THE GALLAI-RAMSEY NUMBER FOR A TREE VERSUS COMPLETE GRAPHS

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Abstract

For a collection of graphs $G_1, G_2, \ldots, G_t$, the Gallai-Ramsey number

$$gr(G_1, G_2, \ldots, G_t)$$

is the least positive integer p such that every t-coloring of the edges of K_p contains a subgraph isomorphic to G_i spanned by edges in color i, for some $1 \leq i \leq t$. This note focuses on the evaluation of the Gallai-Ramsey number

$$gr(T, K_{s_1}, K_{s_2}, \ldots, K_{s_t}),$$

where T is a tree. We offer several exact evaluations that build off of known results and conclude with an overview of critical colorings for such Gallai-Ramsey numbers.

– Dedicated to the memory of Ron Graham.

1. Introduction

Gallai-Ramsey numbers are a common variation of graph Ramsey numbers. Their name is derived from the close connection that rainbow triangle-free colorings share with Gallai's foundational paper [8] on transitively orientable graphs (comparability graphs). An English translation of [8] by F. Maffray and M. Preissmann can be found in [13]. This note focuses on the evaluation of the Gallai-Ramsey number for a tree versus a collection of complete graphs, and a description of the critical colorings associated with this number. We begin with an overview of the terminology and background required for our investigation.

If G is a simple graph (avoiding loops and multiedges), we denote by $V(G)$ and $E(G)$ its vertex and edge sets, respectively. A *t-coloring* of G is a function

$$c : E(G) \longrightarrow \{1, 2, \ldots, t\}.$$

In general, we do not assume that a t-coloring is surjective. A *Gallai t-coloring* is a t-coloring that avoids rainbow triangles. That is, there are no instances of distinct vertices x, y, and z such that $|\{c(xy), c(yz), c(xz)\}| = 3$. When $t = 1$ or $t = 2$, observe that every t-coloring is a Gallai t-coloring.

If $G_1, G_2, \ldots, G_t$ are graphs, then the *Ramsey number* $r(G_1, G_2, \ldots, G_t)$ is defined to be the least positive integer p such that every t-coloring of the complete graph K_p of order p contains a subgraph isomorphic to G_i spanned by edges in color i, for some $1 \le i \le p$. The existence of Ramsey numbers follows from the ubiquitous theorem of Frank Ramsey [14]. Analogously, the *Gallai-Ramsey number* $gr(G_1, G_2, \ldots, G_t)$ is the least positive integer p such that every Gallai t-coloring of K_p contains a subgraph isomorphic to G_i spanned by edges in color i, for some $1 \le i \le t$. Since every Gallai t-coloring is a t-coloring, it follows that

$$gr(G_1, G_2, \ldots, G_t) \le r(G_1, G_2, \ldots, G_t).$$

If $G = G_1 = G_2 = \cdots = G_t$, then we write $gr^t(G)$ for the corresponding t-color Gallai-Ramsey number. Most research on Gallai-Ramsey numbers has focused on the “diagonal” case $gr^t(G)$ (for example, see [2], [5], [7], [9], and [11]). One of the earliest known results in this area is due to Chung and Graham [2], where in 1983, they proved a result equivalent to the statement

$$gr^t(K_3) = \begin{cases} 5^{t/2} + 1 & \text{if } t \text{ is even} \\ 2 \cdot 5^{(t-1)/2} + 1 & \text{if } t \text{ is odd,} \end{cases}$$

whenever $t \ge 2$. This result will prove to be useful to us in Section 2.

Recall that a *tree* T is a connected acyclic graph. Throughout the remainder of this note, assume that T_m is any tree of order m. In 1972, Chvátal and Harary [4] proved a general lower bound for 2-color Ramsey numbers that implied

$$r(T_m, K_n) \ge (m-1)(n-1) + 1.$$

Five years later, Chvátal [3] was able to complete the proof that

$$r(T_m, K_n) = (m-1)(n-1) + 1. \tag{1}$$

Our main result concerns the evaluation of of the $(t+1)$-colored Gallai-Ramsey number $gr(T_m, K_{s_1}, K_{s_2}, \ldots, K_{s_t})$. Specifically, in Theorem 1, we prove that

$$gr(T_m, K_{s_1}, K_{s_2}, \ldots, K_{s_t}) = (m-1)(gr(K_{s_1}, K_{s_2}, \ldots, K_{s_t}) - 1) + 1.$$

Known evaluations of $gr(K_{s_1}, \ldots, K_{s_t})$ then allow us to obtain explicit evaluations. Finally, we consider the critical colorings for $gr(T_m, K_{s_1}, K_{s_2}, \ldots, K_{s_t})$ and discuss the “goodness” of graphs in this setting.

2. The Evaluation of $gr(T, K_{s_1}, K_{s_2}, \ldots, K_{s_t})$

We begin this section with the main result of this note.

Theorem 1. *Let $t \geq 2$ and $m \geq 1$. Then*

$$gr(T_m, K_{s_1}, K_{s_2}, \ldots, K_{s_t}) = (m-1)(gr(K_{s_1}, \ldots, K_{s_t}) - 1) + 1.$$

Proof. Let $n = gr(K_{s_1}, K_{s_2}, \ldots, K_{s_t})$ and fix a Gallai t-coloring of K_{n-1} that avoids a monochromatic copy of K_{s_i} in color i, for all $1 \leq i \leq t$. Replace each of the vertices in this K_{n-1} with complete red copies of K_{m-1} to form a $(t+1)$-colored $K_{(m-1)(n-1)}$. Clearly, no red T_m exists since the largest red component only contains $m-1$ vertices. The largest complete subgraph in colors other than red contain at most one vertex from each K_{m-1}, so this construction lacks monochromatic copies of K_{s_i} in colors $1 \leq i \leq t$. It is also easy to verify that the resulting coloring is a Gallai coloring. It follows that

$$gr(T_m, K_{s_1}, K_{s_2}, \ldots, K_{s_t}) \geq (m-1)(n-1) + 1.$$

To prove the other direction, consider a Gallai $(t+1)$-coloring of $K_{(m-1)(n-1)+1}$. If we identify the last t colors together, we obtain a 2-coloring of $K_{(m-1)(n-1)+1}$. By Equation (1), it follows that there is a red T_m or a copy of K_n spanned by edges using only colors $1 \leq i \leq t$. In the former case, we are done. In the latter case, the K_n is Gallai t-colored, and since $gr(K_{s_1}, K_{s_2}, \ldots, K_{s_t}) = n$, it follows that there is a monochromatic copy of K_{s_i} in color i, for some $1 \leq i \leq t$. Hence,

$$gr(T_m, K_{s_1}, K_{s_2}, \ldots, K_{s_t}) \leq (m-1)(n-1) + 1,$$

completing the proof of the theorem. □

When $t = 2$, observe that $r(K_{s_1}, K_{s_2}) = gr(K_{s_1}, K_{s_2})$. This allows us to apply known nontrivial 2-color classical Ramsey numbers to obtain 3-color Gallai-Ramsey numbers (see Section 2.1 of [12]). A list of these results are contained in Table 1.

Next, we apply Chung and Graham's result [2]:

$$gr^t(K_3) = \begin{cases} 5^{t/2} + 1 & \text{if } t \text{ is even} \\ 2 \cdot 5^{(t-1)/2} + 1 & \text{if } t \text{ is odd.} \end{cases}$$

Theorem 1 implies that the $(t+1)$-color Gallai-Ramsey number satisfies

$$gr(T_m, \underbrace{K_3, \ldots, K_3}_{t \text{ terms}}) = \begin{cases} (m-1)5^{t/2} + 1 & \text{if } t \text{ is even} \\ 2(m-1)5^{(t-1)/2} + 1 & \text{if } t \text{ is odd.} \end{cases}$$

Similarly, the recent evaluation

$$gr^t(K_4) = \begin{cases} 17^{t/2} + 1 & \text{if } t \text{ is even} \\ 3 \cdot 17^{(t-1)/2} + 1 & \text{if } t \text{ is odd,} \end{cases}$$

$r(K_{s_1}, K_{s_2})$	$gr(T_m, K_{s_1}, K_{s_2})$
$r(K_3, K_3) = 6$	$gr(T_m, K_3, K_3) = 5m - 4$
$r(K_3, K_4) = 9$	$gr(T_m, K_3, K_4) = 8m - 7$
$r(K_3, K_5) = 14$	$gr(T_m, K_3, K_5) = 13m - 12$
$r(K_3, K_6) = 18$	$gr(T_m, K_3, K_6) = 17m - 16$
$r(K_3, K_7) = 23$	$gr(T_m, K_3, K_7) = 22m - 21$
$r(K_3, K_8) = 28$	$gr(T_m, K_3, K_8) = 27m - 26$
$r(K_3, K_9) = 36$	$gr(T_m, K_3, K_9) = 35m - 34$
$r(K_4, K_4) = 18$	$gr(T_m, K_4, K_4) = 17m - 16$
$r(K_4, K_5) = 25$	$gr(T_m, K_4, K_5) = 24m - 23$

Table 1: Gallai-Ramsey numbers that follow from the known nontrivial 2-color classical Ramsey numbers compiled in Radziszowski's dynamic survey [12].

by Liu, Magnant, Saito, Schiermeyer, and Shi [11] implies that

$$gr(T_m, \underbrace{K_4, \ldots, K_4}_{t \ terms}) = \begin{cases} (m-1)17^{t/2} + 1 & \text{if } t \text{ is even} \\ 3(m-1)17^{(t-1)/2} + 1 & \text{if } t \text{ is odd.} \end{cases}$$

A well-known conjecture of Fox, Grinshpun, and Pach (Conjecture 1.7 of [6]) states that

$$gr^t(K_n) = \begin{cases} (r(K_n, K_n) - 1)^{t/2} + 1 & \text{if } t \text{ is even} \\ (n-1)(r(K_n, K_n) - 1)^{(t-1)/2} + 1 & \text{if } t \text{ is odd,} \end{cases}$$

which, if proved, would imply a similar result as in the cases $n = 3, 4$.

3. Critical Colorings and Good Graphs

The construction given in the proof of Theorem 1 to obtain the lower bound for $gr(T_m, K_{s_1}, K_{s_2}, \ldots, K_{s_t})$ turns out to be the only such construction. To be precise, if $p = gr(G_1, G_2, \ldots, G_t)$, then a *critical coloring* of K_{p-1} is a t-coloring that lacks a subgraph isomorphic to G_i spanned by edges in color i, for all $1 \leq i \leq t$. To determine a critical coloring for $gr(T_m, K_{s_1}, K_{s_2} \ldots, K_{s_t})$, let $n = gr(K_{s_1}, K_{s_2}, \ldots, K_{s_t})$ and identify the last t-colors together. We know from Theorem 1 and Equation (1) that

$$gr(T_m, K_{s_1}, K_{s_2}, \ldots, K_{s_t}) = (m-1)(n-1) + 1 = r(T_m, K_n).$$

It was proved by Hook and Isaak (Proposition 2.4 of [10]) that the only critical colorings for $r(T_m, K_n)$ are formed by taking a blue K_{n-1} and replacing each of its vertices with a red K_{m-1}. Thus, the only critical colorings for $gr(T_m, K_{s_1}, K_{s_2}, \ldots, K_{s_t})$ are formed by taking a Gallai t-coloring of K_{n-1} that lacks a subgraph isomorphic to K_{s_i} in color i, for all $1 \leq i \leq t$, and replacing each vertex with a red copy of K_{m-1}.

Since every connected graph G contains a spanning tree, it follows that if G has order m, then

$$gr(G, K_{s_1}, K_{s_2}, \ldots, K_{s_t}) \geq (m-1)(gr(K_{s_1}, \ldots, K_{s_t}) - 1) + 1. \tag{2}$$

Building on the concept of "goodness" introduced by Burr and Erdős [1], we say that G is *Gallai-*$\{K_{s_1}, K_{s_2}, \ldots, K_{s_t}\}$*-good* if equality holds in Inequality (2). At the present time, the determination of which G are Gallai-$\{K_{s_1}, K_{s_2} \ldots, K_{s_t}\}$-good is an open problem. A good starting point for investigating this problem is motivated by the work of Chung and Graham [2]: identify the connected graphs G of order m that satisfy

$$gr(G, \underbrace{K_3, \ldots, K_3}_{t\ terms}) = \begin{cases} (m-1)5^{t/2} + 1 & \text{if } t \text{ is even} \\ 2(m-1)5^{(t-1)/2} + 1 & \text{if } t \text{ is odd.} \end{cases}$$

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