ON AN INEQUALITY IN A 1970 PAPER OF R. L. GRAHAM

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Abstract

Having recently come across a 1970 paper of R. L. Graham about cube-numbering and its generalizations, we found that one proof uses an inequality about the summatory function of the sum of binary digits of integers. Graham gave a very elegant and somehow unexpected proof of this inequality. We propose a more "pedestrian" —and somehow more standard— proof of this inequality, as well as questions about possible generalizations.

– Dedicated to the memory of Ron Graham

1. Introduction

R. L. Graham, in a study of cube-numbering and generalizations [6] used a curious inequality for the summatory function of the sum of binary digits which we now describe. Let $w(k)$ be the number of 1's in the binary expansion of the integer k. Let $W(n) := \sum_{0 \leq k \leq n} w(k)$. Then, for all n_1, n_2 with $0 < n_1 \leq n_2$,

$$W(n_1 - 1) + W(n_2 - 1) + n_1 < W(n_1 + n_2 - 1) + 1. \quad (1)$$

The very elegant proof of this inequality given by Graham uses the following lemma.

Lemma 1 (Graham [6]). *Let r and s be nonnegative integers. If φ is a one-to-one map from $[0, r]$ to $[s, s + r]$, define*

$$\delta(\varphi) := \min_{0 \leq k \leq r} \{w(\varphi(k)) - w(k)\}.$$

Then

- *(i) There exists φ such that $\delta(\varphi)) \geq 0$.*
- *(ii) If $s > r$, then there exists φ such that $\delta(\varphi) \geq 1$.*

The proof of this lemma and of the fact that it implies Inequality (1) can be found in [6] —but also see [7]. We could not guess how Graham had the idea of this lemma. Thus we wondered whether there could be a more direct, possibly "pedestrian", proof. The point is that the sum of digit sequence (sequence A000120 in [10]), hence its summatory function (sequence A000788 in [10]), are 2-regular sequences in the sense of [1, 2]. Recall that a sequence $(u(n))_{n\geq 0}$ is called 2-regular if the $\mathbb{Z}$-module generated by its 2-kernel (i.e., the set of subsequences $\{(u(2^k n+a))_{n\geq 0},\ k\geq 0,\ a\in[0,2^k-1]\}$) is a finitely generated $\mathbb{Z}$-module. In our case, this is just a consequence of the fact that $(w(2n))_{n\geq 0}$ and $(w(2n+1))_{n\geq 0}$ are linear combinations of the sequence $(w(n))_{n\geq 0}$ and the constant sequence $(1)_{n\geq 0}$. Namely, for all $n\geq 0$, we have $w(2n)=w(n)$ and $w(2n+1)=w(n)+1$. These equalities imply equalities of a similar type for $W(n)$, and will be the basis of our proof.

2. Proof of Graham's Inequality (1)

First we rewrite Inequality (1) as: for all (n_1,n_2) with $1\leq n_1\leq n_2$,

$$W(n_1-1)+W(n_2-1)+n_1 < W(n_1+n_2-1)+1.$$

Defining $m:=n_1-1$ and $n:=n_2-1$, Inequality (1) is equivalent to Inequality (2): for all (m,n) with $0\leq m\leq n$,

$$W(m)+W(n)+m < W(m+n+1). \tag{2}$$

Lemma 2. *The following equalities hold:*

$$\begin{array}{lrcl} \textit{for all}\ \ n\geq 1, & W(2n) & = & W(n)+W(n-1)+n \\ \textit{for all}\ n\geq 0, & W(2n+1) & = & 2W(n)+n+1. \end{array}$$

Let $A(m,n):=W(m+n+1)-W(m)-W(n)-m$. Then the following equalities hold:

$$\begin{array}{lrcl} \textit{for}\ n\geq 0, & A(n,n) & = & 1\ \ \text{(hence Inequality (2) is sharp)} \\ \textit{for}\ m\geq 1\ \textit{and}\ n\geq 1, & A(2m,2n) & = & A(m-1,n)+A(m,n-1) \\ \textit{for}\ m\geq 0\ \textit{and}\ n\geq 1, & A(2m+1,2n) & = & A(m,n)+A(m,n-1)-1 \\ \textit{for}\ m\geq 1\ \textit{and}\ n\geq 0, & A(2m,2n+1) & = & A(m,n)+A(m-1,n)-1 \\ \textit{for}\ m\geq 0\ \textit{and}\ n\geq 0, & A(2m+1,2n+1) & = & 2A(m,n)-1. \end{array}$$

Proof. We write, for $\ell\geq 1$,

$$\begin{aligned} W(2\ell)=\sum_{k=0}^{2\ell} w(k) \quad &= \sum_{j=0}^{\ell} w(2k)+\sum_{j=0}^{\ell-1} w(2k+1) \\ &= \sum_{j=0}^{\ell} w(k)+\sum_{j=0}^{\ell-1}(w(k)+1) \\ &= W(\ell)+W(\ell-1)+\ell \end{aligned}$$

and, for $\ell \geq 0$,

$$\begin{aligned} W(2\ell+1) = \sum_{k=0}^{2\ell+1} w(k) \quad &= \sum_{j=0}^{\ell} w(2k) + \sum_{j=0}^{\ell} w(2k+1) \\ &= \sum_{j=0}^{\ell} w(k) + \sum_{j=0}^{\ell} (w(k)+1) \\ &= 2W(\ell) + \ell + 1. \end{aligned}$$

Using these relations for $W(2\ell)$ and $W(2\ell+1)$, we obtain successively:

$$\text{For all } n \geq 0,\ A(n,n) = W(2n+1) - 2W(n) - n = 1$$

and the following relations.

- For all (m,n) with $m \geq 1$, $n \geq 1$,

$$\begin{aligned} A(2m,2n) &= W(2m+2n+1) - W(2m) - W(2n) - 2m \\ &= 2W(m+n) - W(m) - W(m-1) \\ &\qquad -W(n) - W(n-1) - 2m + 1 \\ &= A(m-1,n) + A(m,n-1). \end{aligned}$$

- For all (m,n) with $m \geq 0$, $n \geq 1$,

$$\begin{aligned} A(2m+1,2n) &= W(2m+2n+2) - W(2m+1) - W(2n) - 2m - 1 \\ &= W(m+n+1) + W(m+n) - 2W(m) \\ &\qquad -W(n) - W(n-1) - 2m - 1 \\ &= A(m,n) + A(m,n-1) - 1. \end{aligned}$$

- For all (m,n) with $m \geq 1$, $n \geq 0$

$$\begin{aligned} A(2m,2n+1) &= W(2m+2n+2) - W(2m) - W(2n+1) - 2m \\ &= W(m+n+1) + W(m+n) - W(m) \\ &\qquad -W(m-1) - 2W(n) - 2m \\ &= A(m,n) + A(m-1,n) - 1. \end{aligned}$$

- For all (m,n) with $m \geq 0$, $n \geq 0$

$$\begin{aligned} A(2m+1,2n+1) &= W(2m+2n+3) - W(2m+1) - W(2n+1) - 2m - 1 \\ &= 2W(m+n+1) - 2W(m) - 2W(n) - 2m - 1 \\ &= 2A(m,n) - 1. \end{aligned}$$

$\square$

Now we are ready to prove the following proposition which is the result of Graham described above.

Proposition. *Inequality (1) holds.*

Proof. As we have seen, it suffices to prove Inequality (2), namely for all (m, n) with $0 \leq m \leq n$, one has

$$W(m) + W(n) + m < W(m + n + 1).$$

We will prove the statement $(\mathcal{H}_n)$:

$$(\mathcal{H}_n): \quad \text{for all } m \in [0, n],\ A(m, n) = W(m + n + 1) - W(m) - W(n) - m > 0.$$

We will actually check that $(\mathcal{H}_0)$ holds, and prove that, if $(\mathcal{H}_r)$ holds for all $r \in [0, n]$, then $(\mathcal{H}_{2n})$ and $(\mathcal{H}_{2n+1})$ hold. Note that $(\mathcal{H}_0)$ holds trivially. Now suppose that, for some $n \geq 0$, $(\mathcal{H}_r)$ holds for all $r \in [0, n]$, i.e., $A(q, r) > 0$ for all $r \in [0, n]$ and for all $q \in [0, r]$. We look at $A(k, 2n)$ and $A(k, 2n + 1)$ for $k \leq 2n$, resp. $k \leq 2n + 1$.

- $A(k, 2n)$

  We can (and will) suppose that $n > 0$.

  – If $k \in [0, 2n]$ is even, say $k = 2\ell$, we may suppose that $k \in [1, 2n - 1]$, since the case $k = 0$ is trivial, and the case $k = n$ is deduced from $A(2n, 2n) = 1$ as seen above. Thus $\ell \in [1, n - 1]$. We have

  $$A(k, 2n) = A(2\ell, 2n) = A(\ell - 1, n) + A(\ell, n - 1) > 0$$

  using first Lemma 2, then the induction hypothesis.

  – If $k \in [0, 2n]$ is odd, say $k = 2\ell + 1$, we have that $k \in [1, 2n - 1]$. Thus $\ell \in [0, n - 1]$. We have

  $$A(k, 2n) = A(2\ell + 1, 2n) = A(\ell, n) + A(\ell, n - 1) - 1 > 0$$

  using first Lemma 2, then the induction hypothesis.

- $A(k, 2n + 1)$

  – If $k \in [0, 2n + 1]$ is even, say $k = 2\ell$. We may suppose $k \neq 0$ since the case $k = 0$ is trivial. Then $\ell \in [1, n]$. Thus

  $$A(k, 2n + 1) = A(2\ell, 2n + 1) = A(\ell, n) + A(\ell - 1, n) - 1 > 0$$

  using first Lemma 2, then the induction hypothesis.

  – If $k \in [0, 2n + 1]$ is odd, say $k = 2\ell + 1$. Thus $\ell \in [0, n]$. Thus

  $$A(k, 2n + 1) = A(2\ell + 1, 2n + 1) = 2A(\ell, n) - 1 > 0$$

  using first Lemma 2, then the induction hypothesis.

□

Remark 1. It is noted in [7] that $W(n) \leq \frac{1}{2} n \log_2 n$. Actually we know more: from [4] (also see [13]), we have $W(n) = \frac{1}{2} n \log_2 n + nF(\log_2 n)$, where F is a periodic, continuous, nowhere differentiable function with an absolutely convergent Fourier series. The function F, called the Trollope-Delange function, is closely related to the Takagi function [12] (see, e.g., [8, 9], and [11]). The evaluation of $W(n)$, F, and generalizations is still an active subject of research (see, e.g., [5]).

Remark 2. It is possible to obtain an upper bound for $A(m,n)$ by using the inequality $w(m+n) \leq w(m) + w(n)$ for all m, n. This inequality can be found, e.g., in a comment by Shevelev about sequence A000120 [10]. Namely one has $w(m) + w(n) - w(n+m) = v_2(\binom{n+m}{n})$ which is a consequence of Legendre's result [3, p. 10–12]: $w(n) = n - \nu_2(n!)$, where $\nu_2(k)$ is the 2-adic valuation of k. Hence

$$\begin{aligned} A(m,n) &= \sum_{j=0}^{m+n+1} w(j) - \sum_{j=0}^{n} w(j) - \sum_{j=0}^{m} w(j) - m \\ &= \sum_{j=n+1}^{m+n+1} w(j) - \sum_{j=0}^{m} w(j) - m = \sum_{j=0}^{m} w(j+n+1) - \sum_{j=0}^{m} w(j) - m \\ &\leq \sum_{j=0}^{m} w(n+1) - m = (m+1)w(n+1) - m. \end{aligned}$$

Note that this inequality is sharp (e.g., take $m = n = 2^k - 1$ for some $k \geq 1$).

3. Conclusion

A first possible generalization of this inequality is to replace base 2 with base $d \geq 2$, and w with the sum of digits in base d. But, since our proof essentially uses the 2-regularity of the sequence $(w(n))_{n\geq 0}$ and (thus) of its summatory function $(W(n))_{n\geq 0}$, one can ask whether some similar "super-superadditivity" result holds for summatory functions of all 2-regular (resp. d-regular) sequences. A more reasonable question could be whether summatory functions of pattern-counting sequences have such a property: the sequence $(w(n))_{n\geq 0}$ counts the number of 1's in the binary expansion of n, thus one of the first examples to test would be the sequence $(u(n))_{n\geq 0}$ that counts the number of possibly overlapping blocks 11 in the binary expansion of n (this is sequence A014081 in [10]). The difficulty is then that, instead of having the relations $w(2n) = w(n)$ and $w(2n+1) = w(n) + 1$, we have relations for a three-dimensional vector $z(n)$:

$$z(n) := \begin{pmatrix} u(n) \\ u(2n+1) \\ 1 \end{pmatrix} \Rightarrow \ z(2n) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} z(n), \quad z(2n+1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} z(n).$$

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