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ON LACUNARY ARITHMETIC STATISTICAL CONTINUITY FOR DOUBLE SEQUENCES

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Abstract

In this article, we shall introduce the concept of lacunary arithmetic statistical continuity for double sequences and investigate some inclusion relations.

Keywords: Summability; Arithmetic Statistical Convergence; Lacunary Arithmetic Statistical Convergence; Lacunary Arithmetic Statistical Continuity; Double Sequences.

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1. Introduction

The concept of statistical convergence was introduced by Fast [4] and it was further investigated from the sequence space point of view and linked with summability theory by Fridy [2], Connor [3], Fridy and Orhan [1], Šalát [5] and many others.

While the idea of arithmetic convergence was introduced by Ruckle [9]. Yaying and Hazarika [8] used this concept of arithmetic convergence and introduced arithmetic statistical convergence and lacunary arithmetic statistical convergence of single sequence. Also Yaying and Hazarika [8] establish some sequential properties of lacunary arithmetic statistical continuity of single sequence. The concept of statistical convergence of double sequences was introduced by Mursaleen [6]. Using the method of Mursaleen, we shall extend the results of Yaying and Hazarika [8] to double sequences as follows:

2. Lacunary Arithmetic Statistical Continuity *(First we Noted)*

Definition 2.1: (Yaying and Hazarika [2017]) A sequence $x = (x_k)$ is called arithmetically convergent if for each $\varepsilon > 0$ there is an integer l such that for every integer k we have $\left|x_k - x_{\langle k,l\rangle}\right| < \varepsilon$, where the symbol $\langle k, l\rangle$ denotes the greatest common divisor of two integers k and l. We denote the sequence space of all arithmetic convergent sequence by AC.

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Definition 2.2: (Fridy and Orhan [1993]) Let $\theta = (k_r)$ be a lacunary sequence. A number sequence $x = (x_k)$ is said to be lacunary statistically convergent to l or S_θ-convergent to l, if, for each $\varepsilon > 0$,

$\lim_{r\to\infty} \frac{1}{h_r} \, |\{k \in I_r : |x_k - l| \geq \varepsilon\}| = 0$

In this case, one writes $S_\theta - \lim x_k = l$ or $x_k \to (S_\theta)$.The set of all lacunary statistically convergence sequences is denoted by S_θ

Definition 2.3: (Yaying and Hazarika [2017]) A sequence $x = (x_k)$ is said to be arithmetic statistically convergent if for each $\varepsilon > 0$, there is an integer *l* such that

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{k \in n : \left|x_k - x_{\langle k,l\rangle}\right| \geq \varepsilon \right\}\right| = 0$$

We shall use ASC to denote the set of all arithmetic statistical convergent sequences. Thus for $\varepsilon > 0$ and integer l

$$\text{ASC} = \left\{(x_k) : \lim_{n\to\infty} \frac{1}{n} \left|\left\{k \in n : \left|x_k - x_{\langle k,l\rangle}\right| \geq \varepsilon \right\}\right| = 0 \right\}.$$

We shall write $ASC - \lim x_k = x_{\langle k,l\rangle}$ to denote the sequence (x_k) is arithmetic statistically convergent to$x_{\langle k,l\rangle}$.

Definition 2.4: (Yaying and Hazarika [2017]) Let $\theta = (k_r)$ be a lacunary sequence. The number sequence $x = (x_k)$ is said to be lacunary arithmetic statistically convergent if for each $\varepsilon > 0$ there is an integer *l* such that

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r : \left|x_k - x_{\langle k,l\rangle}\right| \geq \varepsilon \right\}\right| = 0$$

We shall write

$$ASC_\theta = \left\{x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r : \left|x_k - x_{\langle k,l\rangle}\right| \geq \varepsilon \right\}\right| = 0 \right\}.$$

We shall write $ASC_\theta - \lim x_k = x_{\langle k,l\rangle}$ to denote the sequence (x_k) is lacunary arithmetic statistically convergent to$x_{\langle k,l\rangle}$.

Definition 2.5: (Yaying and Hazarika [2017]) A function f defined on a subset $E \text{ of } \mathbb{R}$ is said to be lacunary arithmetic statistical continuous if it preserves lacunary arithmetic statistical convergence i.e. if

$$ASC_\theta - \lim x_k = x_{\langle k,l\rangle} \text{ Implies } ASC_\theta - \lim f(x_k) = f(x_{\langle k,l\rangle}).$$

Theorem 2.1: (Yaying and Hazarika [2017]) Let (f_m), $m \in \mathbb{N}$ be sequence of ASC_θ continuous functions defined on a subset of E of $\mathbb{R}$ and f_m , be uniformly convergent to a function f, then f is ASC_θ continuous.

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Theorem 2.2: (Yaying and Hazarika [2017]) The set of all ASC_θ continuous functions defined are on a subset E of $\mathbb{R}$ is a closed subset of all continuous function on E, i.e. $\overline{ASC_\theta(E)} = ASC_\theta(E)$, where $ASC_\theta(E)$ denotes the set of all ASC_θ continuous functions defined on E and $\overline{ASC_\theta(E)}$ denotes the closure of $ASC_\theta(E)$.

We shall now use the concept of statistical convergence to extend above concept and result to double sequences, using Analogy;

3. Lacunary Arithmetic Statistical Continuity For Double Sequences

Definition 3.1: A function f defined on a subset $D \; of \; \mathbb{R}$ is said to be lacunary arithmetic statistical continuous for double sequences if it preserves lacunary arithmetic statistical convergence for double sequences i.e. if

$$ASC_{\theta_{r,s}} - \lim x_{k,m} = x_{\langle \langle k,l \rangle, \langle m,n \rangle \rangle} \text{ Implies} ASC_{\theta_{r,s}} - \lim f(x_{k,m}) = f(x_{\langle \langle k,l \rangle, \langle m,n \rangle \rangle}).$$

Where the symbol $\langle k, l, m, n \rangle$ denotes the greatest common divisor of four integers k , l, m and n. We shall write $ASC_{\theta_{r,s}}$ continuous function to denote lacunary arithmetic statistical continuous for double sequences. It is easy to see that the sum and the difference of two $ASC_{\theta_{r,s}}$ continuous functions is $ASC_{\theta_{r,s}}$ continuous. Also the composition of two $ASC_{\theta_{r,s}}$ continuous functions is again $ASC_{\theta_{r,s}}$ continuous. In the classical case, it is known that the uniform limit of sequentially continuous function is sequentially continuous, now we see that the uniform limit of $ASC_{\theta_{r,s}}$ continuous functions is also $ASC_{\theta_{r,s}}$ continuous.

Theorem 3.1: Let $(f_{k,m})$, $k, m \in \mathbb{N}$ be sequence of $ASC_{\theta_{r,s}}$ continuous functions defined on a subset of D of $\mathbb{R}$ and $f_{k,m}$, be uniformly convergent to a function f, then f is $ASC_{\theta_{r,s}}$ continuous.

Proof 3.1: Let $\varepsilon > 0$ and $(x_{k,m})$ be any $ASC_{\theta_{r,s}}$ convergent sequence on a subset D of $\mathbb{R}$. By the uniform convergence of $f_{k,m}$, there exist $N \in \mathbb{N}$ such that $|f_{k,m}(x) - f(x)| \leq \frac{\varepsilon}{3}$ for all $k, m \in N$ and for all$x \in D$. Since f_N is continuous on D, we have for an integerl, n.

$$\lim_{r,s \to \infty} \frac{1}{h_{r,s}} \left| \left\{ k, m \in I_{r,s}: \left| f_N(x_{k,m}) - f_N(x_{\langle \langle k,l \rangle, \langle m,n \rangle \rangle}) \right| \geq \frac{\varepsilon}{3} \right\} \right| = 0$$

On the other hand, for an integer l, n we have

$$\left\{ k, m \in I_{r,s}: \left| f(x_{k,m}) - f(x_{\langle \langle k,l \rangle, \langle m,n \rangle \rangle}) \right| \geq \frac{\varepsilon}{3} \right\} \subset \left\{ k, m \in I_{r,s}: \left| f_N(x_{k,m}) - f(x_{\langle \langle k,l \rangle, \langle m,n \rangle \rangle}) \right| \geq \frac{\varepsilon}{3} \right\}$$
$$\cup \left\{ k, m \in I_{r,s}: \left| f_N(x_{\langle \langle k,l \rangle, \langle m,n \rangle \rangle}) - f_N(x_{k,m}) \right| \geq \frac{\varepsilon}{3} \right\} \cup \left\{ k, m \in I_{r,s}: \left| f_N(x_{k,m}) - f(x_{k,m}) \right| \geq \frac{\varepsilon}{3} \right\}$$

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Thus it follows from the above inclusion that

$$\lim_{r,s\to\infty}\frac{1}{h_{r,s}}\left|\left\{k,m\in I_{r,s}:\left|f(x_{k,m})-f(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle})\right|\geq\varepsilon\right\}\right| \leq \lim_{r,s\to\infty}\frac{1}{h_{r,s}}\left|\left\{k,m\in I_{r,s}:\left|f_N(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle})-f(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle})\right|\geq\frac{\varepsilon}{3}\right\}\right| + \lim_{r,s\to\infty}\frac{1}{h_{r,s}}\left|\left\{k,m\in I_{r,s}:\left|f_N(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle})-f_N(x_{k,m})\right|\geq\frac{\varepsilon}{3}\right\}\right| + \lim_{r,s\to\infty}\frac{1}{h_{r,s}}\left|\left\{k,m\in I_{r,s}:\left|f_N(x_{k,m})-f(x_{k,m})\right|\geq\frac{\varepsilon}{3}\right\}\right|$$

Thus, f *is* $ASC_{\theta_{r,s}}$ continuous. ■

Theorem 3.2: The set of all $ASC_{\theta_{r,s}}$ continuous functions defined on a subset D of $\mathbb{R}$ is a closed subset of all continuous function on D, i.e. $\overline{ASC_{\theta_{r,s}}(D)} = ASC_{\theta_{r,s}}(D)$, where $ASC_{\theta_{r,s}}(D)$ denotes the set of all $ASC_{\theta_{r,s}}$ continuous functions defined on D and $\overline{ASC_{\theta_{r,s}}(D)}$ denotes the closure of $ASC_{\theta_{r,s}}(D)$.

Proof 3.2: Let f be any element of $\overline{ASC_{\theta_{r,s}}(D)}$ .Then there exist a sequence of points in $ASC_{\theta_{r,s}}(D)$ such that $lim f_{k,m} = f$. Now let $(x_{k,m})$ be any $ASC_{\theta_{r,s}}$ convergent sequence in D. Since $(f_{k,m})$ converges to f, there exist a positive integer N such that

$$\left|f(x)-f_{k,m}(x)\right| \leq \frac{\varepsilon}{3}, \forall\, k,m\geq N \text{ and } \forall\, x\in D$$

Now f_N is $ASC_{\theta_{r,s}}$ continuous on D, so we have for an integer l,n

$$\lim_{r,s\to\infty}\frac{1}{h_{r,s}}\left|\left\{k,m\in I_{r,s}:\left|f_N(x_{k,m})-f_N(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle})\right|\geq\frac{\varepsilon}{3}\right\}\right|=0$$

On the other hand, for an integer l,n we have

$$\left\{k,m\in I_{r,s}:\left|f(x_{k,m})-f(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle})\right|\geq\frac{\varepsilon}{3}\right\}$$

$$\subset \left|\left\{k,m\in I_{r,s}:\left|f_N(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle})-f(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle})\right|\geq\frac{\varepsilon}{3}\right\}\right|$$

$$\cup\left\{k,m\in I_{r,s}:\left|f_N(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle})-f_N(x_{k,m})\right|\geq\frac{\varepsilon}{3}\right\}$$

$$\cup\left\{k,m\in I_{r,s}:\left|f_N(x_{k,m})-f(x_{k,m})\right|\geq\frac{\varepsilon}{3}\right\}$$

From the above inclusion we can write

$$\lim_{r,s\to\infty}\frac{1}{h_{r,s}}\left|\left\{k,m\in I_{r,s}:\left|f(x_{k,m})-f(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle})\right|\geq\varepsilon\right\}\right|$$

$$\leq \lim_{r,s\to\infty}\frac{1}{h_{r,s}}\left|\left\{k,m\in I_{r,s}:\left|f_N(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle})-f(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle})\right|\geq\frac{\varepsilon}{3}\right\}\right|$$

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$$+ \lim_{r,s\to\infty} \frac{1}{h_{r,s}} \left| \left\{ k,m \in I_{r,s} : \left| f_N(x_{\langle \langle k,l \rangle, \langle m,n \rangle \rangle}) - f_N(x_{k,m}) \right| \geq \frac{\varepsilon}{3} \right\} \right|$$

$$+ \lim_{r,s\to\infty} \frac{1}{h_{r,s}} \left| \left\{ k,m \in I_{r,s} : \left| f_N(x_{k,m}) - f(x_{k,m}) \right| \geq \frac{\varepsilon}{3} \right\} \right| = 0$$

Thus f is $ASC_{\theta_{r,s}}$ continuous, so $f \in ASC_{\theta_{r,s}}(D)$ which gives us our required result. ■

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