Exploring heterogeneity and expectations in technical progress – an exercise in agentization

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Abstract

Endogenous technical progress is often modelled through learning by doing, in that technical progress is assumed to evolve proportionally to the capital stock. For the standard Ramsey growth model, an optimal growth path can be obtained assuming a representative agent who maximises intertemporal utility. In a decentralised economy with many agents, it is generally assumed that actors in the economy disregard their contribution to technical progress, leading to a suboptimal outcome as the external effect of investment leading to technical progress is not internalised. This paper presents an agent-based model that builds on a standard Ramsey growth model for investigating further settings. It introduces different investment strategies for agents so that they act like the representative agent or take their contribution to technical progress into account. In the latter case, agents need to form expectations about the investment of others. The paper explores heterogeneity of agents in terms of investment strategy, initial capital stocks, and heuristics for updating expectations. It finds that standard economic results can be reproduced in systems of homogeneous agents, that unequal capital distributions converge to equal ones leading to higher growth, that taking into account one's contribution to technical progress does not qualitatively change the picture, and that the economic performance of a system does not depend on how well agents' expectations match the true values of investment of others.

1 Introduction

Technological progress is generally assumed to be the driving force behind economic growth [Romer, 1990]. Solow [1957] defines technological change as "any kind of shift in the production function" (p.312). The occurrence of endogenous technological change can be explained as a consequence of innovation processes, and the most essential input factor in innovation is knowledge. There are various mechanisms to explain gaining new knowledge, e.g. learning by doing or knowledge spillovers [Dawid, 2006]. In this work, we treat knowledge accumulation implicitly, through a technical progress term that is an input to the production function in a basic Ramsey growth model. Technical progress here evolves with the capital stock, representing a form of learning by doing.

This paper originates in the context of a line of work investigating economic mechanisms behind the possibility of green growth [see Schütze et al., 2017, and references therein]. It directly builds on the model presented by Steudle et al. [2018], who add three mechanisms to a strongly simplified Ramsey growth model structure: technical progress through learning-by-doing, directed technical progress for brown and green capital stock, and a labour market with search. Focusing on just two time steps, the authors provide a proof of concept on how with these three building blocks a coordination game structure arises, where agents can coordinate on a better "green" equilibrium rather than staying in the given "brown" one with lower investment, lower growth, and higher unemployment. Here, we add a dynamic component; in the longer run, this shall enable investigations of transitions between such equilibria. However, to start simple, for now we focus on the first of these mechanisms only: endogenous technical progress with spillover effects through learning by doing.

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In the literature, the standard approach for analysing the Ramsey-Cass-Koopmans growth model uses a representative agent framework, since for heterogeneous agents the problem of an optimal growth path is not tractable [Acemoglu, 2009]. The representative household maximises its dynastic utility, where time can be modelled as discrete or continuous. While the household's investment drives capital accumulation and hence growth, the household's investment and consumption are complementary variables – the produced good is divided among the two purposes of consumption and investment. And as utility depends on consumption, this is generally used as the control variable. In the continuous-time case, the optimisation problem is solved using a Hamiltonian function to obtain a differential equation that describes the optimal consumption path for the representative household [Acemoglu, 2009, Aghion and Howitt, 2009, Barro and Sala-i-Martin, 2004]. In the discrete-time case, the maximisation problem can be solved, e.g., using dynamic programming. In this paper, a discrete time framework is considered. Steudle et al. [2018] use investment as the control variable in solving the agent's optimisation problem. To our knowledge, the use of this control variable is not widespread in the literature, but we adopt it here as well, as we are interested in different investment strategies of agents and their effects on growth.

For decentralised economies, the literature generally assumes that agents neglect their contribution to technical progress, as each agent is small in comparison to the overall economy. This, however, leads to a suboptimal result in that agents invest less than what would be socially optimal; the external effect of technical progress increasing with capital and hence with investment is not internalised. Subsidies for investment are proposed as a political means of obtaining the socially optimal growth path in a decentralised economy. In this paper we want to go beyond this assumption for decentralised economies; we analyse agents who, even in a decentralised economy, invest in the socially optimal way, and agents that do take their contribution to technical progress into account in their investment decision. In the latter case, as will be seen, agents need to be equipped with expectations about how much other agents invest and hence also with heuristics for updating their expectations. One question of particular interest is whether, and if so under which conditions, an economy consisting of agents who use these investment strategies that deviate from the usual "ignoring of the contribution made" can reach the socially optimal pathway laid out by the representative agent's optimal solution. If so, expectation management, e.g. through policies, might provide an alternative to the above-mentioned subsidies. We will investigate these questions by exploring results of an agent-based model (ABM): employing a bottom-up approach, agents are developed using concepts from a standard Ramsey growth model and several investment strategies as well as heuristics for forming expectations are introduced.

We follow the scheme proposed by Guerrero and Axtell [2011] for the method of agentization: a first model represents the "crudest possible" computational implementation of a Ramsey growth model with many agents, where all act according to the well-known investment strategy that neglects their contribution to technical progress. In addition, we define a strategy that corresponds to that of the representative agent but in a multi-agent setting. The only interaction between agents in this setting is the effect of technical progress that arises from their aggregate investment. They experience it through the amount produced. With this basic model, standard economic results, such as Kaldor facts and the above-described case of the decentralised economy not reaching the socially optimal growth path, can be reproduced. It will further be seen that a decentralised economy of agents who invest like a benevolent planner corresponds to the representative agent's path; but only if their initial capital is distributed equally. For unequal economies, we find that they converge to equal ones, increasing their growth rate through doing so – an aspect of the model that seems too good to be true.

We then move to a richer model with agents that consider their contribution to technical progress and form expectations about other agents' investments. We explore several assumptions in this context, for example on different heuristics that agents use to formulate their expectations. Essentially, this investment strategy of agents does not qualitatively change the picture, and perhaps surprisingly, a system of good forecasters does not translate into better macroeconomic performance.

Throughout this work, we remain in a setting that could be called static in the sense that agents do not change their investment strategy – we leave agents that would adapt their strategies to future work.

The paper is organised as follows. Section 2 provides the notation for the underlying Ramsey growth model and introduces agents with different investment strategies. Section 3 introduces the agent-based model developed on this basis. Section 4 presents results from the agent-based model and discusses these against the background of standard results before Section 5 concludes.

2 Agents in a basic Ramsey growth model

The starting point for this paper is a one-sector Ramsey growth model with endogenous technical progress. To focus on key mechanisms and interactions in a multi-agent system, Steudle et al. [2018] further restrict the model to only two-time steps: present ($t = 0$), and future ($t = 1$). In the most basic version, the agents are "farms" that combine activities usually attributed to firms (like production) and households (like labour and consumption). Production is based on capital K and labour L, where the latter is augmented by technical progress η. The produced good is used for both investment I and consumption C. Farms want to maximise utility from consumption; for simplicity, it is assumed that they do not experience disutility from labour, so the maximum labour amount, set equal to 1 here also for simplicity, is always used.

The intertemporal utility is thus given by:

$$U = u(C_0) + \rho u(C_1). \tag{1}$$

where $u : \mathbb{R} \to \mathbb{R}$ is a felicity function, C_t denotes consumption at time $t \in \{0, 1\}$ and $0 < \rho < 1$ is the discount factor for future felicity. As a simple starting point, we use the natural logarithm $u(C_t) = \ln(C_t)$ as the felicity function. Utility is to be maximised under the usual conditions for capital accumulation (with K_t the capital stock at time t and I_t the investment)

$$K_1 = (1 - \delta) \cdot K_0 + I_0$$

and with production used for consumption and investment

$$\begin{aligned} C_0 + I_0 &= f(K_0, \eta_0 L_0) = f(K_0, \eta_0) \\ C_1 &= f(K_1, \eta_1) \end{aligned} \tag{2}$$

where no investment term is needed for $t = 1$, since only two time steps are considered. As mentioned in the introduction, we will consider investment as the decision variable, which is also why we will write it as the argument of the utility function, $U(I_0)$ and analogously, in the following. In order for I_0 to be feasible[1], due to (2), it needs to take a value that is at most as large as the amount produced $f(K_0, \eta_0)$, and, if we allow for negative investment, at most the value of the capital stock after depreciation can be de-invested, so that $-(1 - \delta)K_0 < I_0 < f(K_0, \eta_0)$. It will be seen that this is fulfilled for all cases we consider.

We further assume a Cobb-Douglas production function

$$f(K, \eta) = K^\alpha \cdot \eta^{1-\alpha}, \tag{3}$$

where α is the output elasticity of capital, and we assume that technical progress evolves according to capital accumulation (through "learning by doing" [Arrow, 1962]), in formulæ $\eta_1 = K_1 \cdot \frac{\eta_0}{K_0}$. Given this basic setting, we consider different possible investment strategies for agents, based on the information they have at hand, in the following sections.

2.1 The representative agent or benevolent planner

A standard agent found in many economic models – the representative agent that acts as a "benevolent planner" – is a single agent who owns the economy's total capital stock. In this case, production reduces to the AK-model[2] with $A = (\frac{\eta_0}{K_0})^{1-\alpha}$. There is no need to explicitly consider technical progress in this setting and the optimal investment for the planner is (see Appendix A)

$$I_0^{\mathrm{BP}} = \frac{1}{1+\rho}(A\rho - 1 + \delta)K_0 = \frac{1}{1+\rho}\left(\rho P_0 - \hat{K}_0\right), \tag{4}$$

where we denote with the label "BP" that this is the "benevolent planner" investment strategy, and we use $P_0 = AK_0$ for the agent's production and $\hat{K}_0 = (1-\delta)K_0$ as notational shortcuts that will be useful

[1] At the same time, the following bounds will prevent a negative value inside the natural logarithm in the utility function.

[2] $f(K, \eta) = K^\alpha \left(K \frac{\eta_0}{K_0}\right)^{1-\alpha} = AK$

in the following. The feasibility of this investment is guaranteed by the choice of positive values of initial capital and initial technical progress (see Appendix B). The value of I_0^{BP} is positive when $A\rho > 1-\delta$, that is, $\eta_0 > \left(\frac{1-\delta}{\rho}\right)^{\frac{1}{1-\alpha}} \cdot K_0$. For example, with $\alpha = 0.33, \delta = 0.05, \rho = 0.99$, $A > 0.96$ and η_0 needs to be at least about $0.94 \cdot K_0$. This economy grows, i.e. investment is larger than just replacing depreciated capital, whenever $I_0 > \delta K_0$, that is, $A > \delta + \frac{1}{\rho}$ meaning that $\eta_0 > \left(\delta + \frac{1}{\rho}\right)^{\frac{1}{1-\alpha}} \cdot K_0$. Again, for the same parameter values, we get $A > 1.06$, and roughly $\eta_0 > 1.1 \cdot K_0$ (see also Figure 19 in Appendix A). The fact that this A is considerably larger than empirically observed[3] is due to the short time horizon. As capital is here considered productive only for two time steps, the production needs to be artificially high to create economic growth in this setting. We will, however, disregard this point in the following, as we are here interested in an abstract understanding of mechanisms, not in an empirically grounded model.

2.2 A decentralised economy with knowledge spillovers

Next, consider an economy with many agents, say n of them, labelled $i \in \{1, \dots, n\}$. We denote their variables with superscripts while variables without superscripts denote the total values of the economy, e.g., $K_t = \sum_{i=1}^n K_t^{(i)}$, where we also switch to a generic time index t. This is merely done for later ease in reusing formulæ in a dynamic setting; agents here still consider the current and one future time step only. Since we are considering "farms", that is, households who are at the same time the producers and the consumers of the economy, we still endow each of our agents with a unit amount of labour $L_t^{(i)} = 1$ for all t. We could instead provide them with the share of labour that corresponds to their initial share of capital, so that each farm initially employs capital and labour in the same ratio, that is also the economy-wide ratio, as Frankel [1962] describes. However, this equal ratio would not necessarily remain in place when agents invest differently, hence we would not gain much; also this would clutter the formulæ and introduce differences between the agents (in terms of labour amount) that are not the focal point of this paper. As we do not consider a labour market (or, in fact any markets) in this work,[4] we fix the labour amount available to each agent to 1 and assume it is always fully employed, for simplicity.

In this decentralised economy, we assume "knowledge spillover" effects [Romer, 1990], that is, while each agent has its own capital stock, production, consumption, and investment values, technical progress remains global. It increases with the capital increase due to investment made by all agents and all agents benefit from it. For optimal investment at time t, agent i would need to solve

$$\max_{I_t} U_t^{(i)} = U(I_t^{(i)}) = u(C_t^{(i)}) + \rho u(C_{t+1}^{(i)}) \tag{5}$$

$$\text{given that } K_{t+1}^{(i)} = (1-\delta) \cdot K_t^{(i)} + I_t^{(i)} \tag{6}$$

$$C_t^{(i)} = \left(K_t^{(i)}\right)^{\alpha} \eta_t^{1-\alpha} - I_t^{(i)} \tag{7}$$

$$C_{t+1}^{(i)} = \left(K_{t+1}^{(i)}\right)^{\alpha} \eta_{t+1}^{1-\alpha} \tag{8}$$

$$\eta_{t+1} = K_{t+1} \cdot \frac{\eta_t}{K_t} = \sum_{j=1}^{n} K_{t+1}^{(j)} \cdot \frac{\eta_t}{\sum_{k=1}^n K_t^{(k)}}. \tag{9}$$

This, however, depends on $K_{t+1} = \sum_{j=1}^n K_{t+1}^{(j)}$ and $K_t = \sum_{j=1}^n K_t^{(j)}$, where the present total capital in the economy may be considered known to agents (through a statistics office or similar), but the other agents' future capital $K_{t+1}^{(j)}$ for agents $j \neq i$ depend on their investment decisions in the present time step, placing the problem in the realm of game theory.

[3]For example, dividing output-side real GDP at current PPPs by capital stock at current PPPs for the France, Germany, the UK and the US for the years 2010 to 2019 as reported by the Penn World Tables [Feenstra et al., 2023], suggest values around $\frac{1}{5}$ for the three European countries and almost $\frac{1}{3}$ for the US.

[4]In future work, coming back to the other "building blocks" of green growth [Steudle et al., 2018], a labour market with search will be an important element.

2.3 The standard economic "ignorant" agent

A common assumption in the literature, which avoids the problem of an unknown piece of information in the optimisation problem, is that each agent is small enough to neglect its own contribution to the aggregate capital stock, ignoring equation (9), that is, η_{t+1} is treated as a given constant. Such an "ignorant" agent considers as the utility function to be maximised

$$U^{\text{Ig}}(I_t^{(i)}) = \ln\left[(K_t^{(i)})^\alpha \eta_t^{1-\alpha} - I_t^{(i)}\right] + \rho\alpha \ln\left[(1-\delta)K_t^{(i)} + I_t^{(i)}\right] + \rho(1-\alpha)\ln\left[\eta_{t+1}\right] \tag{10}$$

The last term here shows that the level of utility increases with a larger value of future technical progress η_{t+1}; Figure 1 below illustrates this utility function. However, the optimal investment value for agent i (see Appendix C),

$$I_t^{\text{Ig}(i)} = \frac{1}{1+\alpha\rho}\left(\rho\alpha\left(K_t^{(i)}\right)^\alpha \eta_t^{1-\alpha} - (1-\delta)K_t^{(i)}\right) = \frac{1}{1+\alpha\rho}\left(\rho\alpha P_0^{(i)} - \hat{K}_t^{(i)}\right). \tag{11}$$

is independent of the actual value of η_{t+1}, as can also be seen in Figure 1 from the fact that one sees only one optimal investment line. The ignorant agent's curves with different η_{t+1} but the same settings reach their maxima at the same point. We will consider the constraints of feasibility and positivity of investment, and growth of the economy below (see Section 4.1), in the context of how values for simulations are set.

2.4 A "collaborative" agent

From an agent-based modeller's perspective, the assumption that each single agent ignores its contribution to technical progress is somehow unsatisfactory, when at the same time the modeller herself knows about this externality.[5] The question how an agent should invest if it knows about spillover effects, however, is not completely straightforward to answer.

Considering the technical progress term exogenous to the agent, it will cancel in the optimisation, as seen in (36) in Appendix C. If the agent considers its own contribution to technical progress, it needs to differentiate also η_{t+1} with respect to $I_t^{(i)}$. This case will be seen for the witty agent in Section 2.5 but the result differs structurally from the case of the single representative agent. By structural similarity, a naive definition for a "collaborative agent's" strategy can be given by analogy with the representative agent's result (4), i.e. agent i in a decentralised economy invests, at time t,

$$I_t^{\text{Co(i)}} = \frac{1}{1+\rho}\left(\rho P_t^{(i)} - \hat{K}_t^{(i)}\right) = \frac{1}{1+\rho}\left(\rho\left(K_t^{(i)}\right)^\alpha \eta_t^{1-\alpha} - (1-\delta)K_t^{(i)}\right). \tag{12}$$

That the similarity goes beyond this definition will be considered in Section 4.2. From the point of view of an agent i in an economy of n agents, this result for the optimal investment can be recovered by having the agent assume that all n agents are identical. That is, not only do they have the same current capital as agent i, and hence $K_t = n \cdot K_t^{(i)}$, but also their future capital is assumed equal $K_{t+1} = n \cdot K_{t+1}^{(i)}$. The latter assumption seems a bit contradictory, as agent i is actually deciding about its investment, and hence its future capital $K_{t+1}^{(i)}$, in that same moment. However, with these assumptions, future technical progress reduces to $\eta_{t+1} = n \cdot K_{t+1}^{(i)} \cdot \frac{\eta_t}{n \cdot K_t^{(i)}} = K_{t+1}^{(i)} \cdot \frac{\eta_t}{K_t^{(i)}}$ and the optimisation can be computed exactly in analogy with (30)-(31) in Appendix A. Since with these assumptions the future consumption of a collaborative agent can be rewritten as follows

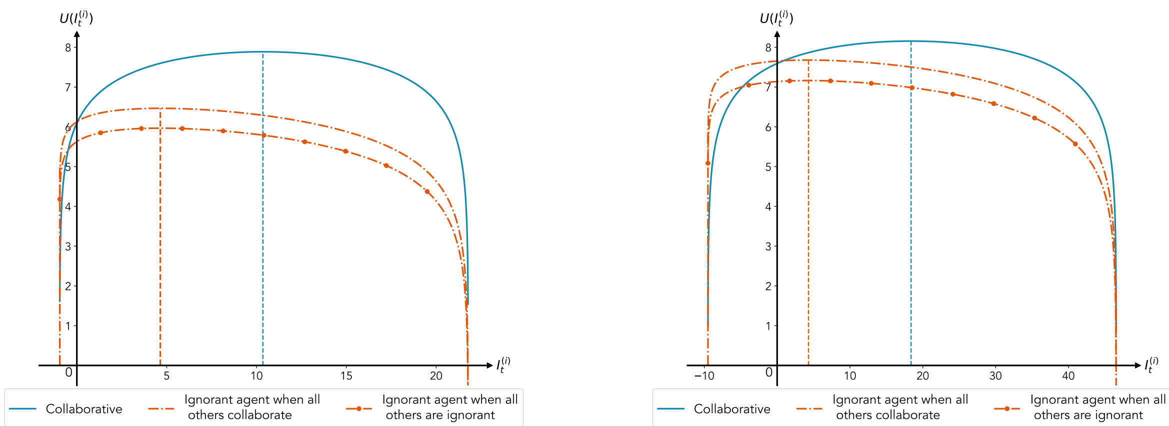
$$C_{t+1}^{(i)} = \left(K_{t+1}^{(i)}\right)^\alpha \left(K_{t+1}^{(i)} \cdot \frac{\eta_t}{K_t^{(i)}}\right)^{1-\alpha} = K_{t+1}^{(i)}\left(\frac{\eta_t}{K_t^{(i)}}\right)^{1-\alpha},$$

the utility function can also be rewritten:

$$U^{\text{Co}}(I_t^{(i)}) = \ln\left[\left(K_t^{(i)}\right)^\alpha \eta_t^{1-\alpha} - I_t^{(i)}\right] + \rho\ln\left[(1-\delta)K_t^{(i)} + I_t^{(i)}\right] + \rho(1-\alpha)\ln\left[\frac{\eta_t}{K_t^{(i)}}\right] \tag{13}$$

[5] Similarly, in Frankel's (1962) consideration of an "ex ante" production function for the single agent, who then "moves along" a different "realised function" may raise the question whether or why agents do not know or learn about the latter function.

Note that this is independent of the system's future technical progress η_{t+1} and the last term mixes system-level technical progress with agent-level capital, both for the current time step. Comparing this with the ignorant agent's utility function, the first term, concerning current consumption, is equal, the second term (which concerns future capital) is weighted with ρ, where the ignorant agent has a prefactor of $\rho\alpha$, and the third term is both weighted differently and contains a different technical progress value. The utility functions of an ignorant and a collaborative agent are illustrated in Figure 1, which also shows the well-known fact that the optimal investment of an ignorant agent is less than that of an agent acting like a benevolent planner.



(a) Agent's capital $K_t^{(i)} = 1.0$.

(b) Agent's capital $K_t^{(i)} = 10.0$.

Figure 1: Intertemporal utilities as functions of investment, where the solid blue line refers to a collaborative agent's utility (12) and the orange lines to the utility of an ignorant agent under the stated assumptions about other agents. The plot is based on example values when the system's total capital is 100 and technical progress is $\eta_t = 99.3$. When the agent's capital is 1.0 (see 1a) the maximum utility ($U_t^{\mathrm{Co}(i)} = 7.88$) for the collaborative farm occurs at $I_t^{\mathrm{Co}(i)} = 10.36$. The maximum utility for the ignorant agent when the other farms collaborate (meaning $\eta_{t+1} = 101.02$ and $U^{\mathrm{Ig}(i)} = 5.97$) or when the other farms are ignorant (leading to $\eta_{t+1} = 52.81$ and $U^{\mathrm{Ig}(i)} = 5.96$) occurs at $I_t^{\mathrm{Ig}(i)} = 4.65$. In (1b), the agent's capital is 10, and the maximum utility ($U_t^{\mathrm{Co}(i)} = 8.16$) for the collaborative farm occurs at $I_t^{\mathrm{Co}(i)} = 18.39$, the maximum utility for the ignorant agent when the other farms collaborate (with $\eta_{t+1} = 103.43$ and $U^{\mathrm{Ig}(i)} = 7.68$) or when the other farms are ignorant ($\eta_{t+1} = 58.12$ and $U^{\mathrm{Ig}(i)} = 7.30$) occurs at $I_t^{\mathrm{Ig}(i)} = 4.30$.

2.5 A well-informed, or "witty" agent

The assumption that agents are aware of the cumulative effects of capital on technical progress, i.e., include Equation (9) to find the optimal investment level, seems not to be discussed in the literature. We consider a well-informed agent one who takes into account that its investment also has an effect (even if a small one) on future technical progress, i.e., it explicitly considers that the future technical progress η_{t+1} contains its own investment $I_t^{(i)}$ in that

$$\begin{aligned}\eta_{t+1} &= \frac{K_{t+1}}{K_t}\eta_t = \sum_j K_{t+1}^{(j)}\frac{\eta_t}{K_t} = \sum_j((1-\delta)K_t^{(j)} + I_t^{(j)})\frac{\eta_t}{K_t} \\ &= \frac{\eta_t}{K_t}\left((1-\delta)K_t^{(i)} + I_t^{(i)} + K_{t+1}^{(\sim i)}\right) \qquad (14)\end{aligned}$$

where the terms of all other agents are summarised in the notation $K_{t+1}^{(\sim i)} = \sum_{j\neq i} K_{t+1}^{(j)}$. Such an agent, that we will call "witty" in the following, thus views the utility function to be maximised as given by

$$U^{\mathrm{Wi}}\left(I_t^{(i)}\right) = \ln\left[(K_t^{(i)})^{\alpha}\eta_t^{1-\alpha} - I_t^{(i)}\right] + \rho\alpha\ln\left[(1-\delta)K_t^{(i)} + I_t^{(i)}\right] + \rho(1-\alpha)\ln\left[\frac{\eta_t}{K_t}\left((1-\delta)K_t^{(i)} + I_t^{(i)} + (1-\delta)K_t^{(\sim i)} + I_t^{(\sim i)}\right)\right]. \quad (15)$$

This depends on the value of the total investment of the other agents, denoted $I_t^{(\sim i)}$. Given an expectation for $I_t^{(\sim i)}$, say $I_t^{(\sim i),e}$ – a topic that will be treated in Section 3.2 – maximising (15) leads to the following quadratic equation (see Appendix D):

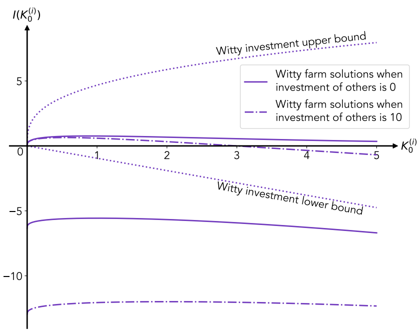
$$-(1+\rho)\left(I_t^{(i)}\right)^2 + \left[\rho P_t^{(i)} - (2+\rho)\hat{K}_t^{(i)} - (1+\rho\alpha)\left(K_{t+1}^{(\sim i),e}\right)\right] I_t^{(i)} + \rho P_t^{(i)}\hat{K}_t^{(i)} + \rho\alpha P_t^{(i)} K_{t+1}^{(\sim i),e} - \left(\hat{K}_t^{(i)}\right)^2 - \hat{K}_t^{(i)} K_{t+1}^{(\sim i),e} = 0 \quad (16)$$

where $K_{t+1}^{(\sim i),e} = \sum_{j\neq i} K_t^{(j)} + I_t^{(\sim i),e}$. The solution of this quadratic equation provides two critical points, of which only one is feasible (again, see Appendix D). It is

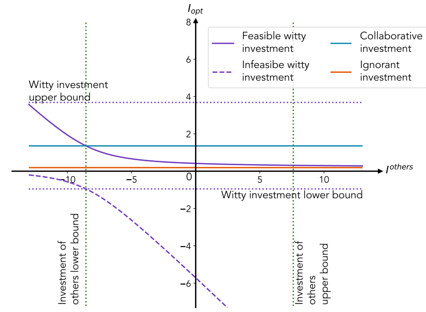
$$I_t^{(i)} = \frac{1}{2(1+\rho)}\left(\rho P_t^{(i)} - (2+\rho)\hat{K}_t^{(i)} - (1+\rho\alpha)K_{t+1}^{(\sim i),e}\right) + \frac{\sqrt{\left(\rho(P_t^{(i)} + \hat{K}_t^{(i)}) + (1+\rho\alpha)K_{t+1}^{(\sim i),e}\right)^2 - 4\rho(1-\alpha)K_{t+1}^{(\sim i),e}\left(P_t^{(i)} + \hat{K}_t^{(i)}\right)}}{2(1+\rho)} \quad (17)$$

For the usual parameter values, its value lies between that of the ignorant and the collaborative strategies (see Figure 20 in Appendix D), and considering the optimal investment as a function of the expected investment of everybody else, it decreases, i.e., the more a well-informed agents expects the others to invest, the less it will invest itself (see Figure 2), where we also see that for the unlikely case of total de-investment of the others, the witty agent's investment may even exceed that of the collaborative agent (Figure 2). In Figure 2b, the dotted lines mark the feasibility constraints of investing at most the total production and de-investing at most the total capital after depreciation: purple for the witty agent, green for all other agents. The interval between the green lines is hence the domain of interest for the other agents' investment. If the others' investment goes to the upper bound, the witty agent's investment approaches the value of the ignorant agent's investment.

That the investment decreases with increasing investment of the others is intuitive as the agent is balancing current and future consumption: if, through a higher investment of the others, future technical progress and hence future production is expected to be higher, the balance will also tend to increase the current consumption and thus decrease the current investment. We can deduce that the witty agent qualitatively confirms the idea of "free-riding" on the other agent's investments. Together with the fact that the smaller the witty agent compared to the overall economy, the less it becomes discernible from the ignorant agent (again, see Figure 20 in Appendix D), this analysis at the level of the individual agent suggests that knowing about one's contribution to technical progress does not make a big difference. What results at the macro-level will be explored in Section 4.7. Before coming to results, however, the agent-based model has to be introduced in the following section.



(a) Optimal investment in terms of the agent's capital in an economy with $\tilde{\eta}_0 = 10$. The agent either expects the others not to invest anything (solid line) or to invest ten (dash doted line).



(b) Optimal investment as a function of other agents' investment in an economy with $\tilde{\eta}_0 = 7$, and the agent's capital is 1.

Figure 2: Two representations of the witty agent's optimal investment in terms of the agents capital 2a and in terms of the other agents' investment 2b with parameters as above for an economy with total capital 10.

3 A dynamic agent-based model

As foreseen by the idea of agentization, we have defined an agent-based model to reproduce the Ramsey growth model with endogenous technical progress as known from the literature and to investigate going beyond it, e.g., by introducing new investment strategies for agents as described above and by considering repeated decisions of agents in a dynamic setting. Heterogeneity of agents will later be explored in terms of their investment strategies and their initial capitals. A detailed description of the ABM and the code is provided by Figueroa Alvarez et al. [2024], here we will give a brief overview, sketching the agents, introducing two mechanisms for updating expectations, and then illustrating the sequence of steps carried out.

3.1 Three types of agents

We consider a set of agents who are farms that produce, invest and consume as described, where the aim of their investment decision is an intertemporal utility maximisation for the current and one future time step. To consider a longer time-horizon, enabling the analysis of dynamics of the system, the ABM is defined in such a way that with a given investment decision, capital is updated for the next time step and with it technical progress, production takes place and the next investment decision will again be an intertemporal optimisation for the next, then current and then future time steps. That is, farms are similarly myopic as before in Section 2. The utility they base their decision on is the anticipated utility of consumption in this and the next time step, as in (5). This utility is however not actually realised in the dynamics described below as the future consumption will generally differ from the anticipated value due to a next investment decision in the meantime. This is also the reason for sticking to short-sighted agents who take into account just the present and one future time step; if several time steps were considered, agents would have to take several future investment decisions, of which, however, only one can be implemented in the current time step. Then updating capital, production, etc, a new investment decision should anyhow be taken rendering the previous decisions already made for this time step obsolete. To consider also the utility that agents actually experience, we will look at the instantaneous utility given by the felicity function of current consumption $u(C_t^{(i)})$, and take stock of these values over time through cumulative utility $U_t^{(i)} = \sum_{k=1}^{t} u(C_k^{(i)})$.

While the background remains the general setting given in (5)-(9) with the diverse assumptions on knowledge of the agents made, we can more precisely define three types of agents:

- **Ignorant agents** solve the maximisation problem for the utility function (10) and hence invest according to (11).

- **Collaborative agents** solve the maximisation problem for the utility function (13) and hence invest according to (12).
- **Witty agents** solve the maximisation problem for the utility function (15) and hence invest according to (17). How they determine the necessary expectations is introduced in Section 3.2.

To complete the set of agents, apart from **a collection of farms**, each with one of the three possible investment strategies, where the number of agents of each type is set on a case to case basis later, the model contains

- A **statistician**. The statistician computes all necessary aggregate variables of the model: it communicates the technical progress of the economy to all farms at each time step and for the observer it aggregates total capital, and computes Gross Domestic Product, growth rate, gross investment, and aggregate utility.

3.2 Expectation update for witty agents

To update their expectations on the other agents' investment, witty farms do a *one-period-ahead forecast*, since they predict the value at the beginning of the period and they will learn the real value in the next period. They use one of two heuristics: a simple adaptive rule or a trend following rule [Palestrini, 2017].

With the **adaptive rule**, the expectation on the the investment of the others at time t is:

$$I_t^{(\sim i),e} = I_{t-1}^{(\sim i),e} + \lambda_A \left(I_{t-1}^{(\sim i)} - I_{t-1}^{(\sim i),e} \right),$$

where $I_{t-1}^{(\sim i),e}$ is the agent's last forecast, $I_{t-1}^{(\sim i)}$ is the last observed value for the investment of the others, and $0 \leq \lambda_A \leq 1$ is the expectation weight factor. When $\lambda_A = 1$ the agent has naive expectations, i.e. the agent's expectation is equal to the previous realisation,

$$I_t^{(\sim i),e} = I_{t-1}^{(\sim i),e}.$$

With the **trend following rule**, the agent is going to use the last value of the investment of the others and adjusts it in the direction of the last change of the values [Anufriev and Hommes, 2012], i.e.

$$I_t^{(\sim i),e} = I_{t-1}^{(\sim i)} + \lambda_T (I_{t-1}^{(\sim i)} - I_{t-2}^{(\sim i)})$$

where $\lambda_T > 0$ is the extrapolation coefficient that measures the strength of the adjustment [Anufriev and Hommes, 2012]. The higher λ_T, the stronger the impact of the trend on expectations [Palestrini, 2017].

A problem that arises with these rules is that there is no macro variable in the system that directly indicates the actual investment of the other farms. But farm i is a well-informed agent, reads the statistician's reports and therefore knows the last time gross investment of the system I_{t-1},

$$I_{t-1} = I_{t-1}^{(\sim i)} + I_{t-1}^{(i)},$$

and can isolate $I_{t-1}^{(\sim i)}$, to obtain

$$I_{t-1}^{(\sim i)} = I_{t-1} - I_{t-1}^{(i)}.$$

Hence, the expectation heuristics can be rewritten as follows,

$$\textbf{Adaptive}: \quad I_t^{(\sim i),e} = (1 - \lambda_A) I_{t-1}^{(\sim i),e} + \lambda_A (I_{t-1} - I_{t-1}^{(i)}) \tag{18}$$

$$\textbf{Trend following}: \quad I_t^{(\sim i),e} = I_{t-1} - I_{t-1}^{(i)} + \lambda_T (I_{t-1} - I_{t-2} - (I_{t-1}^{(i)} - I_{t-2}^{(i)})) \tag{19}$$

Farm i will use these rules to predict the value of the investment of the other agents for obtaining its optimal investment, as seen in the following sequence of events.

3.3 Sequence of events

The model evolution, that is also illustrated in Figure 3, takes place via the following updates in each time step of the simulation:

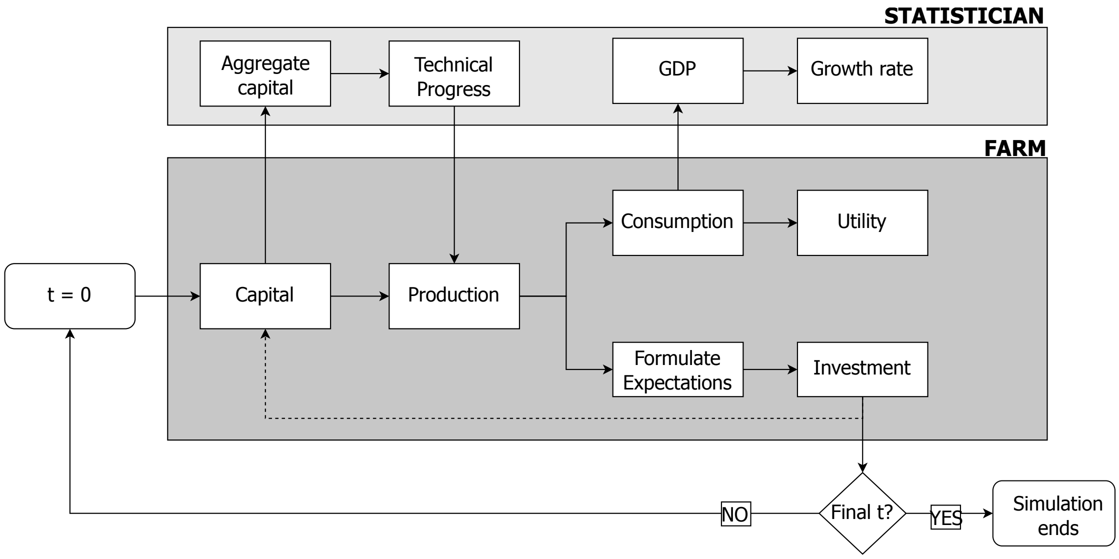


Figure 3: Flow diagram of the model

1. The time index is increased by one, so what was computed as $K_{t+1}^{(i)}$ in the previous time step is now K_t.
2. The statistician aggregates the system's capital by summing over all agents, i.e., $K_t = \sum_i K_t^{(i)}$.
3. From this aggregate capital, the statistician computes the technological progress, η_t, using Equation (9).
4. All farms produce according to Equation (3) using their own capital $K_t^{(i)}$ and the technical progress η_t given by the statistician.
5. Witty farms update their expectations on the others' investment using (18) or (19). Then all farms maximize their anticipated utility, that is, they compute the optimal investment $I_t^{(i)}$ according to their respective strategies, as described in Section 3.1.
6. The farms' current consumption is calculated from (7), i.e., as the difference between their production and investment values.
7. Each farm computes its instantaneous utility $u(C_t^{(i)})$ and adds it to the previous cumulative utility to obtain the current one, $U_t^{(i)}$.
8. The statistician computes the GDP and growth rate of the system. Since the economy is closed, the GDP is the sum of all the farms' output

$$GDP_t = \sum_i P_t^{(i)}. \tag{20}$$

And the *growth rate* g of the economy is given by

$$g_t = \frac{GDP_t - GDP_{t-1}}{GDP_{t-1}}. \tag{21}$$

9. Farms update their capital, using Equation (6) to compute $K_{t+1}^{(i)}$.

In the following section, results obtained from this ABM for various combinations of agents and several distributions of initial capital will be shown.

4 Results

This section presents results along the objectives of agentization, where a first aim is to reproduce standard economic results with an agent-based model. This will be done in Section 4.2, after a few theoretical considerations on suitable parameters for model setup 4.1; then Section 4.3 briefly analyses the conditions under which a prisoner's dilemma structure arises, that is also a standard result. The second goal of agentization then is to go beyond standard results by generalising the ABM. A very first step in this direction is the consideration of a dynamic setting, which we first explore for homogeneous economies in which all agents have the same strategy and the same initial capital, see Section 4.4. We then relax these assumptions one at a time, considering heterogeneous initial capital stocks in Section 4.5 and heterogeneity in strategy in Section 4.6. Finally, in Section 4.7, we turn to economies that contain agents with expectations.

4.1 Feasibility of investment and conditions for economic growth

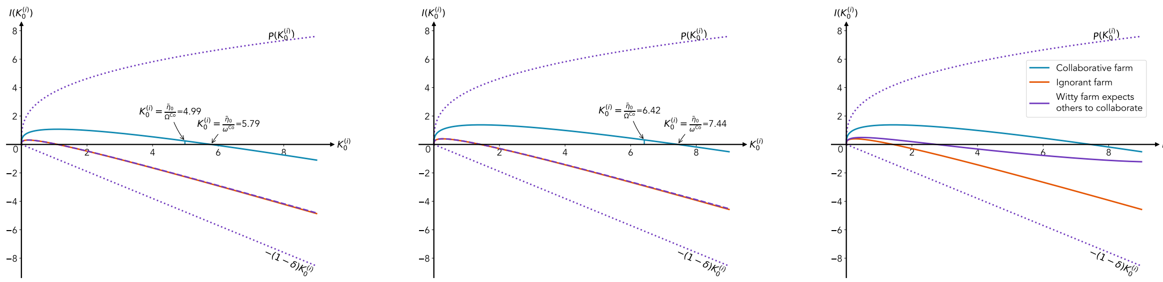
As a very first step, the feasibility of investment values computed needs to be assured. As mentioned when the general setting was sketched, an agent's investment cannot exceed its production, a de-investment cannot exceed its capital after depreciation, i.e., $-(1-\delta)K_t^{(i)} < I_t^{(i)} < P_t^{(i)}$, and this needs to be true for any investment strategy the agent may use. For collaborative and ignorant agents, it is easy to show that this always holds (along the lines of Appendix B). For the witty agent, only one of the solutions will lie in the feasible domain (see Appendix D).

Given that investment is feasible, the next question is whether it leads to growth, stagnation, or de-growth of the agent's capital stock (and, combining this for many agents, of the economy). For the benevolent planner, we saw that the initial technical progress needs to be chosen "large enough" in relation to the initial capital stock, in order for investment to be positive, and "larger" for the economy to grow. The analogous constraints for the initial values are stricter in the case of an ignorant investment strategy: for agent i's investment to be positive, i.e., $I_t^{\mathrm{Ig}(i)} = \frac{1}{1+\alpha\rho}\left(\rho\alpha P_0^{(i)} - \hat{K}_t^{(i)}\right) > 0$, one needs to assure that $\tilde{\eta}_0 > \left(\frac{1-\delta}{\rho\alpha}\right)^{\frac{1}{1-\alpha}} K_0$, which for the same parameter values as above[6] means that $\tilde{\eta}_0$ needs to be larger than about $4.89 \cdot K_0$. For the agent's capital to increase (i.e., investment is larger than capital depreciation), $\eta_0 > \left(\delta + \frac{1}{\rho\alpha}\right)^{\frac{1}{1-\alpha}} K_0$, i.e., for the given parameters the ratio of technical progress to capital needs to exceed about 5.44. For most of the following results, we will choose settings where capital growth is guaranteed for all agents, however, there will also be cases where this may not be the case for agents with a large initial capital stock. Negative investment (i.e., decreasing the capital stock for consumption) will be allowed in such cases.

For the witty agent, the formulæ are more complicated, and they depend on further aspects (such as the other agents' total capital and their aggregate (expected) investment. However, it was seen that for the usual parameter values and several settings in terms of total initial capital and initial technical progress, the optimal investment of the witty agent lies between that of the collaborative and the ignorant ones, so that we can choose values based on the two simpler cases.

The given constraints can also be illustrated by considering the optimal investment (given each of the three strategies) as a function of the initial capital of an agent, for several values of initial technical progress in comparison, as seen in Figure 4.

[6] $\alpha = 0.33, \delta = 0.05, \rho = 0.99.$



(a) $\tilde{\eta}_0 = 5.44$ and $K_0 = 1000$ (b) $\tilde{\eta}_0 = 7.00$ and $K_0 = 1000$ (c) $\tilde{\eta}_0 = 7.00$ and $K_0 = 10$

Figure 4: Optimal investment – for a collaborative, an ignorant, and a witty agent – as a function of the agent's capital when the system's capital is 1000 with initial technical progress 5.44 (Figure 4a) and 7.00 (Figure 4b), and when the system's capital is 10 with initial technical progress 7.00 (Figure 4c).

As can be seen, the interval with positive investment is smaller for the ignorant agent than for the collaborative one with the same values of initial technical progress. The witty agent is practically indistiguishable from the ignorant one when it is a small part of a big economy; when it is a larger part of the total economy, it's investment curve moves closer to that of the collaborative agent.

4.2 Reproducing standard economic results

According to the first objective of agentization, an agent-based model should be able to reproduce standard economic model results. To start doing so, we can remain, for a moment, at the level of the definition of the single agents, without even entering the dynamic setting with many time steps. In the following, we will call a system of farms that all have the same investment strategy a homogeneous economy.

- As was seen above, e.g., in Figure 1, the ignorant agent's optimal investment is smaller than that of a collaborative agent, that was constructed in analogy with a benevolent planner. The fact that the decentralised solution in a system of agents that ignore their contribution to technical progress falls short of the social optimum is well-known: the spillovers not being internalised, ignorant agents invest too little. Remedies proposed in the economic literature include subsidies for investment or production through which the government could generate the social optimum [Acemoglu, 2009].
- Considering further investment strategies in a decentralised economy, one can actually reproduce the benevolent planner's optimum through a homogeneous economy of n collaborative agents, when the total initial capital in the system is the same, it is distributed equally among the agents, and the representative agent's initial technical progress value η_0 is scaled to $\tilde{\eta}_0 = \frac{1}{n}\eta_0$ for the decentralised economy, or the labour amount of the single agents is scaled to $\frac{1}{n}$. Alternatively, the representative agent could be equipped with a labour amount of n instead of 1 while keeping the given technical progress value. However, when considering other capital allocations than an equal distribution, this solution breaks down. The initial production in the decentralised economy $\sum_{i=1}^{n}\left(K_0^{(i)}\right)^{\alpha}\eta_0^{1-\alpha} \geq K_0^{\alpha}\eta_0^{1-\alpha}$ (see Appendix E). Where an obvious aspect is that as we increase the number of agents, all equipped with a labour input of 1, the initial production increases. Besides this, the distribution of the initial capital among the agents also plays a role, as with K^{α} the capital contribution to the production increases less than linearly with increasing capital. To obtain decentralised economies comparable with the corresponding representative agent, we will use an adapted technical progress value

$$\tilde{\eta}_0 = \eta_0 \left(\frac{K_0^{\alpha}}{\sum_{i=1}^{n}\left(K_0^{(i)}\right)^{\alpha}} \right)^{\frac{1}{1-\alpha}} \tag{22}$$

for the former. Note that for an equal distribution of capital, this reduces to $\tilde{\eta}_0 = \frac{1}{n}\eta_0$. Then, $\sum_{i=1}^{N} P_0^{(i)} = P_0$ by definition of $\tilde{\eta}_0$ and the analogous holds for investment, hence for the next step's capital, and hence the next technical progress value (see Appendix F).
Superficially, this answers the question whether a decentralised economy can move towards or along the socially optimal path. However, the construction of the collaborative agent's investment strategy did not actually make sense from the point of view of the single agent. We therefore do not suggest this construction as an answer to this question and will hence relegate potential answers to the analysis of systems with witty agents. The homogeneous economy of collaborative agents with equal initial capital, in its role of decentralised version of a representative agent, will be kept rather as a benchmark to compare other decentralised economies with.

- Due to the fact that a homogeneous economy of collaborative agents with equal capital stocks reproduces a benevolent planner, the system inherits the Kaldor facts, such as a constant growth rate.

4.3 A prisoner's dilemma?

As one thought experiment, before moving on to the dynamic ABM version, we can pose the question, what agents should do, if they knew about both the ignorant and the collaborative strategies, in a simple game theoretic setting, that arises naturally from (5)-(9). In this simplest setting with only two time steps and only the two strategies of the collaborative and the ignorant farm, Steudle et al. [2018] posit that the game's structure is that of a prisoner's dilemma, in the sense that it is convenient for the single agent to invest according to the ignorant strategy and benefit from technical progress induced by others who invest collaboratively.

This is true if the single agent is small compared to the overall economy, which is the natural assumption. Considering only two agents of equal size (i.e., in terms of initial capital), this is not the case; it is then beneficial that they both invest collaboratively, as can be seen in Figure 5 below. Hence, the question arises from which "relative size" of an agent's initial capital compared to the economy's capital it is convenient for this single agent to "free-ride", i.e., that $U(I^{\mathrm{Ig}(i)}) > U(I^{\mathrm{Co}(i)})$, or in words when the ignorant investment strategy produces a higher utility than the collaborative one, given a fixed assumption about everybody else's strategy. As seen in Appendix G, this question leads to rather messy formulæ, but the answer can be illustrated by a simple plot, see Figure 5, that uses the same parameter values used above.

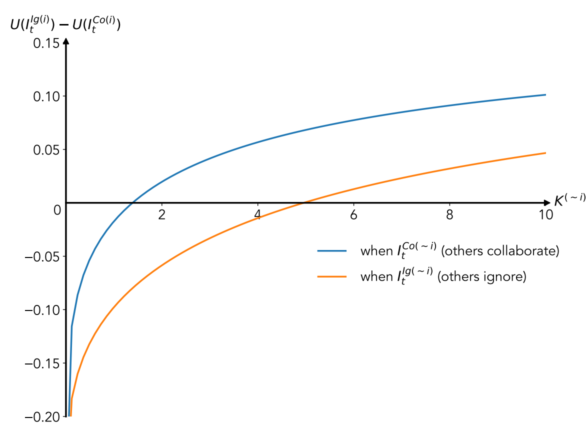


Figure 5: Comparison of strategies for one agent as depending on its relative size: where the difference of utilities $U(I^{\mathrm{Ig}(i)}) - U(I^{\mathrm{Co}(i)})$ is positive, it is beneficial for the agent to act according to the ignorant investment strategy. We consider just two agents, and assume that the second agent acts collaboratively (blue line) or as an ignorant agent (orange). All parameters as before, the agent under consideration has an initial capital of 1.

It can be seen that there is a threshold value of the other agent's capital size beyond which an agent's utility from the ignorant investment strategy exceeds that of the collaborative strategy, i.e., that we get a prisoner's dilemma structure when the single agent is small enough compared to the overall economy. This threshold, providing the concrete definition of "small enough", depends on the investment strategy chosen by the other agent in the intuitive way: the more the other one invests, the more beneficial "free-riding" becomes. Note, however, that this assessment of the situation arises when looking at a single time step: the anticipated utility of the ignorant agent exceeds the anticipated utility of a collaborative one from this threshold on. We will next move to the iterated or dynamic setting of the agent-based model, where this assessment turns out to be very short-sighted (see Figure 8).

4.4 Homogeneous economies of collaborative and ignorant agents

Unlike for many ABM that contain random elements, the ABM presented here is deterministic. We can hence consider single simulation runs instead of having to draw conclusions from ensembles, and, given the already established constancy of growth rates, in most cases very few steps of the model will be sufficient to investigate the system's behaviour.[7]

As a first step for going beyond reproducing standard results, we consider homogeneous economies of collaborative or ignorant agents in such a dynamic setting, where we use the parameters given in Table 1. In most of the following, we will consider an economy that contains $n = 1000$ farms.

Symbol	Description	Value
α	Output elasticity of capital	0.33
δ	Capital depreciation rate	0.05
ρ	Factor to discount future felicity	0.99
$K_0^{(i)}$	Farm's i initial capital	1.00
n	Number of farms	1000

Table 1: Baseline parameters for the ABM simulations

In section 4.1, we derived the conditions for the agents' investment to be positive and to exceed their capital's depreciation, i.e. to prevent capital decay. The threshold values for zero investment are $\omega^{\mathrm{Co}} = \left(\frac{1-\delta}{\rho}\right)^{\frac{1}{1-\alpha}}$ for a collaborative agent and $\omega^{\mathrm{Ig}} = \left(\frac{1-\delta}{\alpha\rho}\right)^{\frac{1}{1-\alpha}}$ for an ignorant one, and those for exactly replacing the capital stock's depreciation and hence keeping the capital stock constant are $\Omega^{\mathrm{Co}} = (\delta + \frac{1}{\rho})^{\frac{1}{1-\alpha}}$ and $\Omega^{\mathrm{Ig}} = (\delta + \frac{1}{\alpha\rho})^{\frac{1}{1-\alpha}}$ for collaborative and ignorant agent, respectively. With the parameters from Table 1, these are $\omega^{\mathrm{Co}} = 0.94, \Omega^{\mathrm{Co}} = 1.091, \omega^{\mathrm{Ig}} = 4.92$ and $\Omega^{\mathrm{Ig}} = 5.44$. For a quick check, we pick values of initial technical progress lying in the various intervals delimited by these thresholds[8], see Table 2, and illustrate in Figure 6 that the homogeneous economies in fact switch from decline to stagnation to growth accordingly.

[7] In fact, while we initially ran the model for many more steps, it turned out that differences in growth rates even necessitate looking at a few steps only as in the longer run exponential growth distorts the visual inspection of all but those trajectories with the highest growth rate as "too small to be seen well".

[8] For completeness, there should be an extra case between cases 6 and 7, i.e. $\omega^{\mathrm{Ig}} \cdot K_0^{(i)} < \tilde{\eta}_0 < \Omega^{\mathrm{Ig}} \cdot K_0^{(i)}$, but we omit it as it repeats for the ignorant economy what happened for the collaborative economy at the case 3, which is not qualitatively different from case 2.

No.	Case	$\tilde{\eta}_0$ **value**	η_0 **value**
1	$\tilde{\eta}_0 < \omega^{\text{Co}} \cdot K_0^{(i)}$	0.50	500.00
2	$\tilde{\eta}_0 = \omega^{\text{Co}} \cdot K_0^{(i)}$	0.94	940.00
3	$\omega^{\text{Co}} \cdot K_0^{(i)} < \tilde{\eta}_0 < \Omega^{\text{Co}} \cdot K_0^{(i)}$	0.99	990.00
4	$\tilde{\eta}_0 = \Omega^{\text{Co}} \cdot K_0^{(i)}$	1.091	1091.00
5	$\Omega^{\text{Co}} \cdot K_0^{(i)} < \tilde{\eta}_0 < \omega^{\text{Ig}} \cdot K_0^{(i)}$	3.00	3000.00
6	$\tilde{\eta}_0 = \omega^{\text{Ig}} \cdot K_0^{(i)}$	4.92	4920.00
7	$\tilde{\eta}_0 = \Omega^{\text{Ig}} \cdot K_0^{(i)}$	5.44	5440.00
8	$\tilde{\eta}_0 > \Omega^{\text{Ig}} \cdot K_0^{(i)}$	7.00	7000.00

Table 2: Initial technical progress values used in simulations together with the corresponding value for the benevolent planner.

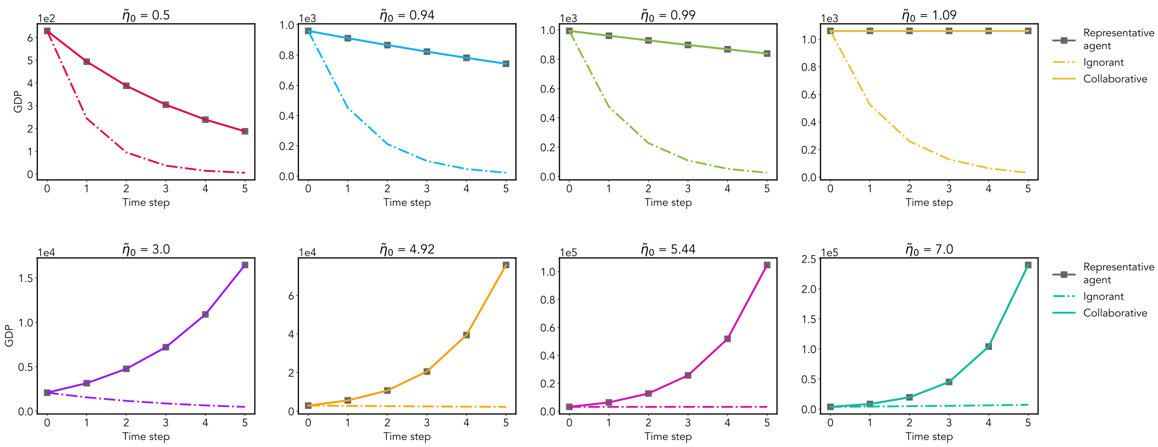


Figure 6: Gross domestic product for homogeneous economies of collaborative and ignorant farms against a representative agent of size n across the different values for $\tilde{\eta}_0$ given in Table 2.

As expected, homogeneous economies with both types of agents decline for the first three cases; the economy of collaborative farms stagnates for the fourth case and grows from case 5 onward, the same holds for the ignorant farms for cases 7 and 8. What is also clearly illustrated is the standard result that the economy of ignorant agents remains below the benevolent planner's growth path, which is identical to that of the economy of collaborative farms.

As the growth rates for the ignorant economies are hard to discern for the last three cases, the numbers for all cases are also summarised in Table 3.

Case No.	$\tilde{\eta}_0$	Average GDP growth rate	
		Collaborative economy	Ignorant economy
1	0.50	-0.24	-0.93
2	0.94	-0.05	-0.74
3	0.99	-0.03	-0.72
4	1.09	0.00	-0.69
5	3.00	0.40	-0.28
6	4.92	0.64	-0.05
7	5.44	0.69	0.00
8	7.00	0.82	0.13

Table 3: GDP growth rates for "standard" homogeneous economies across different initial technical progress values.

To take a closer look, Figure 7 illustrates the agents' investment across the same values for initial technical progress. One can see that for cases of negative investment, where $\tilde{\eta}_0$ is too small in comparison with the agents' initial capital, although the systems' gross investment increases over time, it approaches zero but a positive level of investment is never reached. Rephrasing this situation in terms of the threshold, e.g., for the collaborative agent, $\tilde{\eta}_0 < \omega^{\text{Co}} \cdot K_0^{(i)}$ corresponds to $K_0^{(i)} > \frac{\tilde{\eta}_0}{\omega^{\text{Co}}} = k_0^*$. With negative investment, an agent's capital decreases, meaning that the capital value gets closer to k_0^*, but, as in a homogeneous economy with equal initial capital of all agents the system's capital stock decreases, also the technical progress decreases

$$\eta_{t+1} = \frac{K_{t+1}}{K_t}\eta_t < \eta_t, \text{ since } \frac{K_{t+1}}{K_t} < 1 \tag{23}$$

Consequently, the domain for positive investment is also reduced, keeping the agents' investment at negative levels.

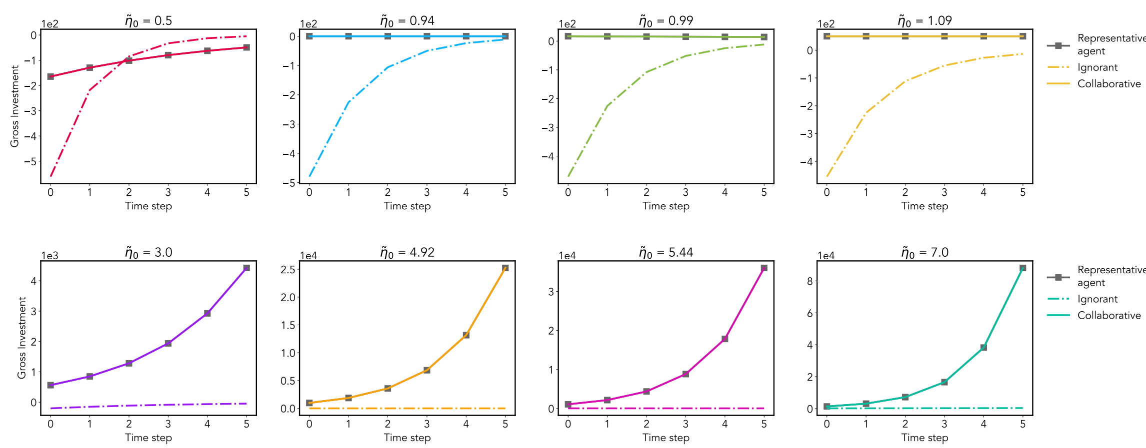
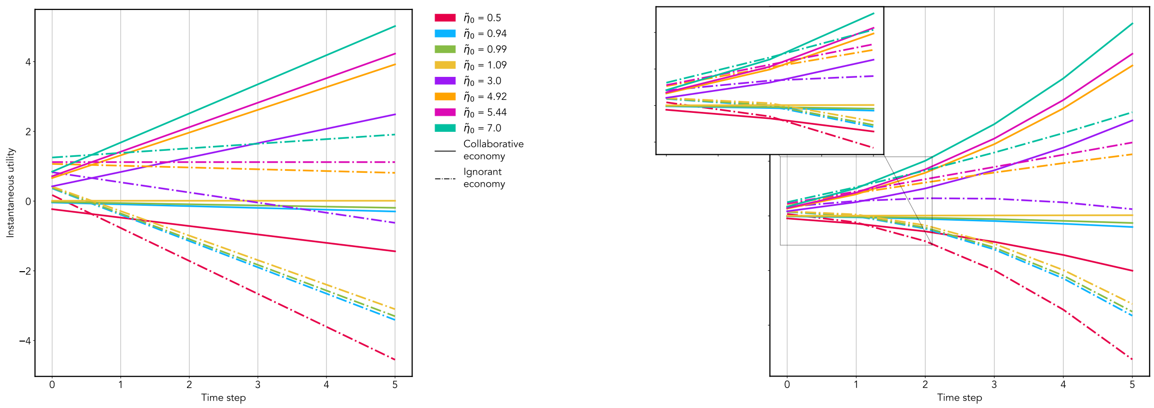


Figure 7: Gross investment for homogeneous economies of collaborative and ignorant farms, and representative agent of size n across different initial technical progress $\tilde{\eta}_0$ given in Table 2.

In the case where investment is positive but insufficient to compensate capital depreciation, one observes a similar phenomenon: the agents' capital decreases, leading to a decrease in technical progress. Consequently, the domain for positive and sufficient investment also shrinks, causing the agents to continue investing less than what is lost to depreciation, losing capital until it is entirely depleted.

For a final result on homogeneous economies of "standard" agents, let us consider utility. It was seen that in terms of anticipated utility, in the static setting with just two time steps and just one investment decision to be taken, it is beneficial to "free-ride" by choosing the ignorant strategy as soon as an agent's capital is small enough in comparison to the total capital of the economy. In the



(a) Instantaneous utility

(b) Economies' cumulative utility

Figure 8: Instantaneous utility per farm for homogeneous economies 8a and economies' cumulative utility 8b across different initial technical progress $\tilde{\eta}_0$ given by Table 2.

dynamic setting, however, this anticipated utility does not actually manifest – in the ABM, agents obtain instantaneous utility from consumption in each time step $u(C_t^{(i)}) = \ln(C_t^{(i)})$, and take stock of their cumulative utility over time. In simulations, we can now observe that, even if ignorant agents fare better at the very beginning, since they invest less and can hence consume more, this picture quickly flips to the contrary: since collaborative agents invest more, their capital stock increases faster than that of ignorant agents, inducing an increase in their future production which also implies that the collaborative agent will consume more than the ignorant one. As a result, in the longer run, which for the parameters used here begins as early as the second investment decision, taken at $t = 1$, the collaborative agent has a better instantaneous utility than the ignorant agent. Its cumulative utility exceeds that of the ignorant agent after two steps and this is independent of the initial technical progress value.

Note that, while the economy of collaborative agents reproduces the corresponding representative agent's dynamics, we cannot generally carry over the result for utility. Summing up the utilities of all agents, $\sum_{j=1}^{N} \ln(C_t^{(j)})$, we generally do not recover the utility of the representative agent $\ln(C_t) = \ln\left(\sum_{j=1}^{N}(C_t^{(j)})\right)$ as the natural logarithm chosen as the felicity function is not linear.

4.5 Exploring heterogeneous initial capital stocks in a homogeneous economy

In this section, we introduce the simplest form of heterogeneity between agents by considering distributions of initial capital among farms that deviate from the equal distribution where $K_0^{(i)} = \frac{1}{n}K_0$. For simplicity, we stick to a homogeneous economy in terms of strategies and focus on the case of collaborative agents because this makes results directly comparable with the case of a single representative agent, or the corresponding homogeneous collaborative economy with equal capital distribution that we refer to as a "benchmark economy".

As Pareto distributions are generally used to model income and wealth distributions [Charpentier and Flachaire, 2022], and the farms' capital can be considered their wealth, we distribute the initial capital of the system using a Pareto type I distribution, where we study three settings: the standard 80-20 case as well as a more equal and a more unequal distribution. The initial capital distribution is obtained by first drawing a random variable $x \geq 0$ from

$$f(x) = \frac{b}{x^{b+1}}, \tag{24}$$

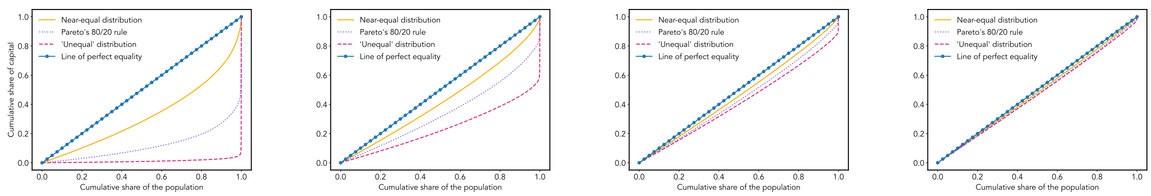
where $b > 0$ is the tail parameter. This is repeated for all n farms in the simulation. Values are then normalized and multiplied with the given total initial capital of the system to obtain each agent's initial capital (see Figure 9a). Parameters used are the following:

1. **Near-equal distribution**: we assume a relatively "equal" distribution of the capital such that 1% of the population holds 10% of the initial capital. The tail parameter for this case is $b = 2.00$.

2. **Pareto's 80-20 rule**: 80% of the wealth is held by 20% of the population [Pareto, 1964](Equivalently: 1% holds 52.82% of the initial capital). For this case, the tail parameter is $b = 1.1608$.

3. **Unequal distribution**: we distribute the initial capital more unequally among the farms, such that 1% of the farms owns 90% of the initial capital. For this case, the tail parameter takes the value $b = 1.02341$.

For more detail on computing the tail parameter see Appendix H. Figure 9a shows the Lorenz curves[9] of the three resulting distributions of initial capital. For comparison, it also plots the Lorenz curve for an economy with equal initial capital stocks as considered in the previous section; the corresponding "line of perfect equality" is the diagonal.



(a) t=0 (b) t=1 (c) t=2 (d) t=3

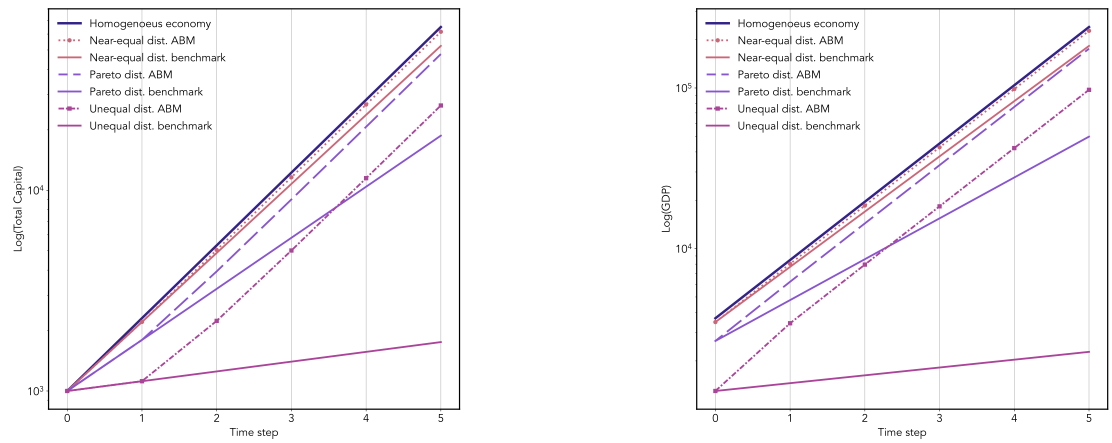
Figure 9: Evolution of the farms' capital distribution for the first four time steps with an initial technical progress value $\tilde{\eta} = 7.00$. The initial capital was distributed equally (blue dotted line), near-equally (yellow solid line), unequally (pink dashed line) and according to Pareto's 80-20 rule (purple dotted line).

Note that different initial capital distributions lead to different levels of initial production, as discussed in Appendix E. To compare economies with different levels of inequality, we hence have to decide whether to equip them with the different values for $\tilde{\eta}_0$ that correspond to a common benchmark economy (BM), given by a representative agent or by the corresponding homogeneous collaborative economy, i.e., compute $\tilde{\eta}_0$ from (22) with the same value of η_0 in all three cases, or to equip economies with the same value for $\tilde{\eta}_0$, meaning that they correspond to benchmark economies with different levels of η_0. In the first case, the more unequal the initial capital distribution, the larger the value of initial technical progress that the single farms encounter, due to a smaller denominator in (22). We opt for the second case to consider farms faced with the same value of technical progress $\tilde{\eta}_0$ here, which conversely means lower levels of η_0 the larger the inequality (see Table 4). As before, we compare several levels of initial technical progress, that for the benchmark economy would lead to decline, stagnation, and growth, respectively, that is, we reuse values from Table 2. For other parameters, we stick to the ones in Table 1.

Farms' initial $\tilde{\eta}_0$	**Benchmark economy's initial η_0**		
	Near-equal distribution	Pareto distribution	Unequal distribution
0.500	460.824	307.802	105.520
1.091	1005.517	671.623	230.246
7.000	6451.531	4309.221	1477.286

Table 4: Values for available initial technical progress for the farms $\tilde{\eta}_0$ and for their respective BM economies. Note that these values are stochastic, but will be similar for several drawings of the initial capital distribution.

[9] A Lorenz curve is a graphical representation of the distribution of wealth that shows cumulated percentages of the population from the poorest to the richest on the x-axis, and along the y-axis the percentage of the total wealth held by these percentages of the population [Lorenz, 1905].



(a) Logarithmic total capital

(b) Logarithmic gross domestic product

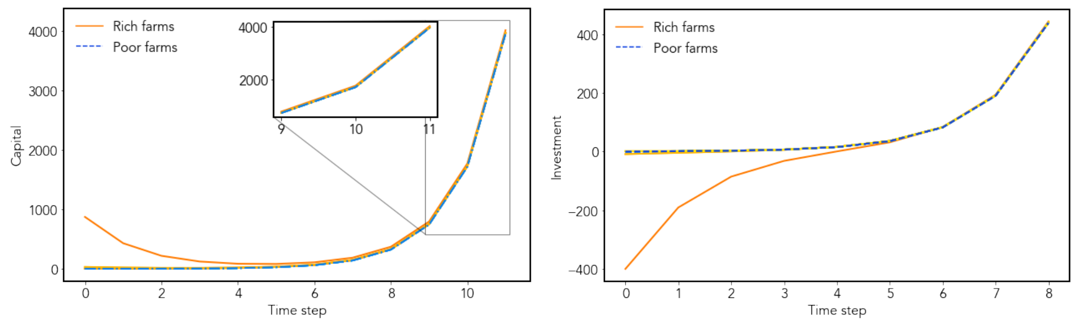
Figure 10: Logarithmic total capital 10a, and gross domestic product 10b for 5 time periods with initial technical progress of 7.00, comparing ABM economies with perfectly equal (homogeneous), near-equal, Pareto, and unequal initial capital distributions to their corresponding benchmark economies (solid lines).

Figure 10a shows a result that may be somewhat unexpected at first sight: the unequal economies outperform their respective benchmark economies, despite the fact that the values for η_0 were computed explicitly to correspond to the different initial production values obtained through the inequality, see also Table 5.

$\tilde{\eta}_0$	**Equal dist.**	**Near-equal dist.**		**Pareto's 80-20 rule**		**Unequal dist.**	
		BM Eco.	ABM Eco.	BM Eco.	ABM Eco.	BM Eco.	ABM Eco.
0.5000	-0.2393	-0.2626	-0.2412	-0.3581	-0.2411	-0.5387	-0.2405
1.0910	0.0000	-0.0284	0.0000	-0.1579	0.0001	-0.4168	0.0006
7.0000	0.8268	0.7903	0.8334	0.5846	0.8334	0.1120	0.8337

Table 5: GDP growth rates across some initial technical progress values $\tilde{\eta}_0$, comparing the ABM collaborative economies against their benchmark economy.

As can be seen in Figures 9a–9d, however, the inequality quickly reduces. With the chosen parameters and initial values, rich farms first de-invest while poor farms have positive investment values, as can be seen in Figure 11a. As the total capital in the system still increases due to the many poor farms, technical progress increases so that eventually the rich farms end up in the range of capital for which optimal investment is positive (see Figure 11b).



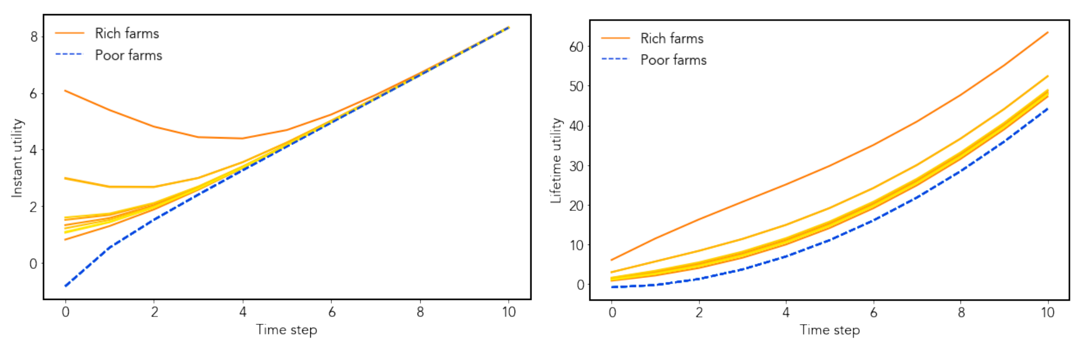
(a) Farms' capital evolution

(b) Farms' investment evolution

Figure 11: Capital for twelve time periods 11a, and investment for nine time periods 11b for the farms' top and bottom 1% from the simulation with unequal distribution of initial capital, when the initial available technical progress $\tilde{\eta}_0 = 7.00$.

However, even when initial technical progress is chosen in such a way that all farms have positive investment values, the initially unequal distribution becomes more equal over time. Due to the sublinear increase of K^α for increasing K, rich farms produce proportionally less than poor farms and consequentially also invest a smaller share of their capital, implying that they grow more slowly than poor farms. With inequality decreasing, at each time step the computation of the benchmark economy's η_t would have to be repeated, and it would turn out that an unequal economy becoming less unequal would now correspond to a benchmark economy with a larger new technical progress value. In fact, in Figure 10a, it can be seen that the growth rate of capital increases from one step to another for the unequal economies. In the long run, the unequal farms converge to the homogeneous collaborative benchmark economy. Constant growth rates as according to the Kaldor facts are hence not a feature of these economies.

Note that rich farms in the beginning experience declining levels of instantaneous utility, due to their decreasing capital and production. However, in terms of cumulative utility, they are still better off than poor farms in the unequal economy (Figure 12b).



(a) Instantaneous utility

(b) Lifetime Uitlity

Figure 12: Instantaneous (12a) and lifetime utility (12b) for 10 time periods when the available initial technical progress is 7.00, comparing the top and bottom 1% of the farms when the initial capital is unequally distributed.

4.6 Exploring combinations of "standard" agents

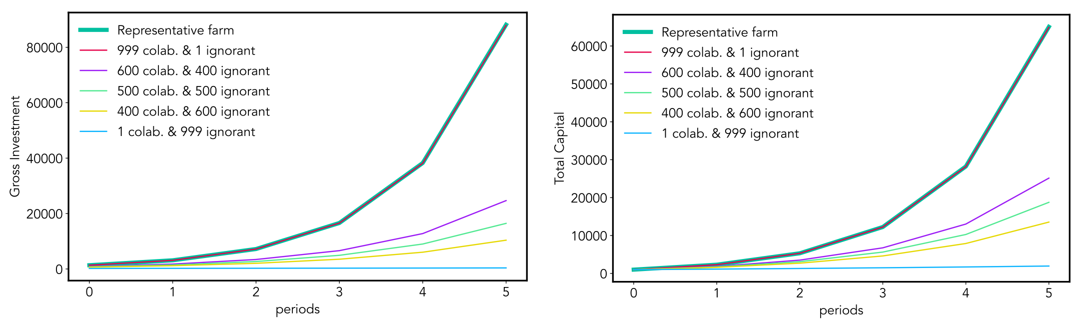
Next, we consider mixed economies of collaborative and ignorant agents, with varying shares of these two strategies among agents. Parameters remain those from Table 1 and initial technical progress is set to $\tilde{\eta}_0 = 7.00$. Table 6 shows the combinations of agents under investigation, the initial capital, $K_0 = 1000$ is now again distributed equally among agents.

Economy No.	Number of farms		Average GDP growth rate
	Collaborative	Ignorant	
Benchmark economy	1000	0	0.83
2	999	1	0.83
3	600	400	0.66
4	500	500	0.60
5	400	600	0.54
6	1	999	0.13
7	0	1000	0.13

Table 6: Combinations of agents for heterogeneous economies of 1000 agents and Average GDP growth rate for these heterogeneous economies.

Economy no 1 corresponds to the benchmark economy with $K_0 = 1000, \eta_0 = 7000$. As we introduce heterogeneity to the system, i.e, replace collaborative farms with ignorant farms, the growth rate performance decreases (see Table 6, where the decrease from case 1 to case 2 is too small to show in 2 digits).

The same holds for total capital and gross investment as can be seen in Figure 13. That is, the fact that collaborative economies outperform ignorant ones, seen in Section 4.4, extends to mixed economies where those with higher shares of collaborative agents outperform those with lower shares. These results confirm what was to be expected, as more ignorant agents imply less investment which leads to less technical progress for all farms.

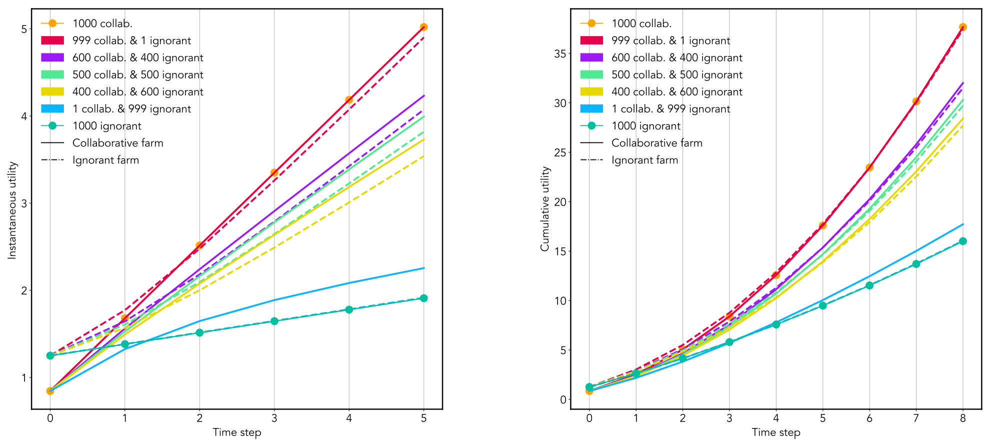


(a) Gross Investment

(b) Total Capital

Figure 13: Macroeconomic variables gross investment 13a and total capital 13b, when initial technical progress is 7, comparing heterogeneous economies of 1000 agents against the corresponding representative agent.

In terms of instantaneous utility, for homogeneous economies we saw a switch from the initial step, where ignorant agents benefit from consuming more, to later steps, where collaborative agents benefit from having more capital. The analogous result holds for economies that differ in shares of collaborative and ignorant agents, as seen in Figure 14.



(a) Instantaneous utilities

(b) Cumulative utilities

Figure 14: Utilities of collaborative farms (solid lines) and ignorant farms (dashed lines) with initial technical progress $\tilde{\eta}_0 = 7.00$, comparing systems with different combinations of agents' strategies and the corresponding benchmark economy.

4.7 Expectations

Finally, we also add witty agents into the economies under consideration. General parameters are still given by Table 1, the values for the expectation weight factor λ_A and the extrapolation coefficient λ_T are set in accordance with the experimental evidence provided by Anufriev and Hommes [2012], see Table 7.

Expectation rule	Description	Value
Adaptive	Adaptive (ADA)	0.65
	Naive	1.00
Trend following	Weak trend follower (WTR)	0.40
	Strong trend follower (STR)	1.30

Table 7: Values for expectation parameters

To initialize the simulation, for lack of previous values such as $I_{-1}^{(\sim i),e(i)}$ for use in (18) or (19), the agents' expectations need to be set. We use several assumptions:

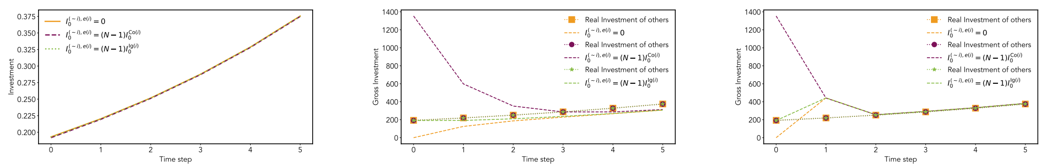
$$I_0^{(\sim i),e} = 0$$

$$I_0^{(\sim i),e} = (N-1) I_0^{\mathrm{Co}(i)} \tag{25}$$

$$I_0^{(\sim i),e} = (N-1) I_0^{\mathrm{Ig}(i)} \tag{26}$$

where in (25) and (26), agent i assumes that everybody else to have the same amount of initial capital and assumes they all invest collaboratively, respectively, according to the ignorant strategy. The first case simply represents the assumption that there is no prior history or information before time zero.

Figure 15 shows that the initialisation of expectations does not make a big difference. Given the small size of an agent compared to the whole economy, the different initialisation values quickly wear off; in Figure system 15c one can see that for trend followers this happens within one time step. Hence, for simplicity, we use initial expectations of zero in the following.



(a) Investment with different initial expectations

(b) ADA Expectations against real value

(c) STR Expectations against real value

Figure 15: Investment under different initial expectations (15a), and agent's expectations against the real value of the investment of the other agents under the ADA rule (15b) and the STR rule (15c). Initial technical progress is $\tilde{\eta}_0 = 7.00$.

4.7.1 Homogeneous economies of witty farms

As in Section 4.4, we first explore the impact of different expectation updating rules within homogeneous economies with an equal distribution of initial capital for the four scenarios from Table 7 (ADA, NAIVE, WTR, STR). Again, we use different values of initial technical progress (see Table 2) to compare these economies with the ones seen previously. This approach focuses on first understanding impacts of the different expectation heuristics on the economic system independently of the other heuristics (following Dosi et al. [2020]). Table 8 provides the average GDP growth rate of the homogeneous economies across the different scenarios.

Initial $\tilde{\eta}_0$	**Average GDP growth rate**					
	Ignorant	ADA	NAIVE	WTR	STR	Collaborative
0.500	-0.94	-0.94501	-0.94613	**-0.94349**	-0.94469	-0.24
1.091	-0.70	-0.70397	**-0.68985**	-0.70136	-0.70262	0.00
3.000	-0.29	**-0.28847**	-0.28862	-0.28871	-0.28887	0.41
5.441	0.00	**0.00153**	0.00153	0.00153	0.00153	0.70
7.000	0.13282	**0.13340**	0.13338	0.13337	0.13335	0.83505
11.000	0.37959	**0.38138**	0.38131	0.38125	0.38114	1.08281

Table 8: GDP growth rate of the four homogeneous witty economies across different initial technical progress $\tilde{\eta}_0$ with initial expectations set to zero (bold lettering is used to indicate the best macroeconomic performance). For comparison, we repeat the values for ignorant and collaborative homogeneous economies.

The expectation rules perform similarly, with the adaptive rule slightly outperforming the others in terms of growth rate. However, this difference is negligible, as it only appears after several decimal places. In most cases, the witty economies are rather close to, but slightly better off than the ignorant economy. None of the homogeneous economies of witty agents thus resolves the well-known problem of too little investment in a decentralised economy.

Given agents with expectations, in addition to the economic performance in terms of growth, we examine the forecast performance of the agents' predictions and its interaction with the macroeconomic performance of their economies. We start the evaluation from time step three, as the farms that use the trend following rule cannot fully apply it before then. To evaluate how well the agents forecast, we compute the forecast error, i.e, the difference between the farm's expectation and the actual value:

$$\text{Error}_t^{(i)} = I_t^{(\sim i)} - I_t^{(\sim i),e}. \tag{27}$$

The sign of the forecast error indicates whether an agent under or over expected, i.e. whether

$$\text{Error}_t^{(i)} > 0 \implies I_t^{(\sim i)} > I_t^{(\sim i),e}, \text{ i.e. agent } i \text{ under expected}$$
$$\text{Error}_t^{(i)} < 0 \implies I_t^{(\sim i),e} > I_t^{(\sim i)}, \text{ i.e. agent } i \text{ over expected.}$$

To consider more than one time step, we compute the *mean squared forecast error (MSFE)*, by averaging a farm's squared forecast error over all time steps, i.e.,

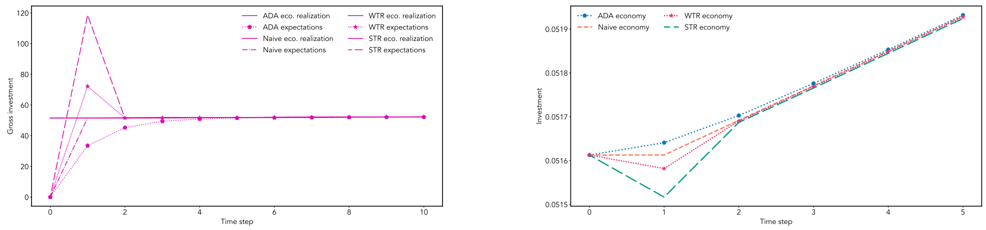
$$\text{MSFE} = \frac{1}{T}\left(\sum_{t=1}^{T}\left(\text{Error}_t^{(i)}\right)^2\right).$$

Table 9 reports the MSFE for $T = 10$ for the four witty economies under consideration and the previously used initial technical progress values for comparing the different economies. While from $\tilde{\eta}_0 = 3$ the adaptive heuristics outperforms the other rules in terms of economic growth, the ADA-witty farms under expect, and they are worse forecasters than the (weak or strong) trend followers.

Overall, we observe that the expectation heuristic used plays a minor role for economic growth, which is mostly determined by the initial technical progress value. The forecast performance of agents does not provide an advantage to one economy of witty agents over another one in terms of growth. Figure 16 illustrates both that the agent who under expects (ADA) invests more than the one who over expects (STR), re-iterating the qualitative free-rider nature of witty agents, and the fact that both expectations and investment levels converge to very similar levels rather quickly.

η_0	**Strategy**	**Mean forecast error**	**MSFE**
0.500	ADA	23.1299	1834.2083
	NAIVE	8.4737	316.4678
	WTR	-0.2628	**0.3323**
	STR	-19.8819	1741.5303
1.091	ADA	24.7039	1728.5980
	NAIVE	11.2707	425.7595
	WTR	2.0666	**13.8971**
	STR	-18.1467	1118.1749
3.000	ADA	17.7929	459.3573
	NAIVE	10.6242	180.8833
	WTR	4.9669	**39.6634**
	STR	-7.6490	91.0308
5.441	ADA	0.4584	0.641
	NAIVE	0.0795	0.0063
	WTR	0.0466	**0.0022**
	STR	-0.0356	0.0026
7.000	ADA	102.3945	11857.7651
	NAIVE	70.2860	5643.3506
	WTR	45.6642	2382.1935
	STR	-9.7006	**107.1201**
11.000	ADA	6423.0403	81e6
	NAIVE	4880.7784	46e6
	WTR	3544.1083	24e6
	STR	545.0876	**5e5**

Table 9: Performance of the farms' expectation heuristics in terms of mean forecast error and MSFE across different levels of initial technical progress values (bold text highlights the most accurate forecasts).



(a) Agent's expectation against the real value (b) Agent's investment

Figure 16: Agent's expectation under different heuristics against the real value of the investment of the other agents (16a), and agent's investment (16b) when initial technical progress is $\tilde{\eta}_0 = 5.44$ for ten time steps.

While our agent-based model would allow to further extend the investigation into several directions at this point, such as homogeneous economies with different initial capital allocations, economies with varying combinations of different strategies with equal or unequal initial capitals, and witty agents with varying heuristics for updating expectations, doing all of these at once would go beyond the scope of this paper. In the following section, we construct a simple example of heterogeneous farms both in terms of investment strategy and in terms of initial capital for some further exploration.

4.7.2 A tale of three farms: a witty farm and a combination of standard agents

For simplicity, we consider only three farms to explore different initial capital settings for agents with different (collaborative, ignorant, and witty) investment strategies. Based on the results from the previous section, where the adaptive updating rule for expectations demonstrated the best macroeconomic performance and the STR rule provided better forecasts, we compare witty farms that use the Adaptive (ADA) or the Strong Trend Following (STR) rule.

We equip the farms with initial capital stocks $K_0^{(1)} > K_0^{(2)} > K_0^{(3)}$ and vary which strategy is used with which initial capital to study twenty-four scenarios (for more detail refer to Table 10) that can be categorized as follows:

1. Twelve scenarios involve two collaborative or two ignorant agents and a witty agent using different heuristics with varying capital allocation for the agents.
2. The other twelve involve one collaborative, one ignorant and one witty agent. In six of these scenarios, the collaborative agent receives more capital than the ignorant agent, while varying capital allocations for the agents.
3. The remaining six scenarios are those where the ignorant agent receives more capital than the collaborative agent, while capital allocations are varied.

The values of initial capital we assign to the agents are: $K_0^{(1)} = 2.00, K_0^{(2)} = 0.7$ and $K_0^{(3)} = 0.3$ These values are selected in such way that we can guarantee a positive investment in almost all cases.

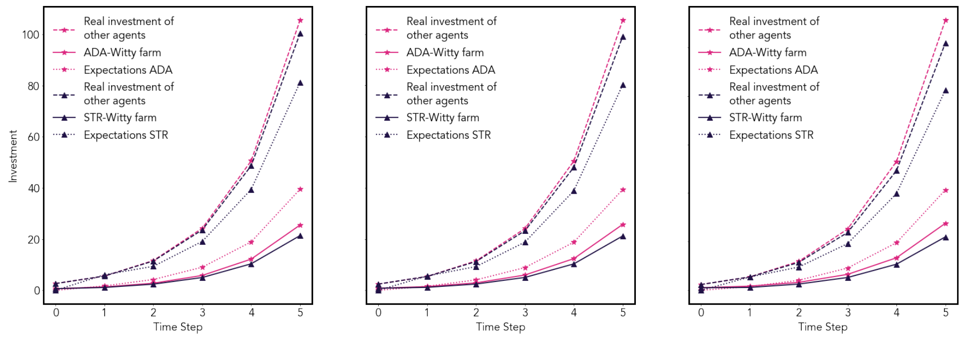
To assess the macroeconomic performance of these systems, we compare their GDP growth rates with the corresponding benchmark economy of three collaborative agents. As to be expected, the best-performing scenarios are those with a majority of collaborative, the worst-performing scenarios those with a majority of ignorant agents. In the direct comparison of the witty agent's heuristics, in all cases, the systems that include a witty farm following an adaptive rule perform better than their counterparts where the witty farm follows a trend-following rule. However, they are all outperformed by the benchmark economy, which has a growth rate of 0.832960 (see Table 10).

As seen before in Table 9 and here below in Figure 17, farms that utilize STR heuristics can forecast other agents' investments with greater precision than their counterparts using ADA heuristics, but this goes along with lower growth.

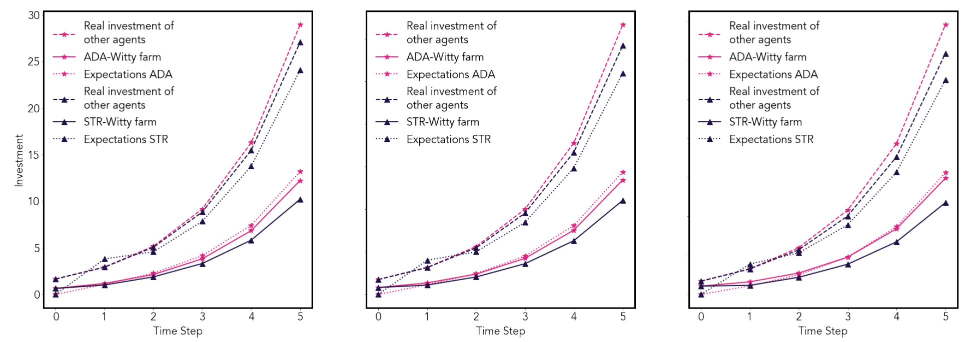
			Capital Combination		
			(2, 0.7, 0.3)	(2, 0.3, 0.7)	(0.7, 0.3, 2.0)
Strategy	2 Collaborative +	ADA	0.734330	0.734333	0.734474
		STR	0.722043	0.722015	0.722085
	1 Collaborative + Ignorant +	ADA	0.575321	0.575548	0.575781
		STR	0.558449	0.558640	0.558787
	1 Ignorant+ 1 Collaborative +	ADA	0.574923	0.574984	0.575620
		STR	0.558036	0.558056	0.558619
	2 Ignorant +	ADA	0.389751	0.390202	0.391100
		STR	0.374152	0.374575	0.375398

Table 10: Growth rates across different scenarios. The order of strategies employed by the agents corresponds to the order of the combinations of initial capital. For example, consider the scenario with 2 collaborative agents and 1 witty agent. In this case, for the capital combination (2, 0.7, 0.3), the two collaborative agents receive capital amounts of 2 and 0.7 respectively, while the witty agent receives 0.3.

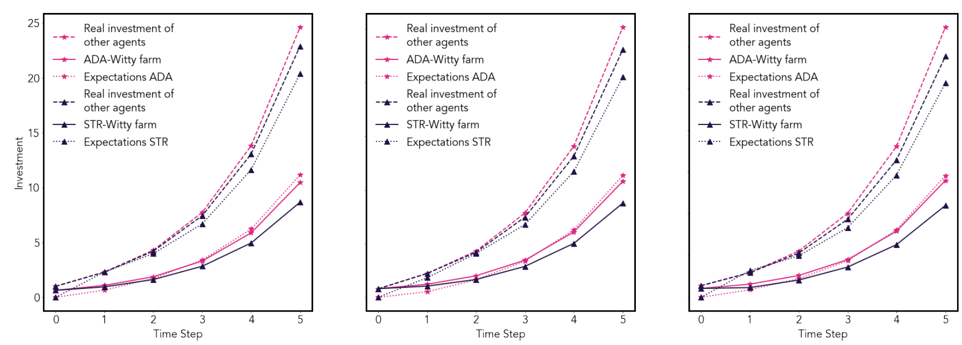
Comparing the initial capital distributions, in most cases the growth rate slightly increases with the initial capital of the witty agent but the differences are marginal. Interestingly, the witty farms manage to achieve comparable levels of instantaneous utility as the collaborative farms, and even surpass them when they start with more initial capital (see Figure 18). This is despite the fact that the witty farms invest systematically less than the collaborative farms (see 17).



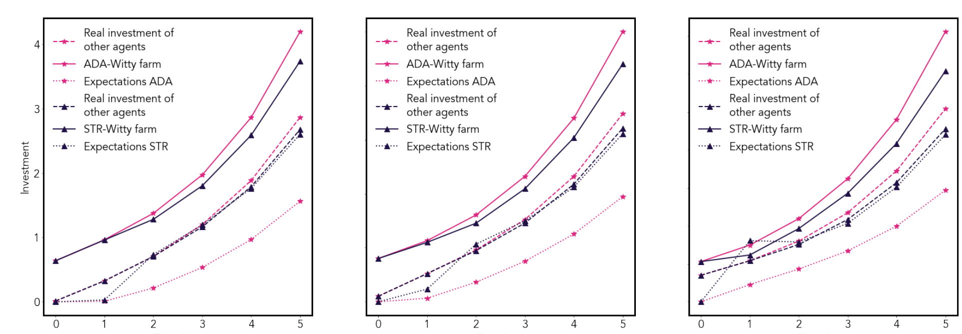
(a) Two collaborative farms and one witty farm



(b) One collaborative farm with initial capital higher than the ignorant farm, and one witty farm

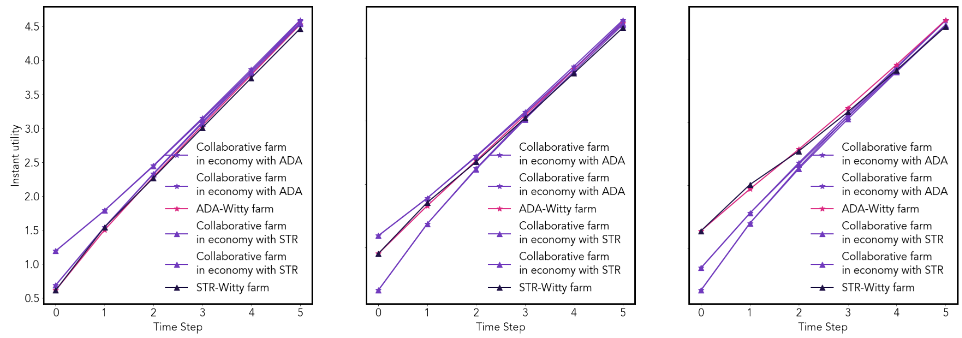


(c) One ignorant farm with initial capital higher than the collaborative farm, and one witty farm

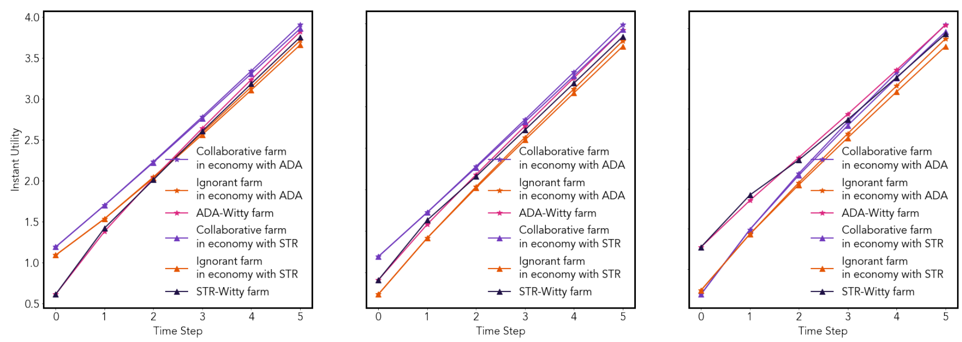


(d) Two ignorant farms and one witty farm

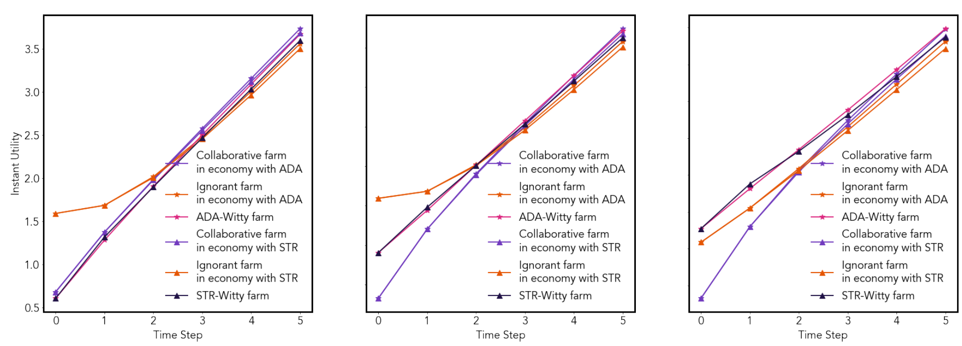
Figure 17: Agents' expectations against the real investment of the other agents across different scenarios for five time steps.



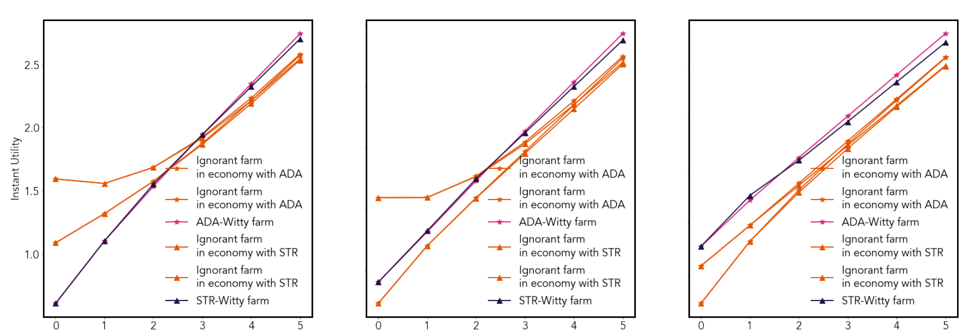
(a) Two collaborative farms and one witty farm



(b) One collaborative farm with initial capital higher than the ignorant farm, and one witty farm



(c) One ignorant farm with initial capital higher than the collaborative farm, and one witty farm



(d) Two ignorant farms and one witty farm

Figure 18: Agent's instantaneous utility across different scenarios for five time steps.

5 Conclusions and outlook

In this paper, we presented an agent-based model in a discrete-time setting to reproduce and explore generalisations of a standard Ramsey growth model with endogenous technical progress. The agents in the model are 'farms' that combine activities typically associated with both firms (such as production) and households (consumption). These agents maximise their intertemporal utility over the current and one future time step and then dynamically iterate in that the investment decision determines the capital for the following time step in which the same decision is repeated for the new level of capital. A statistician agent takes account of the thus evolving capital stock in the system and the resulting technical progress that evolves correspondingly to represent learning-by-doing.

Moving from a representative agent to a decentralised economy, we went beyond the standard assumption that agents ignore their contribution to technical progress – leading to lower investment than the socially optimal solution by a representative agent – and define two further investment strategies of agents. A collaborative agent acts in analogy with the representative one, a witty agent solves the maximisation problem taking into account its contribution to technical progress. We then explored economies populated by such agents introducing heterogeneity in terms of initial capital and of investment strategy.

A main research question behind the work was whether or how a decentralised economy can move to the socially optimal path when relaxing the assumption of the agents' ignorance about technical progress. In a sense, we have obtained two kinds of non-result on this question. On the one hand, a decentralised economy of collaborative agents can reproduce the socially optimal pathway if they all start with the same initial capital, but the construction of the collaborative agent was rather unintuitive at the individual agent's level, so that this result does not provide a satisfying explanation of a decentralised economy. On the other hand, the witty agent does not solve the problem either: it essentially remains an agent with a propensity to "free-ride" on other agents' investment, investing less the more it expects others to invest, and it is noticeably different from the ignorant agent only when owning a comparatively large part of the economy's capital stock. As a convenient side effect, the latter result may be viewed as soothing the troubled mind of the non-economist who wonders why ignorance should be the standard assumption in the first place: if agents are small in comparison to the total economy, which is the case for all but a few in the real world, taking into account their contribution does not actually make much of a difference.

Nevertheless, the explorations carried out have been fruitful in two respects. First, they provided some more and some less expected insights: standard results could be reproduced for standard agents with equal capital distribution. For unequal initial capital distributions in a collaborative economy, the fact that these inevitably become more equal and in terms of growth rate converge to the equal economy, hence outperforming the corresponding benchmark economy, came as a bit of a surprise. Here, the troubled mind of the non-economist looking at the Ramsey model is rather more troubled by this fairy tale of the problem of inequality resolving itself by model construction. The problem could be solved by outsourcing production to a central entity that uses the total capital and labour amounts in the economy, while agents earn a return on capital and a wage [see Asano et al., 2021]. Second, they raised new questions. The ABM, while transferring the Ramsey growth model to an iterative dynamic setting, still has very myopic agents. As seen, the impression that it is beneficial to invest less largely depends on this myopic view; in the longer run, the agents who invest more fare better. How best to include a longer time horizon in the agents' decision making (without simply adding time steps into the maximisation that then require making decisions for several time steps which will become obsolete when time moves on) is one of these questions.

A further interesting extension to the ABM concerns the fact that it is still rather static in the sense that agents interact only via aggregate technical progress and, in the case of the witty agent, expectations on each others' investment. One of the advantages of agent-based models is the possibility to include explicit networks and interactions. Here this should be done to investigate whether forms of cooperation, or more generally, which forms of institutions might help shift the system on the socially optimal growth path. After all, the externality in learning-by-doing can be considered a type of commons problem, suggesting that there is a lot to learn from the respective literature.

Moreover, the model presented was, as mentioned, deterministic. Extending it to agents with stochastic investment decisions, and exploring whether, for example, expectations can then tilt as in the case of opinion dynamics models, and whether this can make the system tilt from a given growth path to another one, is another question worth investigating in future work.

Last but not least, and returning to the context that this work originated in, the question of how to extend agents with expectations to a setting where there are (at least) two types of capital stock and a labour market with search arises.

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A Optimal investment for a benevolent planner

To recap, a benevolent planner, as a single agent corresponding to the whole economy, wants to maximise

$$U = \ln C_0 + \rho \ln C_1 = u(C_0) + \rho u(C_1) \tag{28}$$

such that

$$\begin{aligned}
C_0 &= K_0^\alpha \eta_0^{1-\alpha} - I_0 \\
K_1 &= (1-\delta) \cdot K_0 + I_0 \\
\eta_1 &= K_1 \cdot \frac{\eta_0}{K_0} \\
C_1 &= K_1^\alpha \eta_1^{1-\alpha} = K_1^\alpha \left(K_1 \cdot \frac{\eta_0}{K_0} \right)^{1-\alpha} = \left(\frac{\eta_0}{K_0} \right)^{1-\alpha} K_1
\end{aligned}$$

In this case, η_1 can be eliminated; as mentioned, this corresponds to the AK-model with $A = \left(\frac{\eta_0}{K_0}\right)^{1-\alpha}$, that is, the production is simply AK_t for all t. We can rewrite the utility function for a benevolent planner as follows:

$$U(I_0) = \ln\left[AK_0 - I_0\right] + \rho \ln\left[A\left((1-\delta)K_0 + I_0\right)\right] \tag{29}$$

As the logarithm requires positive input values, $-(1-\delta)K_0 < I_0 < AK_0$ must hold, where the second inequality has the real-world interpretation that investment cannot be larger than production. The constraint provided by the first inequality is relevant only in a setting with reversible investment, i.e., where negative investment is allowed: if capital can be de-invested, the threshold for doing so is given by the value of capital after depreciation.

To maximize this utility, we can differentiate with respect to I_0,

$$\begin{aligned}
\frac{dU}{dC_0} &= -\frac{1}{AK_0 - I_0} + \frac{\rho A}{A((1-\delta)K_0 + I_0)} \\
&= -\frac{1}{C_0} + \frac{\rho}{K_1},
\end{aligned} \tag{30}$$

equate (30) with zero and solve for I_0,

$$\begin{aligned}
\frac{1}{AK_0 - I_0} &= \frac{\rho}{(1-\delta)K_0 + I_0} \\
(1-\delta)K_0 + I_0 &= \rho(AK_0 - I_0) \\
I_0 + \rho I_0 &= \rho A K_0 - (1-\delta)K_0 \\
I_0 &= \frac{1}{1+\rho}(\rho A - (1-\delta))K_0.
\end{aligned} \tag{31}$$

To show that this critical point of (29) is indeed a maximum, the second derivative needs to be negative, but this is the case:

$$\frac{d^2U}{dI_0^2} = -\frac{1}{C_0^2} - \frac{\rho}{K_1^2} < 0, \tag{32}$$

given that the denominators are positive thanks to the quadratic terms.

We can further explore how the optimal investment I_0 behaves as a function of of initial capital K_0. Figure 19 illustrates this function for several values of initial technical progress η_0. The figure shows that the domain where the optimal investment is non-negative increases for larger initial technical progress values; more precisely, $I(K) = \frac{1}{1+\rho}\left(\rho \eta_0^{1-\alpha} K^\alpha - (1-\delta)K\right)$ is positive between

$$\begin{aligned}
&K = 0 \text{ and} \\
&K = \eta_0 \left(\frac{\rho}{1-\delta}\right)^{1/(1-\alpha)}.
\end{aligned}$$

It can further be seen that the function has a maximum and shown that this maximum lies at

$$K^* = \eta_0 \left(\frac{1-\delta}{\alpha\rho}\right)^{1/(\alpha-1)}, \tag{33}$$

we will, however, not further use this maximum in the following.

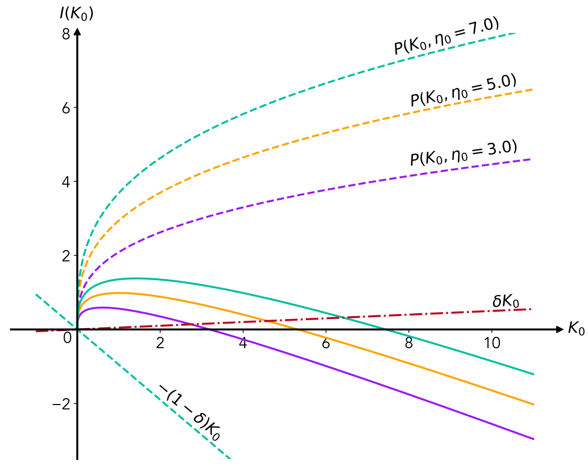


Figure 19: Initial production ($P(K_0, \eta)$) and deinvestment ($-(1-\delta)K_0$) boundaries (dashed curves) and optimal investment as a function of capital (31) for the parameter values in the main text and three different values of initial technical progress: $\eta_0 = 3$ (purple), $\eta_0 = 5$ (yellow), and $\eta_0 = 7$ (teal). The red dash dotted line represents δK. For each of the "investment hills" depicted, we can see that at the right end, before the optimal investment crosses to negative numbers, investment is less than δK, so this is the range where investment is positive but not enough to replace the capital depreciation that will occur to the next time step.

B Feasibility of investment for the benevolent planner

To make sure that the investment of the benevolent planner is always feasible, one needs to show $-(1-\delta)K_0 < I_0 < P_0$ and hence $-(1-\delta)K_0 < \frac{1}{1+\rho}(\rho P_0 - (1-\delta)K_0) < P_0$. The first inequality means

$$\begin{aligned}-(1+\rho)(1-\delta)K_0 &< \rho P_0 - (1-\delta)K_0\\ -\rho(1-\delta)K_0 &< \rho P_0\\ -(1-\delta)K_0 &< P_0\end{aligned}$$

which is always true as both K_0 and η_0 are chosen positive, and hence P_0 also is. For the second inequality, the same holds, as

$$\begin{aligned}\rho P_0 - (1-\delta)K_0 &< (1+\rho)P_0\\ -(1-\delta)K_0 &< P_0.\end{aligned}$$

The benevolent planner can thus always carry out this optimal investment.

C Optimal investment for an ignorant agent

Ignoring equation (9) above, agent i's goal is to maximise (5) under the conditions (6) to (8), where η_{t+1} is considered a given constant.

To find a maximum, we need to set

$$\frac{dU}{dI_t^{(i)}} = \frac{du}{dC_t^{(i)}}\frac{dC_t^{(i)}}{dI_t^{(i)}} + \rho\frac{du}{dC_{t+1}^{(i)}}\frac{dC_{t+1}^{(i)}}{dK_{t+1}^{(i)}}\frac{dK_{t+1}^{(i)}}{dI_t^{(i)}} \tag{34}$$

to zero. The terms are

$$\begin{aligned}
\frac{du}{dC_t^{(i)}} &= \frac{1}{C_t^{(i)}} \text{ and } \frac{du}{dC_{t+1}^{(i)}} = \frac{1}{C_{t+1}^{(i)}} \\
\frac{dC_t^{(i)}}{dI_t^{(i)}} &= -1 \\
\frac{dC_{t+1}^{(i)}}{dK_{t+1}^{(i)}} &= \alpha (K_{t+1}^{(i)})^{\alpha-1} \eta_{t+1}^{1-\alpha}.
\end{aligned}$$

Hence,

$$\frac{dU}{dI_t^{(i)}} = -\frac{1}{C_t^{(i)}} + \frac{\rho}{C_{t+1}^{(i)}} \alpha (K_{t+1}^{(i)})^{\alpha-1} \eta_{t+1}^{1-\alpha} \tag{35}$$

By substituting $C_{t+1}^{(i)}$ and setting the expression equal to zero we get:

$$\frac{1}{C_t^{(i)}} = \frac{\rho}{\left(K_{t+1}^{(i)}\right)^{\alpha} \eta_{t+1}^{1-\alpha}} \alpha \left(K_{t+1}^{(i)}\right)^{\alpha-1} \eta_{t+1}^{1-\alpha}. \tag{36}$$

Note that due to the form of the logarithm's derivative, $\eta_{t+1}^{1-\alpha}$ cancels in (36). That is, even if the agent had information about exogenously given future technical progress, it would not use it here, wherefore there is no room for expectations on future technical progress in this setting.[10] Therefore, the optimal investment is independent of the future technical progress value.

The remaining terms can be further solved by substituting $C_t^{(i)}$ and $K_{t+1}^{(i)}$ as follows:

$$\begin{aligned}
\frac{1}{C_t^{(i)}} &= \frac{\alpha\rho}{K_{t+1}^{(i)}}, \text{ that is, } K_{t+1}^{(i)} = \alpha\rho C_t^{(i)} \\
(1-\delta)\cdot K_t^{(i)} + I_t^{(i)} &= \alpha\rho\left(\left(K_t^{(i)}\right)^{\alpha} \eta_0^{1-\alpha} - I_t^{(i)}\right) \\
I_t^{(i)} + \alpha\rho I_t^{(i)} &= \alpha\rho\left(K_t^{(i)}\right)^{\alpha} \eta_0^{1-\alpha} - (1-\delta)K_t^{(i)} \\
I_t^{(i)} &= \frac{1}{1+\alpha\rho}\left(\alpha\rho\left(K_t^{(i)}\right)^{\alpha} \eta_t^{1-\alpha} - (1-\delta)K_t^{(i)}\right) \qquad (38)
\end{aligned}$$

As for the benevolent planner, it can be shown that this point is a maximum. We will consider feasibility and growth conditions in Section 4.1.

D Optimal investment for a witty agent

Consider an economy of n farms that do not know each others' investment levels. As before, agent i wants to maximize (5), however, now all conditions (6) – (9) are considered. As for the ignorant agent, we need to solve $\frac{dU^{(i)}}{dI_t^{(i)}} = 0$, but now this includes a term $\frac{d\eta_{t+1}}{dI_t^{(i)}}$:

$$\frac{dU^{(i)}}{dI_t^{(i)}} = \frac{du}{dC_t^{(i)}} \frac{dC_t^{(i)}}{dI_t^{(i)}} + \rho\left(\frac{du}{dC_{t+1}^{(i)}} \frac{dC_{t+1}^{(i)}}{dK_{t+1}^{(i)}} \frac{dK_{t+1}^{(i)}}{dI_t^{(i)}} + \frac{du}{dC_{t+1}^{(i)}} \frac{dC_{t+1}^{(i)}}{d\tilde{\eta}_{t+1}} \frac{d\tilde{\eta}_{t+1}}{dI_t^{(i)}}\right) \tag{39}$$

[10] Using another felicity function, such as the Constant Relative Risk Aversion utility function

$$u(C_t^{(i)}) = \begin{cases} \frac{(C_t^{(i)})^{1-\theta}-1}{1-\theta}, & \text{for } \theta \neq 1 \\ \ln C_t^{(i)}, & \text{for } \theta = 1 \end{cases} \tag{37}$$

where $C_t^{(i)}$ is consumption and θ is the coefficient of relative risk aversion, could alleviate this problem, however, this is not the focus of the present analysis, so we leave this point for further research.

Most terms of (39) have been seen before, except:

$$\frac{dC_{t+1}^{(i)}}{d\eta_{t+1}} = (1-\alpha)\left(K_{t+1}^{(i)}\right)^{\alpha}\eta_{t+1}^{-\alpha} \tag{40}$$

$$\frac{d\eta_{t+1}}{dI_t^{(i)}} = \frac{\eta_t}{K_t}, \tag{41}$$

where for the last term we used (14). Again using the short notation for the production,

$$P_t^{(i)} = \left(K_t^{(i)}\right)^{\alpha}\eta_t^{1-\alpha}$$

and substituting all terms into (39), we obtain

$$\begin{aligned}
\frac{dU^{(i)}}{dI_t^{(i)}} &= -\frac{1}{C_t^{(i)}} + \rho\left(\frac{1}{C_{t+1}^{(i)}}\alpha\left(K_{t+1}^{(i)}\right)^{\alpha-1}\eta_{t+1}^{1-\alpha} + \frac{1}{C_{t+1}^{(i)}}(1-\alpha)\left(K_{t+1}^{(i)}\right)^{\alpha}\eta_{t+1}^{1-\alpha}\frac{\eta_t}{K_t}\right)\\
&= -\frac{1}{P_t^{(i)} - I_t^{(i)}} + \frac{\rho}{\left(K_{t+1}^{(i)}\right)^{\alpha}\eta_{t+1}^{1-\alpha}}\left(\alpha\left(K_{t+1}^{(i)}\right)^{\alpha-1}\eta_{t+1}^{1-\alpha} + (1-\alpha)\left(K_{t+1}^{(i)}\right)^{\alpha}\eta_{t+1}^{1-\alpha}\frac{\eta_t}{K_t}\right)\\
&= -\frac{1}{P_t^{(i)} - I_t^{(i)}} + \rho\left(\frac{\alpha}{K_{t+1}^{(i)}} + \frac{1-\alpha}{\eta_{t+1}}\frac{\eta_t}{K_t}\right)\\
&= -\frac{1}{P_t^{(i)} - I_t^{(i)}} + \frac{\rho\alpha}{K_{t+1}^{(i)}} + \frac{\rho(1-\alpha)}{K_{t+1}} = -\frac{1}{P_t^{(i)} - I_t^{(i)}} + \frac{\rho\alpha}{K_{t+1}^{(i)}} + \frac{\rho(1-\alpha)}{K_{t+1}^{(i)} + K_{t+1}^{(\sim i)}}
\end{aligned} \tag{42}$$

The agent can solve (42) = 0 numerically to obtain $I_t^{(i)}$. Further reformulating leads to a quadratic equation that can be solved using the quadratic formula as follows:

$$\begin{aligned}
0 &= -K_{t+1}^{(i)}\left(K_{t+1}^{(i)} + K_{t+1}^{(\sim i)}\right) + \rho\alpha\left(P_t^{(i)} - I_t^{(i)}\right)\left(K_{t+1}^{(i)} + K_{t+1}^{(\sim i)}\right) + \rho(1-\alpha)\left(P_t^{(i)} - I_t^{(i)}\right)K_{t+1}^{(i)}\\
&= K_{t+1}^{(i)}\left[-K_{t+1}^{(i)} + \rho\alpha\left(P_t^{(i)} - I_t^{(i)}\right) + \rho(1-\alpha)\left(P_t^{(i)} - I_t^{(i)}\right)\right] + K_{t+1}^{(\sim i)}\left[-K_{t+1}^{(i)} + \rho\alpha\left(P_t^{(i)} - I_t^{(i)}\right)\right]\\
&= K_{t+1}^{(i)}\left[-K_{t+1}^{(i)} + \rho P_t^{(i)} - \rho I_t^{(i)}\right] + K_{t+1}^{(\sim i)}\left[-K_{t+1}^{(i)} + \rho\alpha P_t^{(i)} - \rho\alpha I_t^{(i)}\right],\\
&\text{which, substituting } K_{t+1}^{(i)} = \hat{K}_t^{(i)} + I_t^{(i)}, \text{ where as before } \hat{K} = (1-\delta)K,\\
&= \left(\hat{K}_t^{(i)} + I_t^{(i)}\right)\cdot\left[-\left(\hat{K}_t^{(i)} + I_t^{(i)}\right) + \rho P_t^{(i)} - \rho I_t^{(i)}\right] + K_{t+1}^{(\sim i)}\left[-\left(\hat{K}_t^{(i)} + I_t^{(i)}\right) + \rho\alpha P_t^{(i)} - \rho\alpha I_t^{(i)}\right]\\
&= \left(\hat{K}_t^{(i)} + I_t^{(i)}\right)\cdot\left[-\hat{K}_t^{(i)} - (1+\rho)I_t^{(i)} + \rho P_t^{(i)}\right] + K_{t+1}^{(\sim i)}\left[-\hat{K}_t^{(i)} - (1+\rho\alpha)I_t^{(i)} + \rho\alpha P_t^{(i)}\right]\\
&= -(1+\rho)\left(I_t^{(i)}\right)^2 + I_t^{(i)}\left[\rho P_t^{(i)} - \hat{K}_t^{(i)} - (1+\rho)\hat{K}_t^{(i)} - (1+\rho\alpha)K_{t+1}^{(\sim i)}\right]\\
&\quad + \rho P_t^{(i)}\hat{K}_t^{(i)} + \rho\alpha P_t^{(i)}K_{t+1}^{(\sim i)} - \left(\hat{K}_t^{(i)}\right)^2 - K_{t+1}^{(\sim i)}\hat{K}_t^{(i)}
\end{aligned}$$

From this we can calculate the roots according to the quadratic formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with

$$a = -(1+\rho)$$

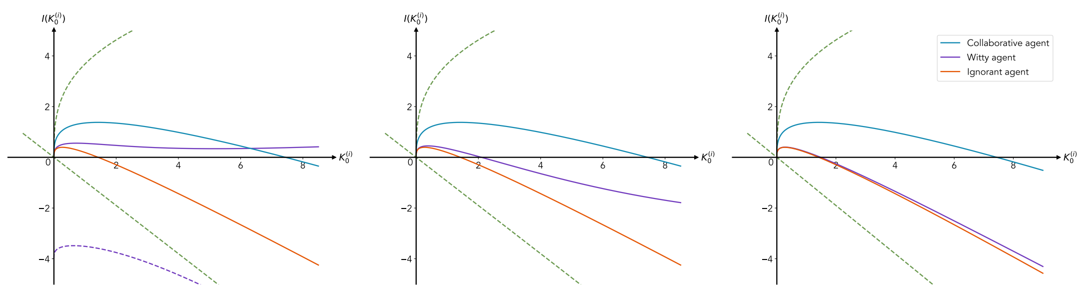
$$b = \rho P_t^{(i)} - (2+\rho)\hat{K}_t^{(i)} - (1+\rho\alpha)K_{t+1}^{(\sim i)} \tag{43}$$

$$c = -\left(\hat{K}_t^{(i)}\right)^2 - \hat{K}_t^{(i)}\left(K_{t+1}^{(\sim i)} - \rho P_t^{(i)}\right) + \rho\alpha P_t^{(i)}K_{t+1}^{(\sim i)} \tag{44}$$

to obtain

$$\left(I_t^{(i)}\right)_{1,2} = \frac{1}{2(1+\rho)}\left(\rho P_t^{(i)} - (2+\rho)\hat{K}_t^{(i)} - (1+\rho\alpha)K_{t+1}^{(\sim i)}\right)$$
$$\mp \frac{\sqrt{\left(\rho P_t^{(i)} - (2+\rho)\hat{K}_t^{(i)} - (1+\rho\alpha)K_{t+1}^{(\sim i)}\right)^2 - 4(1+\rho)\left(\left(\hat{K}_t^{(i)}\right)^2 + \hat{K}_t^{(i)}\left(K_{t+1}^{(\sim i)} - \rho P_t^{(i)}\right) - \rho\alpha P_t^{(i)} K_{t+1}^{(\sim i)}\right)}}{2(1+\rho)} \tag{45}$$

While this formula is not easy to grasp, it helps to look at (43) as a function of the witty agent's capital; this function $b(K_t^{(i)})$ has a negative y-intercept $-(1+\rho\alpha)K_{t+1}^{(\sim i)}$ and while $\rho P_t^{(i)} - (2+\rho)\hat{K}_t^{(i)} > 0$ for small $K_t^{(i)}$, it becomes negative for larger values, so that unless the future capital of all other agents can be assumed very small, the whole term will be negative over the the domain $\mathbb{R}^+$. The term inside the square root is positive so that the first solution turns out to be negative and lies below the potential de-investment threshold. The second one is the solution that will be used for witty agents. Example plots are shown in Figure 20 for a witty agent that is first a larger and then a smaller and smaller part of the total economy. As seen in the figure, the smaller a part of the economy the witty agent is, the closer its optimal investment is to that of the ignorant agent.



(a) Witty agent with one other (b) Witty agent with 9 others (c) Witty agent with 99 others

Figure 20: Witty agent's optimal investment as a function of capital (purple line), together with those of a collaborative (blue) and an ignorant agent's (orange) respective functions. The three plots show the curve for the total economy consisting of two (20a), ten (20b), and a hundred (20c) agents with an equal initial capital of 1. The parameter values are as above, $\eta_0 = 7$. The negative solution (purple dashed line) shifts further down for the larger economies so that it is visible only in the first case.

The latter behaviour of the function can be shown also by looking at the curvature of the function. To do so, we rewrite the solution of the maximization problem as a function $y(x)$ of the agent's capital x, using the quadratic formula terms:

$$y(x) = \frac{1}{2\cdot a}\cdot\left(-b\left(x\right) \; - \; \left(\left(b\left(x\right)\right)^2 \; - 4ac\left(x\right)\right)^{\frac{1}{2}}\right) \tag{46}$$

Its first derivative is:

$$\frac{\partial y}{\partial x} = -\frac{1}{2a}\left(\frac{2b\left(x\right)b'\left(x\right) - 4ac'\left(x\right)}{2\sqrt{b^2\left(x\right) - 4ac\left(x\right)}} + b'\left(x\right)\right) \tag{47}$$

and the second derivative:

$$\frac{\partial^2 y}{\partial x^2} = -\frac{1}{2a}\left(\frac{-4ac''\left(x\right) + 2b\left(x\right)b''\left(x\right) + 2\left(b'\left(x\right)\right)^2}{2\sqrt{b^2\left(x\right) - 4ac\left(x\right)}} - \frac{\left(2b\left(x\right)b'\left(x\right) - 4ac'\left(x\right)\right)^2}{4\left(b^2\left(x\right) - 4ac\left(x\right)\right)^{\frac{3}{2}}} + b''\left(x\right)\right) \tag{48}$$

The curvature K of the function is:

$$K = \frac{|y''(x)|}{[1 + (y'(x))^2]^{3/2}} \tag{49}$$

where $y'(x)$ and $y''(x)$ are first and second derivatives just computed. Plugging in the respective equations (47) and (48), we get:

$$K = \frac{\left| -\frac{1}{2a} \left(\frac{-4ac''(x)+2b(x)b''(x)+2\left(b'(x)\right)^2}{2\sqrt{b^2(x)-4ac(x)}} - \frac{\left(2b(x)b'(x)-4ac'(x)\right)^2}{4(b^2(x)-4ac(x))^{\frac{3}{2}}} + b''(x) \right) \right|}{\left[1 + \left(-\frac{1}{2a} \left(\frac{2b(x)b'(x)-4ac'(x)}{2\sqrt{b^2(x)-4ac(x)}} + b'(x) \right) \right)^2 \right]^{3/2}} \tag{50}$$

Rewriting this in terms of the agents' capital variables, and substituting

$$D = (1-\delta)(K_{t+1}^{(\sim i)} + I_t^{(i)})$$

one gets

$$b'(K_t^{(i)}) = \rho\alpha \left(K_t^{(i)}\right)^{(\alpha-1)} \eta_t^{(1-\alpha)} - (2+\rho)(1-\delta) \tag{51}$$

$$b''(K_t^{(i)}) = \rho\alpha(\alpha-1) \left(K_t^{(i)}\right)^{(\alpha-2)} \eta_t^{(1-\alpha)} \tag{52}$$

$$c'(K_t^{(i)}) = -(1-\delta)2\hat{K}_t^{(i)} + K_{t+1}^{(\sim i)} + \rho P_t^{(i)} + \rho\alpha \left(K_t^{(i)}\right)^{(\alpha-1)} \eta_t^{(1-\alpha)} K_t^{(i)} + \rho\alpha^2 \left(K_t^{(i)}\right)^{(\alpha-1)} \eta_t^{(1-\alpha)} \cdot D \tag{53}$$

$$c''(K_t^{(i)}) = -2(1-\delta)^2 + \rho\alpha \left(K_t^{(i)}\right)^{(\alpha-1)} \eta_t^{(1-\alpha)} + \rho\alpha^2 \left(K_t^{(i)}\right)^{(\alpha-1)} \eta_t^{(1-\alpha)} + \rho\alpha^2(\alpha-1) \left(K_t^{(i)}\right)^{(\alpha-2)} \eta_t^{(1-\alpha)}. \tag{54}$$

Note that only the second derivative of c contains the investment of the others, in the term D. So the relevant part in the curvature of the function is only the term containing $c''(K_t^{(i)})$. We can hence see that when the investment of others goes to infinity, the curvature increases, meaning that the curve turns down more strongly and approaches that of the ignorant agent. If the investment of the others goes to zero, the curvature vanishes.

E Initial technical progress in a decentralised economy

We here consider many agents and distribute the benevolent planner's initial capital among them. As each agent has one unit of labour, the initial production obviously increases with the number of agents, n. Also, and maybe a bit less obviously, the way in which the capital is distributed among them plays a role for how large this economy's initial production is; the more equally the initial capital is distributed, the larger the total initial production. We illustrate this for the simplest case of just two agents and under the assumption that $\alpha = 1/k$ for some $k \in \mathbb{N}$.

In this case, first of all

$$f(K_0, \eta_0) = K_0^\alpha \cdot \eta_0^{1-\alpha} < f(K_0^{(1)}, \eta_0) + f(K_0^{(2)}, \eta_0),$$

where $K_0 = K_0^{(1)} + K_0^{(2)}$ is the total, i.e., the benevolent planner's capital. This corresponds to

$$(K_0^{(1)} + K_0^{(2)})^\alpha \cdot \eta_0^{1-\alpha} < \left(K_0^{(1)}\right)^\alpha \cdot \eta_0^{1-\alpha} + \left(K_0^{(2)}\right)^\alpha \cdot \eta_0^{1-\alpha}$$

Canceling $\eta_0^{1-\alpha}$ and raising both sides to the power of $k = 1/\alpha$, where $k > 1$ because $0 < \alpha < 1$, yields

$$K_0^{(1)} + K_0^{(2)} < (K_0^{(1)\alpha} + K_0^{(2)\alpha})^k. \tag{55}$$

The right hand side of this is

$$\begin{aligned} (K_0^{(1)\alpha} + K_0^{(2)\alpha})^k &= \sum_{i=0}^{k} \binom{k}{i} K_0^{(1)\alpha(k-i)} K_0^{(2)\alpha i} \\ &= K_0^{(1)\alpha k} + \binom{k}{1} K_0^{(1)\alpha(k-1)} K_0^{(2)\alpha} + \ldots + K_0^{(2)\alpha k} \\ &= K_0^{(1)} + \binom{k}{1} K_0^{(1)\alpha(k-1)} K_0^{(2)\alpha} + \ldots + K_0^{(2)} \end{aligned}$$

Thus we have $K_0^{(1)}$ and $K_0^{(2)}$ on each side in equation (55) and they cancel. What remains on the right hand side is positive, proving the claim.

To show that equal distribution leads to the largest production, consider an initial capital of $2K$ for two agents, i.e., we need to show that production is larger if both agents start with initial capital K than when one starts with $K+a$ and the other with $K-a$, where $0 < a < K$.

$$\begin{aligned} 2(K)^\alpha \cdot \eta_0^{1-\alpha} &\geq (K+a)^\alpha \cdot \eta_0^{1-\alpha} + (K-a)^\alpha \cdot \eta_0^{1-\alpha} \\ \Leftrightarrow 2(K)^\alpha &\geq (K+a)^\alpha + (K-a)^\alpha \end{aligned}$$

Again, raising this to the power of $k = 1/\alpha$, we get

$$2^k K \geq ((K+a)^\alpha + (K-a)^\alpha)^k$$

and the right hand side can again be reformulated as above:

$$\begin{aligned} ((K+a)^\alpha + (K-a)^\alpha)^k &= \sum_{i=0}^{k} \binom{k}{i} (K+a)^{\alpha(k-i)} (K-a)^{\alpha i} \\ &= \sum_{i=0}^{k} \binom{k}{i} \left((K+a)^{(k-i)} (K-a)^{i}\right)^\alpha \end{aligned}$$

As $2^k K = \left(\sum_{i=0}^{k} \binom{k}{i}\right) K = \sum_{i=0}^{k} \binom{k}{i} K^{k\alpha}$, what needs to be shown is

$$\sum_{i=0}^{k} \binom{k}{i} K^{k\alpha} \geq \sum_{i=0}^{k} \binom{k}{i} \left((K+a)^{(k-i)} (K-a)^{i}\right)^\alpha \tag{56}$$

In each term of the sum on the right hand side, in $(K+a)^{(k-i)}(K-a)^i$ within the brackets, $((K+a)(K-a))^{\min(k-i,i)}$ can be factored out, and the remaining factor is either $(K+a)^{(k-2i)}$ if $i < k-i$, or $(K-a)^{2i-k}$ if $k-i < i$; in case $k-i = i$, this term is 1 because the exponent vanishes, or in other words, the factors $(K+a)$ and $(K-a)$ appear in equal numbers. The case $k = i$ only occurs when k is an even number. In this case, the number of summands is uneven, and the term $\binom{k}{k/2}$ appears just once in the sum, while for all other terms (and all terms in case of an even k), $\binom{k}{i} = \binom{k}{k-i}$ meaning that we always have two terms beginning with the same factor. As the minimum in the exponent also takes the same value for i and $j = k-i$, the corresponding terms (symmetrically first and last, second and next to last etc. in the sum) can be grouped, so that the right hand side in (57) equals

$$\sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{i} ((K+a)(K-a))^{i\alpha} \left((K+a)^{k-2i} + (K-a)^{k-2i}\right)^\alpha$$

Then, the single terms can be compared: for the $i = k-i$ case, this results in showing

$$\binom{k}{k/2} K^{k\alpha} \geq \binom{k}{k/2} \left(((K+a)(K-a))^{k/2}\right)^\alpha \tag{57}$$

$$K^k \geq \left(K^2 - a^2\right)^{k/2} \tag{58}$$

and this is true because $a < K$ by assumption. For all other cases, it needs to be shown that

$$\begin{aligned} 2\binom{k}{i} K^{k\alpha} &\geq \binom{k}{i} ((K+a)(K-a))^{i\alpha} \left((K+a)^{k-2i} + (K-a)^{k-2i}\right)^\alpha \\ 2K^k &\geq (K^2-a^2)^i \left((K+a)^{k-2i} + (K-a)^{k-2i}\right) \end{aligned}$$

Here, some terms in the last factor on the right hand side cancel, as

$$(K+a)^{(k-2i)} = \sum_{j=0}^{k-2i} \binom{k-2i}{j} K^{(k-2i-j)} a^j$$

$$(K-a)^{(k-2i)} = \sum_{j=0}^{k-2i} \binom{k-2i}{j} K^{(k-2i-j)} (-a)^j$$

That is,

$$(K+a)^{k-2i} + (K-a)^{k-2i} = \sum_{j=0}^{k-2i} \binom{k-2i}{j} \left(K^{(k-2i-j)} a^j + K^{(k-2i-j)} (-a)^j \right)$$

For j even, the last term $K^{(k-2i-j)}a^j + K^{(k-2i-j)}(-a)^j = 2K^{(k-2i-j)}a^j$; for j uneven, this last term is 0. As $a < K$, we know that $2K^{(k-2i-j)}a^j < 2K^{(k-2i)}$ and multiplying this with the other factor $(K^2 - a^2)^i < K^{2i}$, the result follows.

The resulting differences in initial production are illustrated in Figure 21 for different capital distributions for $n = 2$ agents with total initial capital $K_0 = 10$. Note that $K_0^{(1)} = 0$ for one agent means the other agent has $K_0^{(2)} = 10$ and so on. Therefore, the x-axis needs to be plotted only up to the value of 5.

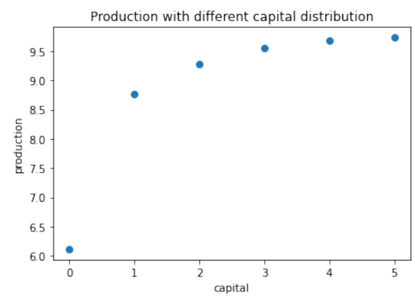


Figure 21: Production with different initial distributions

Economies with different allocations of initial capital are further discussed in Section 4.5.

F A homogeneous economy of collaborative agents reproduces the benevolent planner's dynamics

As discussed in Appendix E, for a decentralised economy, initial technical progress is adapted to produce a system that is comparable with a benevolent planner's economy with the same total initial capital; for the benevolent planner's initial technical progress η_0, we set the initial technical progress value of the decentralised economy to

$$\tilde{\eta}_0 = \eta_0 \left(\frac{K_0^\alpha}{\sum_{i=1}^n \left(K_0^{(i)}\right)^\alpha} \right)^{\frac{1}{1-\alpha}}.$$

This value was chosen in such a way that

$$\sum_{i=1}^n P_0^{(i)} = \sum_{i=1}^n \left(K_0^{(i)}\right)^\alpha \tilde{\eta}_0^{1-\alpha} = \sum_{i=1}^n \left(K_0^{(i)}\right)^\alpha \left(\eta_0 \left(\frac{K_0^\alpha}{\sum_{i=1}^n \left(K_0^{(i)}\right)^\alpha} \right)^{\frac{1}{1-\alpha}} \right)^{1-\alpha}$$

$$= \sum_{i=1}^n \left(K_0^{(i)}\right)^\alpha \eta_0^{1-\alpha} \left(\frac{K_0^\alpha}{\sum_{i=1}^n \left(K_0^{(i)}\right)^\alpha} \right) = \eta_0^{1-\alpha} K_0^\alpha = P_0.$$

Given this equality, also the total investment matches:

$$\sum_{i=1}^n I_0^{(i)} = \sum_{i=1}^n \frac{1}{1+\rho} \left(\rho P_0^{(i)} - (1-\delta) K_0^{(i)} \right) = \frac{1}{1+\rho} \left(\rho \sum_{i=1}^n P_0^{(i)} - (1-\delta) \sum_{i=1}^n K_0^{(i)} \right)$$

$$= \frac{1}{1+\rho} \left(\rho P_0 - (1-\delta) K_0 \right) = I_0$$

and hence

$$\sum_{i=1}^{n} K_1^{(i)} = \sum_{i=1}^{n} \left((1-\delta)K_0^{(i)} + I_0^{(i)}\right) = (1-\delta)\sum_{i=1}^{n} K_0^{(i)} + \sum_{i=1}^{n} I_0^{(i)} = (1-\delta)K_0 + I_0 = K_1$$

which also means that

$$\tilde{\eta}_1 = \sum_{i=1}^{n} K_1^{(i)} \frac{\tilde{\eta}_0}{K_0} = \frac{K_1}{K_0}\tilde{\eta}_0 = \frac{K_1}{K_0}\eta_0 \left(\frac{K_0^\alpha}{\sum_{i=1}^n \left(K_0^{(i)}\right)^\alpha}\right)^{\frac{1}{1-\alpha}} = \eta_1 \left(\frac{K_0^\alpha}{\sum_{i=1}^n \left(K_0^{(i)}\right)^\alpha}\right)^{\frac{1}{1-\alpha}}$$

If the agents' capital is equal, we can repeat this for each time step, so that the behaviour of total capital, production and investment of a homogeneous economy of collaborative agents with equal capital correspond to that of the benevolent planner's economy. For the case of unequal capital distributions, it will be seen in Section 4.5 why this does not work.

G A prisoner's dilemma structure

We consider a single agent, say agent 1 and assume for simplicity a second agent that is much bigger in terms of initial capital. This agent 2 uses the collaborative investment strategy. The question in the main text then becomes: " for which values of $K_0^{(2)}$ is the utility for agent 1 resulting from the ignorant agent's investment strategy larger than the utility resulting from the benevolent planner's investment strategy?". In formulae this is

$$\ln\left(P_0^{(1)} - I_0^{\text{Ig}(1)}\right) + \rho \ln\left(\left(K_0^{(1)}(1-\delta) + I_0^{\text{Ig}(1)}\right)^\alpha \cdot \left(\frac{\left(K_0^{(1)}(1-\delta) + I_0^{\text{Ig}(1)} + K_1^{(2)}\right)\eta_0}{K_0^{(1)} + K_0^{(2)}}\right)^{1-\alpha}\right)$$
$$> \ln\left(P_0^{(1)} - I_0^{\text{Co}(1)}\right) + \rho \ln\left(\left(K_0^{(1)}(1-\delta) + I_0^{\text{Co}(1)}\right)^\alpha \cdot \left(\frac{\left(K_0^{(1)}(1-\delta) + I_0^{\text{Co}(1)} + K_1^{(2)}\right)\eta_0}{K_0^{(1)} + K_0^{(2)}}\right)^{1-\alpha}\right)?$$

Again, $P_0^{(1)}$ is shorthand for production, i.e. $P_0^{(1)} = \left(K_0^{(1)}\right)^\alpha \eta_0^{1-\alpha}$, and $K_1^{(2)}$ is not further detailed because it remains the same (agent 2 invests according to collaborative strategy). We will take a step back and first rewrite the second summand on each side (formally, they look the same when $K_1^{(1)}$ is not further detailed):

$$\begin{aligned}
\rho \ln\left(\left(K_1^{(1)}\right)^\alpha \eta_1^{1-\alpha}\right) &= \rho\alpha \ln\left(K_1^{(1)}\right) + \rho(1-\alpha)\ln(\eta_1) \\
&= \rho\alpha \ln\left(K_1^{(1)}\right) + \rho(1-\alpha)\ln\left(\frac{K_1^{(1)} + K_1^{(2)}}{K_0^{(1)} + K_0^{(2)}}\eta_0\right) \\
&= \rho\alpha \ln\left(K_1^{(1)}\right) + \rho(1-\alpha)\ln\left(K_1^{(1)} + K_1^{(2)}\right) + \rho(1-\alpha)\ln\left(\frac{\eta_0}{K_0^{(1)} + K_0^{(2)}}\right) \quad (59)
\end{aligned}$$

The last term in (59) is independent of agent 1's investment strategy. So, the question above reduces to asking for which values of $K_0^{(2)}$

$$\begin{aligned}
&\ln\left(P_0^{(1)} - I_0^{\text{Ig}(1)}\right) + \rho\alpha\ln\left(K_1^{\text{Ig}(1)}\right) + \rho(1-\alpha)\ln\left(K_1^{\text{Ig}(1)} + K_1^{(2)}\right) \\
&> \ln\left(P_0^{(1)} - I_0^{\text{Co}(1)}\right) + \rho\alpha\ln\left(K_1^{\text{Co}(1)}\right) + \rho(1-\alpha)\ln\left(K_1^{\text{Co}(1)} + K_1^{(2)}\right) \\
\Leftrightarrow &\ln\left(\frac{P_0^{(1)} - I_0^{\text{Ig}(1)}}{P_0^{(1)} - I_0^{\text{Co}(1)}}\right) + \rho\alpha\ln\left(\frac{\hat{K}_0^{(1)} + I_0^{\text{Ig}(1)}}{\hat{K}_0^{(1)} + I_0^{\text{Co}(1)}}\right) + \rho(1-\alpha)\ln\left(\frac{\hat{K}_0^{(1)} + I_0^{\text{Ig}(1)} + \hat{K}_0^{(2)} + I_0^{\text{Co}(2)}}{\hat{K}_0^{(1)} + I_0^{\text{Co}(1)} + \hat{K}_0^{(2)} + I_0^{\text{Co}(2)}}\right) > 0
\end{aligned}$$

The terms inside the first logarithm can be simplified as follows

$$\frac{P_0^{(1)} - I_0^{\mathrm{Ig}(1)}}{P_0^{(1)} - I_0^{\mathrm{Co}(1)}} = \frac{P_0^{(1)} - \frac{1}{1+\rho\alpha}\left(\rho\alpha P_0^{(1)} - \hat{K}_0^{(1)}\right)}{P_0^{(1)} - \frac{1}{1+\rho}\left(\rho P_0^{(1)} - \hat{K}_0^{(1)}\right)} \tag{60}$$

$$= \frac{\frac{P_0^{(1)}(1+\rho\alpha) - \left(\rho\alpha P_0^{(1)} - \hat{K}_0^{(1)}\right)}{1+\rho\alpha}}{\frac{P_0^{(1)}(1+\rho) - \left(\rho P_0^{(1)} - \hat{K}_0^{(1)}\right)}{1+\rho}} = \frac{P_0^{(1)} - \hat{K}_0^{(1)}}{P_0^{(1)} - \hat{K}_0^{(1)}} \cdot \frac{1+\rho}{1+\rho\alpha} = \frac{1+\rho}{1+\rho\alpha} \tag{61}$$

and in the same way, one obtains

$$\frac{\hat{K}_0^{(1)} + I_0^{\mathrm{Ig}(1)}}{\hat{K}_0^{(1)} + I_0^{\mathrm{Co}(1)}} = \frac{(1+\rho)\alpha}{1+\rho\alpha}. \tag{62}$$

However, the third logarithm-term does not simplify as nicely, which is why we resort to a plot for standard example values in the main text.

H About Pareto's principle and the choice of the tail parameter b

The Italian economist Vilfredo Pareto (1848 - 1923) proposed that the number of persons N whose income (capital for our case) is above x, can be modeled as follows [Pareto, 1964]:

$$\log N = \log A - b \log x, \tag{63}$$

where $A, b > 0$ are parameters. (63) doesn't hold for when $N \to \infty$ as $x \to 0$. This implies that we can have an infinite population, although for any $x > 0$ only a finite share of the population would have an income exceeding x [Hardy, 2010]. If we set N_{tot} for the total population and x_{min} for the minimum income of the population, and then we subtract $\log N_{tot} = \log A - b \log x_{min}$ from (63), we can write both equations in proportionate terms:

$$\log\left(\frac{N}{N_{tot}}\right) = -b \log\left(\frac{x}{x_{min}}\right)$$

$$\frac{N}{N_{tot}} = \left(\frac{x}{x_{min}}\right)^{-b}. \tag{64}$$

Let $n = \frac{N}{N_{tot}}$ be the proportion of those whose income is greater than x in the population, and normalize $x_{min} = 1$, we have in this terms "Pareto's Principle" (Isaac):

$$n = x^{-b}. \tag{65}$$

We can find the proportion of the population whose wealth lies after x, by integrating (65). First, we integrate to obtain the area under the curve of (65) from $x_{mint} = 1$ to infinity, we have:

$$\int_{x_{mint}=1}^{\infty} x^{-b} dx = \lim_{\tilde{x}\to\infty} \int_1^{\tilde{x}} x^{-b} dx = -\frac{1}{1-b} = \frac{1}{b-1}.$$

and to find the the area that lies before x, we integrate from $x_{min} = 1$ to x:

$$\int_{x_{min}=1}^{x} \hat{x}^{-b} d\hat{x} = -\frac{1}{b-1}(x^{1-b} - 1). \tag{66}$$

Therefore, the area that lies after x is given by the total area minus the area before x, i.e.

$$-\frac{1}{b-1} + \frac{x^{1-b} - 1}{b-1} = \frac{x^{1-b}}{b-1}. \tag{67}$$

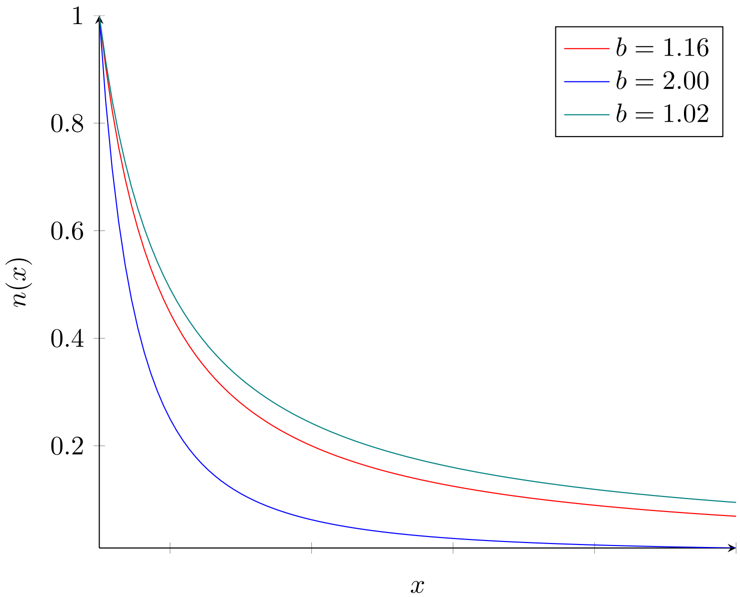


Figure 22: Pareto's principle curve for different tail parameters

So the proportion of the area that lies after any point x is x^{1-b}. Thus, we can write that the share of total income of those with income higher than x is:

$$S = x^{1-b}, \tag{68}$$

from Pareto's principle we can write: $x = n^{-\frac{1}{b}}$, so we find that the top-share of income for a proportion n of the population is given by

$$S = n^{1-\frac{1}{b}}. \tag{69}$$

With this equation we can find Pareto's distribution's tail parameter b, that we can use for distributing the initial capital.

$$b = \frac{\ln n}{\ln n - \ln S}. \tag{70}$$

For example, we have that for the Pareto's 80/20 rule: 20% holds 80% of the wealth, we have that,

$$b_{80/20} = \frac{\ln 0.20}{\ln 0.20 - \ln 0.80} \approx 1.16096.$$

If 1% holds 10% of the wealth, we have that

$$b = \frac{\ln 0.01}{\ln 0.01 - \ln 0.10} = 2.$$

And if 1% of the population holds 90% of the capital, we have:

$$b_{un} = \frac{\ln 0.01}{\ln 0.01 - \ln 0.90} \approx 1.02341.$$