$H\vec{\phi} = G\vec{q} \tag{36}$$

where H and G are two matrices N x N, $\vec{\phi}$ is a vector of size N and $\vec{q}$ is a unit vector of dimension N.

Being part of all elements of the matrices H and G corresponding to the unknown boundary conditions on the left and those corresponding to the known boundary conditions on the right side, and multiplying the matrices on the right, formed the following system of equations

$$A\vec{y} = \vec{b} \tag{37}$$

where $\vec{y}$ is the vector of unknown boundary values of $\vec{\phi}$. The vector $\vec{b}$ is found by multiplying the columns of H or G at the known values of $\vec{\phi}$ and $\vec{q}$.

After solving the boundary is possible to calculate the internal value of any potential or its derivative.

VI. Numerical Method

From the numerical formulation presented was made through the implementation of the MATLAB software and the results were compared with the numerical results obtained [6].

TABLE II
Comparison of Numerical Results

Freq. (Hz)	Analytic [6]	Analytic (MatLab)	Abs. Error	Rel. Error
62,5	0,770955817	0,771765586	-0,000809769	-0,001050344
125	0,419302101	0,425843833	-0,006541732	-0,015601477
250	0,215899747	0,216669203	-0,000769456	-0,00356395
500	0,108885018	0,107506492	0,001378526	0,012660383

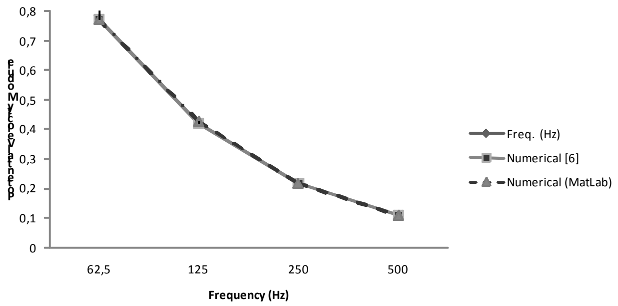

Fig. 3 Comparison of numerical results implemented

The numerical results compare favorably with those obtained by [6] and the absolute error introduced due to the fact that the treatment of the uniqueness of the solution found in the diagonal of the matrix G of the linear system. The singularity in question was resolved using the method of increasing the number of Gauss points for the solution of the Hankel function of containing the singularity. This is a procedure often used to solve some singularities.

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