Periodic solutions for a third-order p-Laplacian functional differential equation

Yanling Zhu and Kai Wang

Abstract—By means of Mawhin's continuation theorem, we study a kind of third-order p-Laplacian functional differential equation with distributed delay in the form:

$$\left(\varphi_p(x'(t))\right)'' = g\left(t, \int_{-\tau}^{0} x(t+s)\,d\alpha(s)\right) + e(t),$$

some criteria to guarantee the existence of periodic solutions are obtained.

Keywords—p–Laplacian, Distributed delay, Periodic solution, Mawhin's continuation theorem

I. INTRODUCTION

IN the last several years, the existence of periodic solutions for p-Laplacian differential equations has been extensively studied, we refer the readers to [1-5, 7-11] and the references cited therein. Very recently, Jin and Lu's [6], and Cheng and Ren's [12] have studied the existence of periodic solutions for fourth-order p-Laplacian Liénard equations as follows, respectively

$$(\varphi_p(u''(t)))'' + f(u(t)) + g(t, u(t), u(t - \tau(t))) = e(t)$$

and

$$(\varphi_p(x''(t)))'' + f(t, x'(t - \sigma(t))) + g(t, x(t - \tau(t))) = e(t).$$

The criteria they presented for guaranteeing the existence of periodic solutions are beautiful. However, the delays in [1-2, 6-12] are all discrete, few people has considered the existence of periodic solutions for p-Laplacian differential equations with distributed delay which is more realistic than the discrete ones in the real world. Moreover, as far as we know the corresponding problem of the third-order p-Laplacian equation has not been studied, especially for the third-order p-Laplacian differential equation with distributed delay.

Simulated by the above reasons, in this paper we investigate the existence of periodic solutions for a kind of third-order p-Laplacian equation with distributed delay as follows

$$(\varphi_p(x'(t)))'' = g\left(t, \int_{-\tau}^{0} x(t+s)\,d\alpha(s)\right) + e(t), \quad (1)$$

where $p > 1$ is a constant, $\varphi_p : R \to R, \varphi_p(x) = |x|^{p-2}x$ for $x \neq 0$ and $\varphi_p(0) = 0$; $g \in C(R^2, R)$ with $g(t+T, \cdot) \equiv g(t, \cdot)$; $e \in C(R, R)$ with $e(t+T) \equiv e(t)$ and $\int_0^T e(t)\,dt = 0$; $T > 0$ is a given positive constant. α is a bounded variation function with $\int_{-\tau}^{0} d\alpha(s) = 1$; $\tau > 0$ is a real-valued number.

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As the way to estimate *a priori bounds* in [1, 6-12] cannot be applied directly to study Eq.(1), because the crucial equality in [1, 6-12] is

$$\int_0^T (\varphi_p(x'(t))')x(t)dt = \int_0^T |x'(t)|^{p-1}dt$$

or

$$\int_0^T (\varphi_p(x''(t))'')x(t)dt = \int_0^T |x''(t)|^{p-1}dt,$$

which doesn't hold any more for $(\varphi_p(x'(t))''$. In this paper, by using some new analysis techniques we estimate the *priori bounds* of all periodic solutions of Eq.(1), and obtain new results to guarantee the existence of periodic solutions for Eq.(1) by applying Mawhin's continuation theorem.

II. LEMMAS

In order to use Mawhin's continuation theorem to study the existence of periodic solutions for Eq.(1), we rewrite first Eq.(1) in the following form:

$$\begin{cases} x_1'(t) = \varphi_q(x_2(t)), \\ x_2''(t) = g(t, \int_{-\tau}^{0} x_1(t+s)\,d\alpha(s)) + e(t), \end{cases} \quad (2)$$

where $1/p + 1/q = 1$. Obviously, the existence of periodic solutions to Eq.(1) is equivalent to the existence of periodic solutions to Eqs.(2). Thus, the problem of finding a $T-$periodic solution for Eq.(1) reduces to finding one for Eqs.(2).

Let $C_T = \{u : u \in C(R, R) \mid u(t + T) \equiv u(t)\}$ with the norm $|u|_\infty = \max_{t \in [0,T]} |u(t)|$, $X = Y = \{x = (x_1(.), x_2(.))^\top \in C(R, R^2) \mid x(t+T) \equiv x(t)\}$ with the norm $||x|| = \max\{|x_1|_\infty, |x_2|_\infty\}$. Clearly, X and Y are Banach spaces. Meanwhile, we define two operators L and N in the following form, respectively

$$L : D(L) \subset C_T \to C_T, \quad Lx = \begin{pmatrix} x_1' \\ x_2'' \end{pmatrix}$$

and $N : C_T \to C_T$,

$$Nx = \begin{pmatrix} \varphi_q(x_2(t)) \\ g\left(t, \int_{-\tau}^{0} x_1(t+s)\,d\alpha(s)\right) + e(t) \end{pmatrix}, \quad (3)$$

where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

One can see easily that Eqs.(2) can be converted to then following abstract equation

$$Lx = Nx.$$

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Moreover, the definition of L yields

$$\text{Ker}L = R^2, \quad \text{Im}L = \{y : y \in Y, \int_0^T y(s)ds = 0\},$$

thus L is a Fredholm operator with index zero.

Set project operators P and Q in the form:

$$P : X \to \text{Ker}L, \quad Px = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

and

$$Q : Y \to \text{Im}Q \subset R^2, \quad Qy = \frac{1}{T}\int_0^T y(s)\, ds,$$

and denote by L_P^{-1} the inverse of $L|_{KerP \cap D(L)}$. Obviously,

$$\text{Ker}L = \text{Im}Q = R^2$$

and

$$[L_P^{-1}y](t) = \begin{pmatrix} \int_0^t y_1(s)\, ds \\ \int_0^T G(t,s)\, y_2(s)\, ds \end{pmatrix}, \qquad (4)$$

where $y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ and

$$G(t,s) = \begin{cases} \frac{s(t-T)}{T}, & 0 \le s < t \le T, \\ \frac{t(s-T)}{T}, & 0 \le t \le s \le T. \end{cases}$$

From (3) and (4), we know that N is $L-$compact on $\overline{\Omega}$, where Ω is an arbitrary open bounded subset of X.

Lemma 2.1 (*See [4]*) Let X and Y be two Banach spaces, $L : \text{Dom}(L) \subset X \to Y$ be a Fredholm operator with index zero, $\Omega \subset Y$ be an open bounded set, and $N : \overline{\Omega} \to X$ be $L-$ compact on $\overline{\Omega}$. If all the following conditions hold:

[1] $Lx \ne \lambda Nx$, for $x \in \partial\Omega \cap \text{Dom}(L)$, $\lambda \in (0,1)$;
[2] $Nx \notin \text{Im}L$, for $x \in \partial\Omega \cap \text{Ker}L$;
[3] $deg\{JQN, \Omega \cap \text{Ker}L, 0\} \ne 0$, where $J : \text{Im}Q \to \text{Ker}L$ is an isomorphism,

then equation $Lx = Nx$ has a solution on $\overline{\Omega} \bigcap \text{Dom}(L)$.

III. MAIN RESULTS

Theorem 3.1 Suppose there exist positive constants d_1, d_2 and $r \ge 0$ such that the following conditions hold:

[A_1] $g(t,x) > 0$ for $(t,x) \in ([0,T] \times R)$ with $x > d_1$, and $g(t,x) < 0$ for $(t,x) \in ([0,T] \times R)$ with $x < -d_2$;
[A_2] $\lim\limits_{x \to -\infty} \sup_{t \in R} |\frac{g(t,x)}{x^{p-1}}| \le r$.

Then Eq.(1) has at least one $T-$periodic solution, if $\frac{r\,T^{p+2}}{2^{p+1}} < 1$.

Proof. Consider the operator equation

$$L\,x = \lambda\,Nx, \quad \lambda \in (0,1).$$

Set $\Omega_1 = \{x : \ Lx = \lambda Nx, \ \lambda \in (0,1)\}$. If $x(t) = (x_1(t),\, x_2(t))^\top \in \Omega_1$, then

$$\begin{cases} x_1'(t) = \lambda\,\varphi_q(x_2(t)), \\ x_2''(t) = \lambda\,g(t, \int_{-\tau}^0 x_1(t+s)\, d\alpha(s)) + \lambda\,e(t). \end{cases} \qquad (5)$$

The second equation of (5) and $x_2(t) = \frac{1}{\lambda^{p-1}}\,\varphi_p(x_1'(t))$ imply

$$(\varphi_p(x_1'(t)))'' = \lambda^p g\left(t, \int_{-\tau}^0 x_1(t+s)\, d\alpha(s)\right) + \lambda^p e(t). \quad (6)$$

We first claim that there must be a constant $t^* \in [0,T]$ such that

$$|x_1(t^*)| \le D, \qquad (7)$$

where $D = \max\{d_1,\, d_2\}$.

Integrating both sides of Eq.(6) from 0 to T, we get

$$\int_0^T g\left(t, \int_{-\tau}^0 x_1(t+s)\, d\alpha(s)\right) dt = 0. \qquad (8)$$

By integral mean value theorem, we know there is a constant $\xi \in [0,T]$ such that

$$g\left(\xi, \int_{-\tau}^0 x_1(\xi+s)\, d\alpha(s)\right) = 0,$$

which together with [A_1] gives

$$-d_2 \le \int_{-\tau}^0 x_1(\xi+s)\, d\alpha(s) \le d_1.$$

If let $D = \max\{d_1,\, d_2\}$, then

$$\left|\int_{-\tau}^0 x_1(\xi+s)\, d\alpha(s)\right| \le D.$$

By the properties of Riemann-Stieltjes integral, we know that there must be a constant $\eta \in (-\tau, 0)$ such that $|x_1(\xi+\eta)| \le D$, *i.e.*, there exists a constant $t^* \in [0,T]$ such that

$$|x_1(t^*)| \le D.$$

This proves (7).

Furthermore, it follows from (7) that

$$|x_1(t)| \le D + \int_{t^*}^t |x_1'(s)|\, ds \quad \text{for } t \in [t^*, t^*+T]$$

and

$$|x_1(t-T)| \le D + \int_{t-T}^{t^*} |x_1'(s)|\, ds \quad \text{for } t \in [t^*, t^*+T],$$

i.e.,

$$\begin{aligned} |x_1|_\infty &= \max_{t \in [t^*, t^*+T]} |x_1(t)| \\ &\le D + \frac{1}{2}\left(\int_{t^*}^t |x_1'(s)|ds + \int_{t-T}^{t^*} |x_1'(s)|ds\right) \\ &\le D + \frac{1}{2}\int_0^T |x_1'(s)|ds \\ &\le D + \frac{T}{2}|x_1'|_\infty. \end{aligned} \qquad (9)$$

Note that $\frac{r\,T^{p+2}}{2^{p+1}} < 1$ yields there is a small constant $\varepsilon > 0$ such that

$$(r+\varepsilon)\frac{T^{p+2}}{2^{p+1}} < 1. \qquad (10)$$

For the sake of convenience, we let $X(t,\alpha) = \int_{-\tau}^0 x_1(t+s)\, d\alpha(s)$. For the above ε, from [A_2] and the properties of

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bounded variation function, we know that there must be a constant $\rho > d_2$ such that

$$
\begin{aligned}
|g(t, X(t, \alpha))| &\leq (r + \varepsilon) |X(t, \alpha)|^{p-1} \\
&\leq (r + \varepsilon) |x_1|_\infty^{p-1} \text{ for } X(t, \alpha) < -\rho.
\end{aligned}
\tag{11}
$$

Set $\triangle_1 = \{t \in [0, T] | X(t, \alpha) > \rho\}$, $\triangle_2 = \{t \in [0, T] | -d_2 \leq X(t, \alpha) \leq \rho\}$, $\triangle_3 = \{t \in [0, T] | X(t, \alpha) < -d_2\}$. We get from (8) and (11) that

$$
\left(\int_{\triangle_1} + \int_{\triangle_2} + \int_{\triangle_3} \right) g(t, X(t, \alpha)) \, dt = 0,
$$

i.e.,

$$
\begin{aligned}
\int_{\triangle_1} &|g(t, X(t, \alpha))| \, dt \\
&= - \left(\int_{\triangle_2} + \int_{\triangle_3} \right) g(t, X(t, \alpha)) \, dt \\
&\leq \left(\int_{\triangle_2} + \int_{\triangle_3} \right) |g(t, X(t, \alpha))| \, dt \\
&\leq \left((r + \varepsilon) |x_1|_\infty^{p-1} + g_d \right) T.
\end{aligned}
\tag{12}
$$

where $g_d = \max_{t \in \triangle_2} |X(t, \alpha)|$.

From $x_2'(0) = x_2'(T)$, we know that there exists a constant $\xi_1 \in [0, T]$ such that $x_2''(\xi_1) = 0$. It follows from the second equation of (5), (9) and (12) that

$$
\begin{aligned}
|x_2''(t)| \\
\leq |x_2''(\xi_1)| + \frac{1}{2} \int_0^T |g(t, X(t, \alpha)) + e(t)| dt \\
\leq \frac{1}{2} \int_0^T |g(t, X(t, \alpha))| dt + \frac{T}{2} |e|_\infty \\
\leq (r + \varepsilon) T |x_1|_\infty^{p-1} + \frac{T}{2} |e|_\infty + g_d T \\
\leq (r + \varepsilon) T \left(D + \frac{T}{2} |x_1'|_\infty \right)^{p-1} + M_{eg},
\end{aligned}
\tag{13}
$$

where $M_{eg} = \frac{T}{2} |e|_\infty + g_d T$.

The first equation of (5) yields

$$
|x_1'|_\infty \leq |x_2|_\infty^{q-1}.
\tag{14}
$$

Substitution of (14) into (13) gives

$$
|x_2''(t)| \leq (r + \varepsilon) T \left(D + \frac{T}{2} |x_2|_\infty^{q-1} \right)^{p-1} + M_{eg}.
\tag{15}
$$

By the classical elementary inequality, there is a constant $\delta > 0$, which is only dependent on n, such that

$$
(1 + x)^n \leq 1 + (1 + n) x \quad \text{for } x \in (0, \delta].
\tag{16}
$$

If $|x_2|_\infty = 0$, then (14) and (9) imply

$$
|x_1|_\infty \leq D.
$$

If $\frac{2D}{T |x_2|_\infty^{q-1}} \geq \delta$, then

$$
|x_2|_\infty \leq \left(\frac{2D}{T\delta} \right)^{1/(q-1)} := M_1,
$$

which together with (14) and (9) yields

$$
|x_1|_\infty \leq D + \frac{D}{\delta} := M_2.
$$

Otherwise, $\frac{2D}{T |x_2|_\infty^{q-1}} < \delta$, it follows from (15) and (16) that

$$
\begin{aligned}
|x_2''(t)| \\
\leq (r + \varepsilon) \frac{T^p}{2^{p-1}} |x_2|_\infty \left(1 + \frac{2D}{T |x_2|_\infty^{q-1}} \right)^{p-1} + M_{eg} \\
\leq (r + \varepsilon) \frac{T^p}{2^{p-1}} |x_2|_\infty \left(1 + \frac{2p \, D}{T |x_2|_\infty^{q-1}} \right) + M_{eg} \\
\leq (r + \varepsilon) \frac{T^p}{2^{p-1}} |x_2|_\infty + (r + \varepsilon) \frac{p \, D T^{p-1}}{2^{p-2}} |x_2|_\infty^{2-q} + M_{eg}.
\end{aligned}
\tag{17}
$$

On the other hand, $x_1(0) = x_1(T)$ imply that there exists $\xi_2 \in [0, T]$ such that $x_1'(\xi_2) = 0$, which together with the first equation of (5) gives

$$
x_2(\xi_2) = \varphi_p(x_1'(\xi_2)) = \varphi_p(0) = 0.
$$

From the above equalities we have

$$
\begin{aligned}
|x_2(t)| &\leq |x_2(\xi_2)| + \frac{1}{2} \left(\int_{\xi_2}^t |x_2'(s)| ds + \int_{t-T}^{\xi_2} |x_2'(s)| ds \right) \\
&\leq \frac{1}{2} \int_0^T |x_2'(s)| \, ds \\
&\leq \frac{T}{4} \int_0^T |x_2''(s)| \, ds \\
&\leq \frac{T^2}{4} |x_2''|_\infty,
\end{aligned}
$$

it follows from (17) that

$$
\begin{aligned}
|x_2(t)| \leq &(r + \varepsilon) \frac{T^{p+2}}{2^{p+1}} |x_2|_\infty + (r + \varepsilon) \frac{p \, D T^{p-1}}{2^{p-2}} |x_2|_\infty^{2-q} \\
&+ \frac{T^3}{4} \left(g_d + \frac{1}{2} |e|_\infty \right),
\end{aligned}
$$

which together with (10) implies that there is a positive constant M_3 such that

$$
|x_2|_\infty \leq M_3.
\tag{18}
$$

Combination of (18), (14) and (9) yields that there must be a positive constant M_4 such that

$$
|x_1|_\infty \leq M_4.
$$

Let $M_0 = \max\{M_1, M_2, M_3, M_4\} + 1$, $\Omega = \{x = (x_1, x_2)^\top : \|x\| < M_0\}$ and $\Omega_2 = \{x : x \in \partial\Omega \cap \mathrm{Ker}L\}$, then

$$
QNx = \frac{1}{T} \int_0^T \left(\begin{array}{c} \varphi_q(x_2(t)) \\ g\left(t, \int_{-\tau}^0 x_1(t+s) \, d\alpha(s)\right) \end{array} \right) dt.
$$

If $x \in \Omega_2$, then $x_2 = 0, x_1 = M_0$ or $-M_0$. Thus, from $[A_1]$ we can get easily

$$
QNx \neq 0,
$$

i.e.,

$$
Nx \notin \mathrm{Im}L \quad \text{for } x \in \Omega.
$$

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So conditions [1] and [2] of Lemma 2.1 are satisfied. In order to show that condition [3] of Lemma 2.1 is also satisfied, we define an isomorphism J in the following form:

$$J : \mathrm{Im}Q \to \mathrm{Ker}L, \quad J\left(\begin{array}{c} x_1 \\ x_2 \end{array}\right) = \left(\begin{array}{c} x_2 \\ x_1 \end{array}\right).$$

Let

$$H(\mu, x) = \mu x + (1 - \mu)JQNx \quad \text{for } (\mu, x) \in [0, 1] \times \Omega,$$

then

$$H(\mu, x) = \left(\begin{array}{c} \mu x_1 + \frac{(1-\mu)}{T}\int_0^T g(t, \int_{-\tau}^0 x_1(t+s)\, d\alpha(s))\, dt \\ \mu x_2 + (1-\mu)|x_2|^{q-2}x_2 \end{array}\right),$$

it follows from condition $[A_1]$ that

$$xH(\mu, x) > 0.$$

Hence

$$\begin{aligned} \deg\{JQN, \Omega \cap \mathrm{Ker}L, 0\} &= \deg\{H(0, x), \Omega \cap \mathrm{Ker}L, 0\} \\ &= \deg\{H(1, x), \Omega \cap \mathrm{Ker}L, 0\} \\ &= \deg\{I, \Omega \cap \mathrm{Ker}L, 0\} \\ &\neq 0. \end{aligned}$$

Thus, condition [3] of Lemma 2.1 is satisfied. So by applying Lemma 2.1, we conclude that equation $Lx = Nx$ has a solution $x(t) = (x_1(t), x_2(t))^\top$ on $\overline{\Omega} \cap D(L)$. So Eq.(1) has a $T-$periodic solution $x_1(t)$. The proof of Theorem 3.1 is now finished.

Corollary 3.1 Suppose there exist positive constants d_1, d_2 and $r \geq 0$ such that the following conditions hold:

$[B_1]$ $g(t, x) < 0$ for $(t, x) \in ([0, T] \times R)$ with $x > d_1$, and $g(t, x) > 0$ for $(t, x) \in ([0, T] \times R)$ with $x < -d_2$;
$[B_2]$ $\lim\limits_{x \to +\infty} \sup_{t \in R} \left|\frac{g(t,x)}{x^{p-1}}\right| \leq r$.

Then Eq.(1) has at least one $T-$periodic solution, if $\frac{r\,T^{p+2}}{2^{p+1}} < 1$.

As an application, we consider the following equation

$$(\varphi_4(x'(t)))'' = g\left(t, \int_{-\tau}^0 x(t+s)\, d\alpha(s)\right) + \cos 2\pi t, \quad (19)$$

where $g\left(t, \int_{-\tau}^0 x(t+s)\, d\alpha(s)\right) = (2 + e^{\beta(x(t))} - \sin 2\pi t)\beta(x(t))\,(\beta(x(t)) - 1)\,(\beta(x(t)) + 2)$ with $\beta(x(t)) = \int_{-1}^0 x(t+s)\, ds$.

Corresponding to Eq.(1), we have $p = 4$, $T = 1$, $g(t, x) = (2 + e^x - \sin \pi t)\, x\,(x - 1)(x + 2)$, $\alpha(s) = s$, $e(t) = \cos 2\pi t$.

If choose $d_1 = 1$, $d_2 = 2$ and $r = 3$, then

$$\lim_{x \to -\infty} \sup_{t \in R} \left|\frac{g(t,x)}{x^{p-1}}\right| \leq r \text{ and } \frac{r\,T^{p+2}}{2^{p+1}} = \frac{3}{32} < 1.$$

So by applying Theorem 3.1 we know that Eq.(19) has at least one 1-periodic solution.

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