SPACES OF BESOV-SOBOLEV TYPE AND A PROBLEM ON NONLINEAR APPROXIMATION

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Abstract. We study fractional variants of the quasi-norms introduced by Brezis, Van Schaftingen, and Yung in the study of the Sobolev space $\dot{W}^{1,p}$. The resulting spaces are identified as a special class of real interpolation spaces of Sobolev-Slobodeckiĭ spaces. We establish the equivalence between Fourier analytic definitions and definitions via difference operators acting on measurable functions. We prove various new results on embeddings and non-embeddings, and give applications to harmonic and caloric extensions. For suitable wavelet bases we obtain a characterization of the approximation spaces for best n-term approximation from a wavelet basis via smoothness conditions on the function; this extends a classical result by DeVore, Jawerth and Popov.

1. Introduction and statements of results

For $d \geq 1$, $b \in \mathbb{R}$ and a locally integrable function $f \in L^1_{\rm loc}(\mathbb{R}^d)$ consider the difference quotient

$$\mathcal{D}_b f(x,y) = \frac{f(x)-f(y)}{|x-y|^b}, \qquad (x,y) \in \mathbb{R}^d \times \mathbb{R}^d = \mathbb{R}^{2d}. \tag{1.1}$$

Haïm Brezis and two of the authors [8] discovered that for $f \in C_c^\infty(\mathbb{R}^d)$ and $1 \leq p < \infty$, the Marcinkiewicz quasi-norm $[\mathcal{D}_{1+d/p} f]_{L^{p,\infty}(\mathbb{R}^{2d})}$ is comparable to the Gagliardo-seminorm $\|\nabla f\|_{L^p(\mathbb{R}^d)}$ (see also [46], [10] for related results). Using this equivalence, they considered in [9] certain borderline Gagliardo-Nirenberg interpolation inequalities that fail, and proved substitutes such as $[\mathcal{D}_{s+d/p} f]_{L^{p,\infty}(\mathbb{R}^{2d})} \lesssim \|f\|^{1-s}_{L^\infty(\mathbb{R}^d)} \|\nabla f\|^s_{L^1(\mathbb{R}^d)}$ for $s = 1/p$ and $1 < p < \infty$, raising the natural question of what can be said about the class of functions for which $[\mathcal{D}_{s+d/p} f]_{L^{p,\infty}(\mathbb{R}^{2d})}$ is finite for $0 < s < 1$. This class was also considered in the papers by Poliakovsky [48] who asked about a more specific relation to Besov spaces, and in the work by Domínguez and Milman [23] who considered abstract versions of [8]. As a special case of our main results

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we show that the above fractional variant arises as a real interpolation space of a family of homogeneous Sobolev-Slobodeckiĭ spaces $\dot{W}^{s,p}$. Henceforth, for $0 < s < 1$ and $1 < p < \infty$, the space $\dot{W}^{s,p}$ consists of all equivalent classes of measurable, finite a.e. functions f (modulo equality a.e. and additive constants) for which $\mathcal{D}_{s+d/p}f \in L^p(\mathbb{R}^{2d})$, with semi-norm $\|f\|_{\dot{W}^{s,p}} = \|\mathcal{D}_{s+d/p}f\|_{L^p(\mathbb{R}^{2d})}$; this space can be naturally identified the diagonal Besov space $\dot{B}^s_{p,p}$ (see e.g. the case $r = p$ in Theorem 1.3 below). We will show that for $p_0, p_1 \in (1, \infty)$ such that $p_0 < p < p_1$ and $0 < s + \frac{d}{p} - \frac{d}{p_i} < 1$ the norm on the interpolation space $[\dot{W}^{s+\frac{d}{p}-\frac{d}{p_0},p_0}, \dot{W}^{s+\frac{d}{p}-\frac{d}{p_1},p_1}]_{\theta,\infty}$ is equivalent with the quasi-norm $\|\mathcal{D}_{s+\frac{d}{p}}f\|_{L^{p,\infty}(\mathbb{R}^{2d})}$.

The class of functions for which $\|\mathcal{D}_{s+d/p}f\|_{L^{p,\infty}(\mathbb{R}^{2d})}$ is finite was labelled BSY^s_p in [23]. Here we shall denote it by $\dot{\mathcal{B}}^s_p(d,\infty)$ as it will arise as a member of a natural and more general scale of spaces $\dot{\mathcal{B}}^s_p(\gamma, r)$. We begin by giving a Fourier analytic definition of the spaces $\dot{\mathcal{B}}^s_p(\gamma, r)$, which extends the classical definition of the homogeneous Besov space $\dot{B}^s_{p,p}$; in fact $\dot{\mathcal{B}}^s_p(\gamma, r)$ all coincide with $\dot{B}^s_{p,p}$ when $r = p$ (regardless of the value of γ). We have learned in the final stage of preparation of this paper that V.L. Krepkogorskiĭ had already introduced the inhomogeneous variants of these classes in a little noticed paper [36] in 1994 and proved that they occur as interpolation spaces for Sobolev and other spaces; see Remark 1.2 and the comments before Theorem 1.15 below.

Variants of Besov-Sobolev spaces. We let $\varphi \in C^\infty_c(\mathbb{R}^d)$ be a radial function with

$$\operatorname{supp}(\varphi) \subset \{\xi : 3/4 < |\xi| < 7/4\}, \tag{1.2a}$$

$$\varphi(\xi) = 1 \text{ for } 7/8 \le |\xi| \le 9/8, \tag{1.2b}$$

$$\sum_{k\in\mathbb{Z}} \varphi(2^{-k}\xi) = 1 \text{ for all } \xi \neq 0. \tag{1.2c}$$

It is easy to check that the three requirements can be achieved. For a tempered distribution f we define the frequency localizations $L_k f$ via the Fourier transform by

$$\widehat{L_k f}(\xi) = \varphi(2^{-k}\xi)\widehat{f}(\xi).$$

We recall the definition of the diagonal homogeneous Besov spaces $\dot{B}^s_{p,p}$. Consider the space $\mathcal{S}_\infty(\mathbb{R}^d)$ of Schwartz functions whose Fourier transforms vanish to infinite order at 0; this space carries the natural Fréchet topology inherited from the space of Schwartz functions. We let $\mathcal{S}'_\infty(\mathbb{R}^d)$ denote the dual space; it can be identified with the space of tempered distributions modulo polynomials. The space $\dot{B}^s_{p,p}$ is defined as the subspace of $f \in \mathcal{S}'_\infty(\mathbb{R}^d)$ for which

$$\|f\|_{\dot{B}^s_{p,p}} := \Big(\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^d} \big|2^{ks}L_k f(x)\big|^p \,\mathrm{d}x\Big)^{1/p}$$

is finite.

We will now define various Lorentz versions of these spaces where a Lorentz norm is taken on the space $\mathbb{R}^d \times \mathbb{Z}$. Recall that if (Ω, μ) is a measure space and $0 < p, r < \infty$, the Lorentz space $L^{p,r}(\Omega, \mu)$ is defined as the space of measurable functions g on Ω for which

$$[g]_{L^{p,r}(\Omega,\mu)} = \Big(r \int_0^\infty \lambda^r \mu(\{x \in \Omega : |g(x)| > \lambda\})^{r/p} \frac{\mathrm{d}\lambda}{\lambda} \Big)^{1/r}$$

is finite. For $r = \infty$ we set $[g]_{L^{p,\infty}(\Omega,\mu)} = \sup_{\lambda>0} \lambda \mu(\{|g| > \lambda\})^{1/p}$. The space $L^{p,r}$ is normable when $1 < p < \infty$, $1 \le r \le \infty$, and for simplicity we will only consider these parameter ranges. The precise expression for the norm is not important for this paper; a suitable choice ([33]) is

$$\|g\|_{L^{p,r}} = \Big(\int_0^\infty [t^{1/p} g^{**}(t)]^r \frac{\mathrm{d}t}{t} \Big)^{1/p}$$

where $g^{**}(t) = t^{-1} \int_0^t g^*(s)\, \mathrm{d}s$ and g^* denotes the nonincreasing rearrangement of g.

Definition 1.1. *Let $\gamma \in \mathbb{R}$.*

(i) For a measurable subset E of $\mathbb{R}^d \times \mathbb{Z}$ let $\mathbb{1}_E$ be the indicator function of E and

$$\mu_\gamma(E) = \sum_{k\in\mathbb{Z}} 2^{-k\gamma} \int_{\mathbb{R}^d} \mathbb{1}_E(x,k)\, \mathrm{d}x.$$

(ii) For $b \in \mathbb{R}$ define $P^b f : \mathbb{R}^d \times \mathbb{Z} \to \mathbb{C}$ by

$$P^b f(x,k) = 2^{kb} L_k f(x).$$

(iii) For $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le r \le \infty$, let $\dot{\mathcal{B}}^s_p(\gamma, r)$ be the space of $f \in \mathcal{S}'_\infty(\mathbb{R}^d)$ such that the function $P^{s+\frac{\gamma}{p}} f$ belongs to the Lorentz space $L^{p,r}(\mathbb{R}^d \times \mathbb{Z}; \mu_\gamma)$ and let

$$\|f\|_{\dot{\mathcal{B}}^s_p(\gamma,r)} = \big\| P^{s+\frac{\gamma}{p}} f \big\|_{L^{p,r}(\mu_\gamma)}. \tag{1.3}$$

Unravelling the definition, with $\operatorname{meas} A$ denoting the Lebesgue measure of $A \subset \mathbb{R}^d$, if $1 \le r < \infty$ we get the following equivalence

$$\|f\|_{\dot{\mathcal{B}}^s_p(\gamma,r)} \approx \Big(r \int_0^\infty \lambda^r \Big[\sum_{k\in\mathbb{Z}} 2^{-k\gamma} \operatorname{meas}\big\{ x \in \mathbb{R}^d : |L_k f(x)| > \lambda 2^{-k(s+\frac{\gamma}{p})} \big\} \Big]^{r/p} \frac{\mathrm{d}\lambda}{\lambda} \Big)^{1/r} \tag{1.4}$$

whereas

$$\|f\|_{\dot{\mathcal{B}}^s_p(\gamma,\infty)} \approx \sup_{\lambda>0} \lambda \Big[\sum_{k\in\mathbb{Z}} 2^{-k\gamma} \operatorname{meas}\big\{ x \in \mathbb{R}^d : |L_k f(x)| > \lambda 2^{-k(s+\frac{\gamma}{p})} \big\} \Big]^{1/p}. \tag{1.5}$$

It is easy to check that we always have $\mathcal{S}_\infty(\mathbb{R}^d) \subseteq \dot{\mathcal{B}}^s_p(\gamma, r)$. Note that a simple Fubini-type argument gives

$$\dot{\mathcal{B}}^s_p(\gamma, p) = \dot{B}^s_{p,p}, \text{ for all } \gamma \in \mathbb{R}. \tag{1.6}$$

In contrast, for $r \neq p$ the spaces $\dot{\mathcal{B}}^s_p(\gamma, r)$ depend on γ (see Theorem 1.6 (ii) below).

Remark 1.2 (Inhomogeneous versions). We may also consider inhomogeneous versions of the above spaces. Define

$$Ł_k = L_k \text{ for } k > 0, \qquad Ł_0 := \mathrm{Id} - \sum_{k>0} Ł_k. \tag{1.7}$$

For $E \subset \mathbb{R}^d \times \mathbb{N}_0$ let $\widetilde{\mu}_\gamma(E) = \sum_{k=0}^\infty 2^{-k\gamma} \int \mathbb{1}_E(x, k)\,\mathrm{d}x$. Define $\Pi^b f(x, k) = 2^{kb} Ł_k f(x)$ for $k = 0, 1, 2, \dots$ We may then define $\mathcal{B}^s_p(\gamma, r)$ to be the space of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{\mathcal{B}^s_p(\gamma,r)} := \|\Pi^{s+\gamma/p} f\|_{L^{p,r}(\mathbb{R}^d \times \mathbb{N}_0, \widetilde{\mu}_\gamma)} \tag{1.8}$$

is finite. These spaces have already been defined by Krepkogorskiĭ [36], who used the notation $BL^{s,k}_{p,q}$. The space $\mathcal{B}^s_p(\gamma, r)$ corresponds to $BL^{s,-\gamma}_{p,r}$ in the notation of [36].

Characterizations via difference operators. In order to explore the relation to the characterization of Sobolev spaces via weak-type quasinorms for difference operators used in [8, 10] we seek equivalent definitions of the spaces $\dot{\mathcal{B}}^s_p(\gamma, r)$ to spaces defined via difference operators, at least for $s > 0$. Let

$$\Delta_h f(x) = f(x + h) - f(x),$$

and define for $M \geq 2$ inductively $\Delta_h^M = \Delta_h \Delta_h^{M-1}$. These operations extend to tempered distributions. We define a measure ν_γ on Lebesgue measurable subsets of $\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$ by

$$\nu_\gamma(E) = \iint_E \mathrm{d}x \frac{\mathrm{d}h}{|h|^{d-\gamma}}.$$

Also define, for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $h \neq 0$,

$$\mathcal{Q}_{M,b} f(x, h) = \frac{\Delta_h^M f(x)}{|h|^b}.$$

We say that g is a *tempered function* if $g \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and if there exists an $N < \infty$ such that

$$\int_{\mathbb{R}^d} |g(x)|(1 + |x|)^{-N}\,\mathrm{d}x < \infty. \tag{1.9}$$

The space of tempered functions will be denoted by $\mathcal{T}$; the Fréchet topology on $\mathcal{T}$ is defined by the seminorms (1.9).

Let $\mathcal{P}_{M-1}$ denote the set of polynomials of degree less than M. We wish to characterize $\dot{\mathcal{B}}^s_p(\gamma, r)$ in terms of the operators $\mathcal{Q}_{M,b}$ which annihilate $\mathcal{P}_{M-1}$.

As $\dot{\mathcal{B}}^s_p(\gamma, r) \subset \mathcal{S}'_\infty$, every element $f \in \dot{\mathcal{B}}^s_p(\gamma, r)$ is actually an equivalent class $[f]$ of tempered distributions modulo *all* polynomials. Using the following theorem, if $0 < s < M$ and $M \in \mathbb{N}$, we determine, for each $f \in \dot{\mathcal{B}}^s_p(\gamma, r)$, a subset of $[f]$, so that all elements of this subset differ by a polynomial in $\mathcal{P}_{M-1}$. Each element of this subset will be called a *representative* of f modulo $\mathcal{P}_{M-1}$. This is often useful in practice, because then it makes sense to define, for example, any derivative of f of order $\geq M$, and to define the convolution of f with any Schwartz function that has M vanishing moments. For the classical Besov and Triebel-Lizorkin spaces (in particular $\dot{B}^s_{p,p}$) this is already addressed in Bourdaud's theory of *realized spaces* [7], in fact for $\dot{\mathcal{B}}^s_p(\gamma, p) \equiv \dot{B}^s_{p,p}$ an essential part of the theorem is subsumed in [7].

Theorem 1.3. *Let $0 < s < M$, $1 < p < \infty$, $1 \leq r \leq \infty$ and $\gamma \in \mathbb{R}$. There exist positive constants C_1, C_2 so that the following holds.*

(i) Let $f \in \dot{\mathcal{B}}^s_p(\gamma, r)$. Then there exists a tempered function $f_\circ$ such that

$$\langle f, \phi \rangle = \int_{\mathbb{R}^d} f_\circ(x)\phi(x)\, \mathrm{d}x \text{ for all } \phi \in \mathcal{S}_\infty \tag{1.10}$$

and

$$\|\mathcal{Q}_{M,s+\frac{\gamma}{p}} f_\circ\|_{L^{p,r}(\nu_\gamma)} \leq C_1 \|f\|_{\dot{\mathcal{B}}^s_p(\gamma,r)}. \tag{1.11}$$

The a.e. equivalent class of the function $f_\circ$ is unique modulo $\mathcal{P}_{M-1}$; we refer to the function $f_\circ$ as a representative *of f modulo $\mathcal{P}_{M-1}$.*

(ii) Suppose $f : \mathbb{R}^d \to \mathbb{C}$ is a measurable function satisfying

$$\mathcal{Q}_{M,s+\frac{\gamma}{p}} f \in L^{p,r}(\nu_\gamma).$$

Then f is a tempered function, and under the natural identification in $\mathcal{S}'_\infty$, we have $f \in \dot{\mathcal{B}}^s_p(\gamma, r)$ with

$$\|f\|_{\dot{\mathcal{B}}^s_p(\gamma,r)} \leq C_2 \|\mathcal{Q}_{M,s+\frac{\gamma}{p}} f\|_{L^{p,r}(\nu_\gamma)}.$$

Theorem 1.3 will be proved in §3, where a more abstract equivalent statement is also given (Theorem 3.2).

Remark 1.4. We point out that in previous works on homogeneous Besov spaces there is the a priori assumption $f \in L^1_{\mathrm{loc}}$ for the bound by difference operators. One way in which our result differs is that we show this assumption is superfluous: the function f in Theorem 1.3 (ii) is a priori only assumed to be measurable and we show that it is locally integrable.

Embeddings and non-embeddings. We establish various embedding relations which sharpen previous results. We relate our classes to standard homogeneous Besov and Triebel-Lizorkin spaces and their Lorentz-space counterparts $\dot{B}^s_q[L^{p,r}]$ and $\dot{F}^s_q[L^{p,r}]$. These are defined as the subspaces of

$f \in \mathcal{S}'_\infty(\mathbb{R}^d)$ for which

$$\|f\|_{\dot B^s_q[L^{p,r}]} = \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} \|L_k f\|^q_{L^{p,r}(\mathbb{R}^d)}\Big)^{1/q} \tag{1.12}$$

$$\|f\|_{\dot F^s_q[L^{p,r}]} = \Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq} |L_k f|^q\Big)^{1/q}\Big\|_{L^{p,r}} \tag{1.13}$$

are finite, respectively. The inhomogeneous analogues $B^s_q[L^{p,r}]$, $F^s_q[L^{p,r}]$ are defined analogously using the frequency localizations $Ł_k$, $k \geq 0$ in (1.7).

For the standard Besov and Triebel-Lizorkin spaces one works with the underlying L^p metric, i.e. they are recovered by setting $r = p$ and we have $\dot B^s_{p,q} = \dot B^s_q[L^p]$, and $\dot F^s_{p,q} = \dot F^s_q[L^p]$. For embedding relations among them one may consult [52] (however some care is needed since the results in [52] are formulated for the inhomogeneous versions $B^s_q[L^{p,r}]$, $F^s_q[L^{p,r}]$).

Theorem 1.5. *The following statements hold for all $s \in \mathbb{R}$, $p \in [1,\infty)$.*

(i) For all $\gamma \in \mathbb{R}$,

$$\begin{aligned} \dot{\mathcal{B}}^s_p(\gamma, r) &\hookrightarrow \dot B^s_r[L^{p,r}], \quad p \leq r \leq \infty, \\ \dot B^s_r[L^{p,r}] &\hookrightarrow \dot{\mathcal{B}}^s_p(\gamma, r), \quad 1 \leq r \leq p. \end{aligned}$$

(ii) Let $\gamma \neq 0$. Then,

$$\begin{aligned} \dot F^s_{p,r} &\hookrightarrow \dot{\mathcal{B}}^s_p(\gamma, r), \quad p \leq r \leq \infty, \\ \dot{\mathcal{B}}^s_p(\gamma, r) &\hookrightarrow \dot F^s_{p,r}, \quad 1 \leq r \leq p. \end{aligned}$$

This will be proved in §4. The statements can be extended by combining them with the three trivial embeddings for $q_1 \leq q_2$, $r_1 \leq r_2$, namely $\dot{\mathcal{B}}^s_p(\gamma, r_1) \hookrightarrow \dot{\mathcal{B}}^s_p(\gamma, r_2)$, $\dot B^s_{q_1}[L^{p,r_1}] \hookrightarrow \dot B^s_{q_2}[L^{p,r_2}]$ and $\dot F^s_{q_1}[L^{p,r_1}] \hookrightarrow \dot F^s_{q_2}[L^{p,r_2}]$. Part (ii) of the theorem is an improvement and generalization over Theorem 1.3 in [30] which (in conjunction with our Theorem 1.3) yields that $\dot F^s_{p,2} \hookrightarrow \dot{\mathcal{B}}^s_p(d,\infty)$ for $0 < s < 1$. Part (ii) also covers the embedding $\dot C^s_p \hookrightarrow BSY^s_p \equiv \dot{\mathcal{B}}^s_p(d,\infty)$ for the homogeneous Calderón-Campanato (or DeVore-Sharpley) spaces in [19], [11] which was obtained in [23, Theorem 4.1] for $0 < s < 1$; indeed from [51] we know that $\dot C^s_p = \dot F^s_{p,\infty}$ for $0 < s < 1$. For every $p \in (1,\infty)$ Theorem 1.5 also recovers the known embeddings $\dot F^s_{p,r} \hookrightarrow \dot B^s_r[L^{p,r}]$ if $p \leq r$, and $\dot B^s_r[L^{p,r}] \hookrightarrow \dot F^s_{p,r}$ if $r \leq p$; *cf.* [52, Theorem 1.2(iv), Theorem 1.1(iv)].

In view of the case $r = \infty$ of the embedding in part (ii) of Theorem 1.5 it is natural to ask whether in the embedding $\dot F^s_{p,\infty} \hookrightarrow \dot{\mathcal{B}}^s_p(\gamma,\infty)$ the Triebel-Lizorkin space $\dot F^s_{p,\infty}$ can be replaced by the larger Besov space $\dot B^s_{p,\infty}$; this was implicitly suggested in [48]. Part (i) of the following theorem implies a negative answer, and in fact a stronger result.

Theorem 1.6. *Let $s \in \mathbb{R}$, $1 < p < r \leq \infty$. Then the following hold.*

(i) For all $\gamma \in \mathbb{R}$,

$$\dot B^s_{p,r} \setminus \dot{\mathcal{B}}^s_p(\gamma,\infty) \neq \emptyset.$$

(ii) For all $\beta, \gamma \in \mathbb{R}$ with $\beta \neq \gamma$,

$$\dot{\mathcal{B}}_p^s(\beta, r) \setminus \dot{\mathcal{B}}_p^s(\gamma, \infty) \neq \emptyset.$$

This will be proved in §5, along with corresponding versions for the inhomogeneous spaces.

Since $\dot{\mathcal{B}}_p^s(\gamma, p) = \dot{B}_{p,p}^s$ for all $\gamma \in \mathbb{R}$ (see (1.6)) it is clear that the assumption $r > p$ is necessary in Theorem 1.6. We also address the case $\gamma = 0$ in part (ii) of Theorem 1.5; the following result shows that the condition $\gamma \neq 0$ is necessary for those statements.

Theorem 1.7. *Let $s \in \mathbb{R}$ and $1 < p < \infty$. For the case $\gamma = 0$ the following hold.*

(i) For all $r > p$

$$\dot{F}_{p,r}^s \setminus \dot{\mathcal{B}}_p^s(0, \infty) \neq \emptyset.$$

(ii) For all $r < p$

$$\dot{\mathcal{B}}_p^s(0, 1) \setminus \dot{F}_{p,r}^s \neq \emptyset.$$

Remark 1.8. By part (ii) of Theorem 1.6 we know that for $0 < s < M$ and $\gamma_1 \neq \gamma_2$ the seminorms $\|\mathcal{Q}_{M,s+\gamma_i/p} f\|_{L^{p,\infty}(\nu_{\gamma_i})}$, $i = 1, 2$ are not equivalent on the space of Schwartz functions. This is in striking contrast with the limiting result for $\mathcal{D}_{1+\gamma/p}$, by Brezis and three of the authors [10], where it is shown that for $1 < p < \infty$, and all $\gamma \neq 0$ the semi-norms $\|\mathcal{D}_{1+\gamma/p} f\|_{L^{p,\infty}(\nu_\gamma)}$ are equivalent with the Gagliardo semi-norm $\|\nabla f\|_p$. Moreover, for $p = 1$ one has $\|f\|_{\dot{BV}} \approx \|\mathcal{D}_{1+\gamma} f\|_{L^{1,\infty}(\nu_\gamma)}$ provided that $\gamma \in \mathbb{R} \setminus [-1, 0]$ (and this additional assumption is necessary). These equivalences hold under the a-priori assumption that f is locally integrable.

An embedding result involving $\dot{BV}$. Denote by $V^\infty = V^\infty(\mathbb{R}^d)$ the quotient space of L^∞ by additive constants, with norm

$$\|f\|_{V^\infty} = \inf_{c \in \mathbb{C}} \|f - c\|_\infty.$$

Denote by $[\cdot, \cdot]_{\theta,r}$ the real interpolation spaces for the Peetre $K_{\theta,r}$ method [4, Section 3.1]. The following embedding result involves a real interpolation space between $\dot{BV}$ and V^∞. It will be used below to study solutions of harmonic and caloric functions on $\mathbb{R}_+^{d+1}$.

Theorem 1.9. *Let $\gamma \in \mathbb{R} \setminus [-1, 0]$ and $1 < p < \infty$. Then*

$$[V^\infty, \dot{BV}]_{\frac{1}{p},1} \hookrightarrow \dot{\mathcal{B}}_p^{1/p}(\gamma, \infty).$$

The case $\gamma = d$ of Theorem 1.9 has its roots in [9, Theorem 1.4]. Its full generality is based on an estimate in [10]. It complements interpolation results in [15, Theorem 1.4], and extends an embedding theorem by Greco and Schiattarella [28] for functions of bounded variation on the unit circle.

Harmonic and caloric functions in the upper half space. We now formulate some consequences of the embedding in Theorem 1.9. The original motivation of the space $\dot{\mathcal{B}}_2^{1/2}(1,\infty)$, defined in terms of difference operators, came from the study of harmonic extension of functions of bounded variation in [28] (see also an earlier result by Iwaniec-Martin-Sbordone [34] for circle homeomorphisms). For a function such that $\int_{\mathbb{R}^d}|f(x)|(1+|x|)^{-d-1}\,\mathrm{d}x<\infty$, the harmonic extension to the upper half space $\mathbb{R}^{d+1}_+$ through the Poisson kernel is given by

$$\mathcal{P}f(x,t)=\frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}}\int_{\mathbb{R}^d}\frac{t}{(|x-y|^2+t^2)^{\frac{d+1}{2}}}f(y)\,\mathrm{d}y.$$

In order to state our result let

$$\mathcal{K}^b f(x,t)=t^{1-b}\nabla\mathcal{P}f(x,t) \tag{1.14}$$

where $\nabla\mathcal{P}$ denotes the (x,t)-gradient, for $t>0$, i.e.

$$\nabla\mathcal{P}f(x,t)=\frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}}\int_{\mathbb{R}^d}\frac{((d+1)t(x-y),|x-y|^2-dt^2)}{(|x-y|^2+t^2)^{\frac{d+3}{2}}}(f(y)-f(x))\,\mathrm{d}y.$$

This last expression makes sense for $f\in V^\infty+\dot{BV}$. Define the measure λ_γ on Lebesgue measurable sets of $\mathbb{R}^{d+1}_+$ by

$$\lambda_\gamma(E)=\iint_E \mathrm{d}x\frac{\mathrm{d}t}{t^{1-\gamma}}. \tag{1.15}$$

Corollary 1.10. *Let* $1<p<\infty$, $\gamma\in\mathbb{R}\setminus[-1,0]$. *Then*

$$\mathcal{K}^{\frac{\gamma+1}{p}}:[V^\infty,\dot{BV}]_{\frac{1}{p},1}\to L^{p,\infty}(\lambda_\gamma)$$

is bounded. In particular

$$\nabla\mathcal{P}:[V^\infty,\dot{BV}]_{\frac{1}{2},1}\to L^{2,\infty}(\mathrm{d}x\,\mathrm{d}t)$$

is bounded.

Remark 1.11. When $d=1$ we have $\dot{BV}(\mathbb{R})\hookrightarrow V^\infty(\mathbb{R})$ and thus we recover the upper half plane analogue of Theorem 4.2 of [28], saying that $\nabla\mathcal{P}f\in L^{2,\infty}(\mathbb{R}^2_+)$ for $f\in\dot{BV}(\mathbb{R})$.

Another corollary is about solutions $u(x,t)=Uf(x,t)=e^{t\Delta}f(x)$ of the initial value problem for the heat equation in the upper half space,

$$\frac{\partial u}{\partial t}=\Delta u,\quad u|_{t=0}=f. \tag{1.16}$$

For $b\in\mathbb{R}$, $t>0$, define $\mathcal{H}^b=(\mathcal{H}^b_1,\dots,\mathcal{H}^b_{d+1})$ by

$$\mathcal{H}^b_j f(x,t)=t^{\frac{1}{2}-b}\frac{\partial}{\partial x_j}Uf(x,t),\quad j=1,\dots,d$$

$$\mathcal{H}^b_{d+1}f(x,t)=t^{1-b}\frac{\partial}{\partial t}Uf(x,t).$$

Corollary 1.12. *Let $\beta \in \mathbb{R} \setminus [-\frac{1}{2}, 0]$, and $1 < p < \infty$. Then*

(i) $\mathcal{H}^{\frac{2\beta+1}{2p}} : [V^\infty, \dot{BV}]_{\frac{1}{p},1} \to L^{p,\infty}(\lambda_\beta)$ *is bounded.*

(ii) Let $u = Uf$ solve the problem (1.16) for $t > 0$. Then

$$f \in [V^\infty, \dot{BV}]_{\frac{2}{3},1} \implies \frac{\partial u}{\partial t} = \Delta_x u \in L^{\frac{3}{2},\infty}(\mathbb{R}^{d+1}_+, \mathrm{d}x\,\mathrm{d}t),$$
$$f \in [V^\infty, \dot{BV}]_{\frac{1}{3},1} \implies \nabla_x u \in L^{3,\infty}(\mathbb{R}^{d+1}_+, \mathrm{d}x\,\mathrm{d}t).$$

When $d = 1$ we obtain a caloric analogue of the result in [28], for boundary values in $\dot{BV}(\mathbb{R})$.

Corollary 1.13. *Let $f \in \dot{BV}(\mathbb{R})$ and let u solve the initial value problem $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $u(x, 0) = f(x)$. Then $\frac{\partial u}{\partial t} \in L^{\frac{3}{2},\infty}(\mathbb{R}^2_+)$ and $\frac{\partial u}{\partial x} \in L^{3,\infty}(\mathbb{R}^2_+)$.*

Remark 1.14. It would be interesting to upgrade the results of Theorem 1.9 and/or the corollaries to other interpolation spaces of V^∞ and $\dot{BV}$. A related question in dimension $d = 1$ is whether such inequalities can be proved for functions in the Wiener spaces V^p of bounded p-variation. Note that $V^1 = \dot{BV}$ and that for $1 < p < \infty$ we have $[V^\infty, V^1]_{\frac{1}{p},p} \subset V^p$, see [5]. If $V^{p,\infty}$ denotes the space of f for which the numbers $\mathfrak{N}(f, \alpha)$ of α-jumps satisfy $\sup_{\alpha>0} \alpha \mathfrak{N}(f, \alpha)^{1/p} < \infty$ then, by [43], $V^p \subset V^{p,\infty} = [V^\infty, V^1]_{1/p,\infty}$. See also [13] for a related result on the K-functional for the couple (V^∞, V^1).

Interpolation. We review the problem of interpolation of Besov spaces. Recall the definition of the homogeneous Besov space $\dot{B}^s_{p,q} \equiv \dot{B}^s_q(L^p)$ as the subspace of $f \in \mathcal{S}'_\infty(\mathbb{R}^d)$ for which $\|f\|_{\dot{B}^s_{p,q}} := (\sum_{k\in\mathbb{Z}} \|2^{ks} L_k f\|_p^q)^{1/q}$ is finite. Regarding real interpolation, the case for fixed p and varying s is well known. Suppose $s_0, s_1 \in \mathbb{R}$ with $s_0 \neq s_1$. If $1 \le p, r \le \infty$, one has [4, Theorem 6.4.5(i)] $[\dot{B}^{s_0}_{p,p}, \dot{B}^{s_1}_{p,p}]_{\theta,r} = \dot{B}^s_{p,r}$ if $s = (1-\theta)s_0 + \theta s_1$, $\theta \in (0, 1)$; see also [21]. For the case $p_0 \neq p_1$ the spaces $\dot{\mathcal{B}}^s_p(\gamma, r)$ arise as interpolation spaces for the $K_{\theta,r}$-method. The following theorem and corollary were already known to Krepkogorskiĭ [36] who considered the inhomogeneous variants. For an extension to the quasi-Banach range see [37]. For a description of the interpolation spaces via wavelet coefficients see also the recent work by Besoy, Haroske, and Triebel [6].

Theorem 1.15. *Let $1 \le p_0, p_1, r \le \infty$, $p_0 \neq p_1$, $s_0, s_1 \in \mathbb{R}$. Let*

$$\gamma = -\frac{s_0 - s_1}{\frac{1}{p_0} - \frac{1}{p_1}}, \tag{1.17}$$

let $0 < \theta < 1$ and

$$(\tfrac{1}{p}, s) = (1-\theta)(\tfrac{1}{p_0}, s_0) + \theta(\tfrac{1}{p_1}, s_1). \tag{1.18}$$

Then

$$[\dot{B}^{s_0}_{p_0,p_0}, \dot{B}^{s_1}_{p_1,p_1}]_{\theta,r} = \dot{\mathcal{B}}^s_p(\gamma, r), \tag{1.19}$$

with equivalence of (quasi-)norms. □

Corollary 1.16. *Let* $1 < p_0, p_1 < \infty$, $p_0 \neq p_1$, $s_0, s_1 \in \mathbb{R}$, $1 \le q_0, q_1, r_0, r_1 \le \infty$ *and* $1 \le r \le \infty$. *Suppose that* (1.17) *and* (1.18) *hold with* $0 < \theta < 1$. *Then*

$$[\dot{\mathcal{B}}^{s_0}_{p_0}(\gamma, r_0), \dot{\mathcal{B}}^{s_1}_{p_1}(\gamma, r_1)]_{\theta,r} = \dot{\mathcal{B}}^{s}_{p}(\gamma, r). \tag{1.20a}$$

Moreover, if $s_0 \neq s_1$,

$$[\dot{F}^{s_0}_{p_0,q_0}, \dot{F}^{s_1}_{p_1,q_1}]_{\theta,r} = \dot{\mathcal{B}}^{s}_{p}(\gamma, r). \tag{1.20b}$$

Note that for $s \in \mathbb{N} \cup \{0\}$ and $1 < p < \infty$, the space $\dot{F}^s_{p,2}$ is identified with the Sobolev space $\dot{W}^{s,p}$. Thus, if s_0, s_1 are non-negative integers, $s_0 \neq s_1$, and $1 < p_0, p_1 < \infty$ with $p_0 \neq p_1$, then for $0 < \theta < 1$ and $1 \le r \le \infty$ we get in particular

$$[\dot{W}^{s_0,p_0}, \dot{W}^{s_1,p_1}]_{\theta,r} = \dot{\mathcal{B}}^{s}_{p}(\gamma, r)$$

where $(\frac{1}{p}, s)$ and γ are given by (1.18) and (1.17).

For completeness we shall sketch in §8 the standard proofs based on the Fourier analytic definition which are very much analogous to [36]. More interestingly, for $M = 1$ and $s_0, s_1 \in (0,1)$, an alternative approach to the interpolation result (1.19) will be given in §9, based directly on the characterization via first order differences.

Nonlinear wavelet approximation. Our results can be obtained to prove new results on best approximation via n terms in a wavelet basis, relating it to suitable regularity properties of the given function.

To fix ideas we first recall basic notation in wavelet theory. Let $u \in \mathbb{N}$, $\phi \in C^u(\mathbb{R})$ be a univariate scaling function associated with the univariate wavelet $\psi \in C^u(\mathbb{R})$. Let $\psi^0 := \phi$ and $\psi^1 := \psi$. If E denote the set of the $2^d - 1$ non-zero vertices of $[0,1]^d$, given $e = (e_1, \dots, e_d) \in E$, we define the d-variate wavelets $\psi^e(x) = \prod_{i=1}^d \psi^{e_i}(x_i)$. As in [40] we assume certain decay and nonvanishing moment conditions on the ψ^e, namely

$$\sup_{x \in \mathbb{R}^d} (1 + |x|)^M |D^\alpha \psi^e(x)| < \infty, \qquad |\alpha| \le u, \qquad e \in E, \tag{1.21a}$$

and

$$\int_{\mathbb{R}^d} x^\alpha \psi^e(x)\, \mathrm{d}x = 0, \qquad |\alpha| < u, \qquad e \in E, \tag{1.21b}$$

for u, M satisfying

$$u > |s|, \quad M > d + u. \tag{1.21c}$$

If one works with L^p-based Besov spaces and allows the parameter range to be $p > 0$, then one needs to require $u > \max\{\frac{d}{\min\{1,p\}} - d - s, s\}$, $M > \max\{\frac{d}{\min\{1,p\}}, d + u\}$.

Let, for $j \in \mathbb{Z}$ and $m \in \mathbb{Z}^d$,

$$\psi^e_{j,m}(x) := 2^{\frac{jd}{2}} \psi^e(2^j x - m). \tag{1.22}$$

We assume that the system

$$\Psi = \{\psi^e_{j,m} : j \in \mathbb{Z},\, m \in \mathbb{Z}^d,\, e \in E\} \tag{1.23}$$

forms an *orthonormal basis in* $L^2(\mathbb{R}^d)$, see e.g. [16] for an introduction to wavelet theory.

Let $1 < q < \infty$. Consider now the *best* n*-term approximation of* $f \in L^q(\mathbb{R}^d)$, *with respect to* Ψ, measured in the $L^q(\mathbb{R}^d)$ norm; i.e.,

$$\sigma_n(f)_q = \inf\Big\{\Big\| f - \sum_{\psi_\nu \in \Lambda \subset \Psi} c_\nu \psi_\nu \Big\|_{L^q(\mathbb{R}^d)} : \#(\Lambda) \le n,\, c_\nu \in \mathbb{C}\Big\}.$$

Let $\alpha > 0$ and $0 < r \le \infty$. The related *approximation space* $\mathcal{A}^\alpha_r(L^q, \Psi)$ is defined as the set of functions $f \in L^q(\mathbb{R}^d)$ for which

$$\|f\|_{\mathcal{A}^\alpha_r(L^q,\Psi)} = \begin{cases} \Big(\sum_{n=1}^\infty \big[n^\alpha \sigma_n(f)_q\big]^r \frac{1}{n}\Big)^{\frac{1}{r}} & \text{if } r < \infty \\ \sup_n n^\alpha \sigma_n(f)_q & \text{if } r = \infty \end{cases}$$

is finite.

It is well known that $\mathcal{A}^\alpha_r(L^q(\mathbb{R}^d), \Psi)$ can be characterized in terms of a certain interpolation space between $L^q(\mathbb{R}^d)$ and Besov spaces. Specifically, let $1 < q < \infty$, $0 < r \le \infty$, and $0 < s < \sigma$. Then

$$\mathcal{A}^{s/d}_r(L^q, \Psi) = [L^q, \dot B^\sigma_{u,u}]_{\theta,r} \quad \text{if } \theta = \frac{s}{\sigma} \text{ and } \frac{1}{u} = \frac{1}{q} + \frac{\sigma}{d}; \tag{1.24}$$

see DeVore's survey [17, (7.41)] and also [47, page 223] for related results on spline approximation with $d = 1$. We specialize (1.20b) with $s_0 = 0$, $p_0 = q$, $q_0 = 2$, $q_1 = p_1 = u$, $s_1 = \sigma$, hence $\gamma = -\frac{s_1 - s_0}{p_1^{-1} - p_0^{-1}} = -d$. We thus see that for θ, u as in (1.24) the space $[L^q, \dot B^\sigma_{u,u}]_{\theta,r}$ coincides with $\dot{\mathcal{B}}^s_p(-d, r)$ if $\frac{1}{p} = \frac{1}{q} + \frac{s}{d}$. Combining this with (1.24), we have verified

Theorem 1.17. *Let* $1 < q < \infty$, $0 < s < d(1 - \frac{1}{q})$, *and let* $\frac{1}{p} = \frac{1}{q} + \frac{s}{d}$. *Then, for* $1 \le r \le \infty$,

$$\mathcal{A}^{s/d}_r(L^q, \Psi) = \dot{\mathcal{B}}^s_p(-d, r).$$

For $r = p$, $0 < s < d(1 - \frac{1}{q})$ we recover $\mathcal{A}^{s/d}_p(L^q, \Psi) = \dot B^s_{p,p}$ for $\frac{1}{p} = \frac{1}{q} + \frac{s}{d}$, which is a result proved by DeVore, Jawerth and Popov [18]. Together with our characterization in Theorem 1.3 we achieve a new interpretation via difference operators of some results in [17, 20, 29, 35] where the spaces $\mathcal{A}^\alpha_r(L^q, \Psi)$ are characterized in terms of wavelet coefficients.

For $r = \infty$ the spaces $\mathcal{A}^{s/d}_\infty(L^q, \Psi)$ are of special interest in applications, see for example [32, 14, 31]. In the statistics literature these spaces are sometimes referred to as 'weak-Besov spaces' (see [3, 50] and the references within). In view of Theorem 1.17, these weak-Besov spaces coincide with $\dot{\mathcal{B}}^s_p(-d, \infty)$, with $s = d(1/p - 1/q)$, i.e. $p = \frac{dq}{d+sq}$. Putting $\alpha = s/d$ and combining Theorem 1.3 and Theorem 1.17 we obtain

Corollary 1.18. *Let $1 < q < \infty$, $0 < \alpha < 1 - \frac{1}{q}$. Then, for $M > \alpha d$,*

$$\sup_{n \geq 1} n^{\alpha} \sigma_n(f)_q \approx \sup_{\lambda > 0} \lambda \Big(\int \operatorname{meas}\Big(\Big\{ x : |h|^{\frac{d}{q}} |\Delta_h^M f(x)| > \lambda \Big\} \Big) \frac{\mathrm{d}h}{|h|^{2d}} \Big)^{\frac{1}{q} + \alpha}.$$

Remark. There are suitable extensions of the definitions of this paper, and many of the results, to certain parameter ranges in the quasi-Banach setting (that is, to the cases $r < 1$ and $p \leq 1$); we intend to pursue these elsewhere. In particular it is interesting to extend Theorem 1.17 to values of $s \geq d(1 - 1/q)$ and $r > 0$; this requires consideration of the spaces $\dot{\mathcal{B}}_p^s(-d, r)$ in the range $p \leq 1$.

Notation. We denote by $\mathcal{L}^d(E)$ the Lebesgue measure of a Lebesgue measurable set in $\mathbb{R}^d$, and also write $\operatorname{meas} E$ for $\mathcal{L}^d(E)$ when the dimension is clear from the context. A measurable function $f : \mathbb{R}^d \to \mathbb{C}$ will always be assumed to be defined almost everywhere. We use $\widehat{f}(\xi) = \int f(y) e^{-i\langle y, \xi \rangle} \, \mathrm{d}y$ as definition of the Fourier transform. For a function m on $\widehat{\mathbb{R}}^d$ we define $m(D)$ to be the convolution operator with Fourier multiplier m, i.e. it is given by $\widehat{m(D)f}(\xi) = m(\xi)\widehat{f}(\xi)$. We let C_c^∞ be the space of compactly supported C^∞-functions, $\mathcal{S}$ be the space of Schwartz functions, and $\mathcal{S}_M$ be the subspace of $\mathcal{S}$ consisting of those Schwartz functions whose moments up to order $M - 1$ vanish. Also let $\mathcal{S}_\infty = \bigcap_{M \in \mathbb{N}} \mathcal{S}_M$. We denote by $\mathcal{S}'$ the space of tempered distributions and by $\mathcal{S}'_M$, $\mathcal{S}'_\infty$ the dual spaces of $\mathcal{S}_M$ and $\mathcal{S}_\infty$, respectively. We let, for $k \in \mathbb{Z}$, $L_k = \varphi(2^{-k}D)$ and $\widetilde{L}_k = \widetilde{\varphi}(2^{-k}D)$ be operators in frequency localizing Littlewood-Paley decompositions, satisfying $\widetilde{L}_k L_k = L_k$. The functions φ, $\widetilde{\varphi}$ are radial and the relevant properties are defined in (1.2) and (2.2), respectively. For a set E with positive measure, the slashed integral $⨍_E f$ is used to denote the average of f over E.

Structure of the paper. In §2 we prove a rudimentary form of the characterization in Theorem 1.3 just for $\mathcal{S}_\infty$ functions. The full proof of Theorem 1.3 will be given in §3. The embedding results in Theorem 1.5 are proved in §4. Various counterexamples establishing Theorem 1.6 are discussed in §5. In §6 we give the proof of Theorem 1.9 and in §7 the proof of Corollaries 1.10 and 1.12. In §8 we include a proof of Theorem 1.15 based only on the Fourier analytic definition of $\dot{\mathcal{B}}_p^s(\gamma, r)$. A different proof of the interpolation result, just for parameters $s_i \in (0, 1)$ and based on a retraction argument using first order differences is given in §9.

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2. Norm equivalences for $\mathcal{S}_\infty$-functions

Before giving the full proof of Theorem 1.3 we give a proof of the norm equivalence for functions in the class $\mathcal{S}_\infty(\mathbb{R}^d)$. Note that for $f \in \mathcal{S}_\infty(\mathbb{R}^d)$ we have $f = \sum_{k\in\mathbb{Z}} L_k f$ with convergence in the topology of $\mathcal{S}(\mathbb{R}^d)$.

Proposition 2.1. *Let $M \in \mathbb{N}$, $1 < p < \infty$, $1 \le r \le \infty$, $\gamma \in \mathbb{R}$ and $0 < s < M$. For $f \in \mathcal{S}_\infty(\mathbb{R}^d)$,*

$$\|f\|_{\dot{\mathcal{B}}^s_p(\gamma,r)} \approx \|\mathcal{Q}_{M,s+\gamma/p} f\|_{L^{p,r}(\nu_\gamma)}. \tag{2.1}$$

Let $\tilde\varphi \in C_c^\infty(\mathbb{R}^d)$ be such that

$$\text{supp }(\tilde\varphi) \subset \{\xi : 1/2 < |\xi| < 2\} \text{ and } \tilde\varphi(\xi) = 1 \text{ for } 3/4 \le |\xi| \le 7/4. \tag{2.2}$$

This implies $\tilde\varphi\varphi = \varphi$. Let $\widetilde{L}_k = \tilde\varphi(2^{-k}D)$ so that $L_k = \widetilde{L}_k L_k = L_k \widetilde{L}_k$.

To bound $\|\mathcal{Q}_{M,s+\gamma/p} f\|_{L^{p,r}(\nu_\gamma)}$ in terms of $\|f\|_{\dot{\mathcal{B}}^s_p(\gamma,r)}$, we use the following lemma.

Lemma 2.2. *Let $M \in \mathbb{N}$, $1 < p < \infty$, $1 \le r \le \infty$, $b, \gamma \in \mathbb{R}$ with $0 < b - \frac{\gamma}{p} < M$. Then the operator*

$$T_b g(x,h) := \sum_{k\in\mathbb{Z}} \frac{\Delta_h^M \tilde L_k g(x,k)}{(2^k|h|)^b}, \quad (x,h) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$$

defines a bounded linear map from $L^{p,r}(\mu_\gamma)$ to $L^{p,r}(\nu_\gamma)$.

Proof. By real interpolation it suffices to consider the case $r = p$. From the elementary inequality

$$\|\Delta_h^M \tilde L_k\|_{L^p \to L^p} \lesssim \min\{1, (2^k|h|)^M\},$$

we obtain

$$\begin{aligned}
\|T_b g\|_{L^p(\nu_\gamma)} &\lesssim \Big(\int \Big[\sum_{k\in\mathbb{Z}} \frac{\|\Delta_h^M \tilde L_k g(\cdot,k)\|_{L^p(\mathrm{d}x)}}{(2^k|h|)^b} \Big]^p \frac{\mathrm{d}h}{|h|^{d-\gamma}} \Big)^{1/p} \\
&\lesssim \Big(\int \Big[\sum_{k\in\mathbb{Z}} \min\{(2^k|h|)^{-(b-\frac{\gamma}{p})}, (2^k|h|)^{M-(b-\frac{\gamma}{p})}\} 2^{-k\frac{\gamma}{p}} \|g(\cdot,k)\|_p \Big]^p \frac{\mathrm{d}h}{|h|^d} \Big)^{1/p} \\
&\simeq \Big(\sum_{j\in\mathbb{Z}} \Big[\sum_{k\in\mathbb{Z}} \min\{(2^{k-j})^{-(b-\frac{\gamma}{p})}, (2^{k-j})^{M-(b-\frac{\gamma}{p})}\} 2^{-k\frac{\gamma}{p}} \|g(\cdot,k)\|_p \Big]^p \Big)^{1/p}
\end{aligned}$$

and the desired conclusion follows since if $\alpha, \beta > 0$ then the convolution on $\mathbb{Z}$ with the sequence $\{\min\{2^{-k\alpha}, 2^{k\beta}\}\}_{k\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$ is bounded on $\ell^p(\mathbb{Z})$. ☐

To apply the lemma note that under the hypotheses of Proposition 2.1 we have

$$f(x) = \sum_{k\in\mathbb{Z}} 2^{-k(s+\frac{\gamma}{p})} \tilde{L}_k P^{s+\frac{\gamma}{p}} f(x,k) \tag{2.3}$$

with the convergence in $\mathcal{S}_\infty(\mathbb{R}^d)$ (in particular pointwise for every $x \in \mathbb{R}^d$). Hence for every $(x,h) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$ we have

$$\frac{\Delta_h^M f(x)}{|h|^{s+\frac{\gamma}{p}}} = T_{s+\frac{\gamma}{p}} P^{s+\frac{\gamma}{p}} f(x,h).$$

Lemma 2.2 with $b := s + \frac{\gamma}{p}$ (which satisfies $0 < b - \frac{\gamma}{p} < M$) and $g := P^{s+\frac{\gamma}{p}} f$ yields the inequality

$$\Big\| \Big\{ \frac{\Delta_h^M f(x)}{|h|^{s+\frac{\gamma}{p}}} \Big\} \Big\|_{L^{p,r}(\nu_\gamma)} \lesssim \|P^{s+\frac{\gamma}{p}} f\|_{L^{p,r}(\mu_\gamma)} \tag{2.4}$$

and thus the following corollary.

Corollary 2.3. *Let $M \in \mathbb{N}$, $1 < p < \infty$, $1 \le r \le \infty$, $\gamma \in \mathbb{R}$ and $0 < s < M$. Then for $f \in \mathcal{S}_\infty(\mathbb{R}^d)$*

$$\Big\| \Big\{ \frac{\Delta_h^M f(x)}{|h|^{s+\frac{\gamma}{p}}} \Big\} \Big\|_{L^{p,r}(\nu_\gamma)} \lesssim \|f\|_{\dot{\mathcal{B}}^s_p(\gamma,r)}.$$

For the converse inequality we like to consider an operator acting on $F(x,h) = |h|^{-b} \Delta_h^M f$, for $b = s + \gamma/p$, and then we are faced with the task of "dividing out" the difference operator. To achieve this we work with the partition of unity of the annulus $\{\xi \in \mathbb{R}^d : 1/2 < |\xi| < 2\}$. Alternative Fourier arguments can be found e.g. in [45, 5.2.1].

Let $\varepsilon < (10M)^{-1}$. We use a finite partition $\{\chi_\kappa\}_{\kappa=1}^N$ of unity on the support of φ, so that $\chi_\kappa \in C_c^\infty$ is supported on the ball $B^d(u_\kappa, \varepsilon)$. Let $w_\kappa = \frac{\pi u_\kappa}{2|u_\kappa|^2}$ and then we have, for $\xi \in \text{supp}\,(\chi_\kappa)$ and $|w - w_\kappa| \le \varepsilon$,

$$|\langle \xi, w\rangle - \frac{\pi}{2}| \le |\langle \xi, w - w_\kappa\rangle| + |\langle \xi - u_\kappa, w_\kappa\rangle| + |\langle u_\kappa, w_\kappa\rangle - \tfrac{\pi}{2}| \le 2\varepsilon + 2\varepsilon + 0.$$

We may then write

$$\varphi(\xi) = \sum_{\kappa=1}^N m_\kappa(\xi) \int_{|h-w_\kappa|\le\varepsilon} (e^{i\langle \xi, h\rangle} - 1)^M \,\mathrm{d}h \tag{2.5a}$$

where

$$m_\kappa(\xi) = \varphi(\xi) \frac{\chi_\kappa(\xi)}{\int_{|h-w_\kappa|\le\varepsilon} (e^{i\langle\xi,h\rangle} - 1)^M \,\mathrm{d}h}. \tag{2.5b}$$

Since the denominator is bounded away from 0 on the support of χ_κ we get $|\partial^\alpha m_\kappa(\xi)| \le C_\alpha$ for all multiindices α, and thus the L^1 norms of the Fourier inverse transforms of the m_κ are finite. We then get

$$L_j f = \sum_{\kappa=1}^{N} m_\kappa(2^{-j}D) \int_{|h-2^{-j}w_\kappa|\le\varepsilon 2^{-j}} \Delta_h^M f \, \frac{\mathrm{d}h}{2^{-jd}}. \tag{2.6}$$

Lemma 2.4. *Let m be the Fourier transform of a bounded Borel measure, with $L^1 \to L^1$ multiplier norm $\|m\|_{M_1}$. Let $w \in \mathbb{R}^d$ such that $1/2 \le |w| \le 2$ and $\varepsilon \in (0, \frac{1}{2})$. For $b, \gamma \in \mathbb{R}$, $1 < p < \infty$, $1 \le r \le \infty$, and $F \in L^{p,r}(\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}), \nu_\gamma)$ define $V^b_{m,w,\varepsilon}F$ by*

$$\begin{aligned} V^b_{m,w,\varepsilon}F(\cdot,k) &\equiv V^{b,k}_{m,w,\varepsilon}F \\ &= m(2^{-k}D) \int_{|h-2^{-k}w|\le\varepsilon 2^{-k}} (2^k|h|)^b F(\cdot,h) \frac{\mathrm{d}h}{2^{-kd}}. \end{aligned} \tag{2.7}$$

Then $V^b_{m,w,\varepsilon}$ maps $L^{p,r}(\nu_\gamma)$ to $L^{p,r}(\mu_\gamma)$ and we have

$$\|V^b_{m,w,\varepsilon}F\|_{L^{p,r}(\mu_\gamma)} \le C\|m\|_{M^1}\|F\|_{L^{p,r}(\nu_\gamma)} \tag{2.8}$$

where C only depends on p, r, b, γ.

Proof. Since $(F, m) \mapsto V^b_{m,w,\varepsilon}F$ is bilinear we may normalize and assume that $\|m\|_{M_1} = 1$. Again by real interpolation it suffices to prove the theorem for $p = r$, $1 \le p \le \infty$. Since $\|m(2^{-k}D)\|_{L^p\to L^p} \le 1$ for $1 \le p \le \infty$ we obtain

$$\begin{aligned} \|V^b_{m,w,\varepsilon}F\|_{L^p(\mu_\gamma)} &\lesssim \Big(\sum_{k\in\mathbb{Z}} 2^{-k\gamma}\Big\| \int_{|h-2^{-k}w|\le\varepsilon 2^{-k}} (2^k|h|)^b F(\cdot,h)\frac{\mathrm{d}h}{2^{-kd}}\Big\|_p^p\Big)^{1/p} \\ &\lesssim \Big(\sum_{k\in\mathbb{Z}} 2^{-k\gamma}\Big\| \int_{2^{-k-1}\le|h|\le 2^{-k+1}} |F(\cdot,h)|\frac{\mathrm{d}h}{|h|^d}\Big\|_p^p\Big)^{1/p} \\ &\lesssim \Big(\int_{\mathbb{R}^d} \|F(\cdot,h)\|_p^p \frac{\mathrm{d}h}{|h|^{d-\gamma}}\Big)^{1/p} = \|F\|_{L^p(\nu_\gamma)} \end{aligned}$$

which completes the proof of the lemma. ☐

To apply the lemma it is beneficial to express $P^b f(\cdot,k) = 2^{kb}L_k f$ as

$$P^b f(x,k) = \sum_{\kappa=1}^{N} m_\kappa(2^{-k}D) \int_{|h-2^{-k}w_\kappa|\le\varepsilon 2^{-k}} (2^k|h|)^b \frac{\Delta_h^M f(x)}{|h|^b} \frac{\mathrm{d}h}{2^{-kd}} \tag{2.9a}$$

and thus we get

$$P^b f = \sum_{\kappa=1}^{N} V^b_{m_\kappa,w_\kappa,\varepsilon}F, \text{ with } F(x,h) = \frac{\Delta_h^M f(x)}{|h|^b}. \tag{2.9b}$$

Now, setting $b = s + \gamma/p$, Lemma 2.4 yields

Corollary 2.5. *Let $M \in \mathbb{N}$, $1 < p < \infty$, $1 \le r \le \infty$, $s, \gamma \in \mathbb{R}$. For $f \in \mathcal{S}_\infty(\mathbb{R}^d)$,*

$$\|f\|_{\dot{\mathcal{B}}^s_p(\gamma,r)} \lesssim \|\mathcal{Q}_{M,s+\frac{\gamma}{p}}f\|_{L^{p,r}(\nu_\gamma)}.$$

Proposition 2.1 is just the combination of Corollaries 2.3 and 2.5.

3. Norm equivalences for all measurable functions

We give the proof of Theorem 1.3. We begin by rephrasing it in a more abstract way which allows us to keep in mind the distinction between equivalence classes modulo all polynomials and modulo polynomials of degree $< M$. Let $\mathcal{M}$ denote the space of (Lebesgue almost everywhere equivalence classes of) measurable functions on $\mathbb{R}^d$ and let $\mathcal{P}_M$ denote the space of (almost everywhere equivalence classes of) functions which are almost everywhere equal to a polynomial of degree at most M. Let $\mathcal{M}_M \coloneqq \mathcal{M}/\mathcal{P}_{M-1}$ and let $\pi_M : \mathcal{M} \to \mathcal{M}_M$ denote the projection map. Since the operators $\mathcal{Q}_{M,s+\gamma/p}$ annihilate polynomials of degree $\le M-1$ we can make the following definition.

Definition 3.1. *For $M \in \mathbb{N}$, $s \in \mathbb{R}$, $1 < p < \infty$, and $1 \le r \le \infty$, we define an extended norm*[1] *on $\mathcal{M}_M$ by*

$$\|\pi_M f\|_{\mathfrak{B}_{M,s,p}(\gamma,r)} \coloneqq \|\mathcal{Q}_{M,s+\frac{\gamma}{p}} f\|_{L^{p,r}(\nu_\gamma)}$$

and let $\mathfrak{B}_{M,s,p}(\gamma,r)$ be the subspace of $\mathcal{M}_M$ for which $\|\pi_M f\|_{\mathfrak{B}_{M,s,p}(\gamma,r)}$ is finite.

Recall that $\mathcal{T} \subseteq \mathcal{M}$ denotes the space of (Lebesgue almost everywhere equivalence classes of) tempered functions on $\mathbb{R}^d$, and let $\mathcal{T}_M \coloneqq \mathcal{T}/\mathcal{P}_{M-1}$. $\pi_M\colon \mathcal{M} \to \mathcal{M}_M$ restricts to a map $\pi_M\colon \mathcal{T} \to \mathcal{T}_M$. We let ι_M denote the natural map $\mathcal{T}_M \to \mathcal{S}'_\infty(\mathbb{R}^d)$ which assigns to $\pi_M(f)$ (with $f \in \mathcal{T}$) the linear functional $\iota_M(\pi_M(f)) : \phi \mapsto \int_{\mathbb{R}^d} f(x)\phi(x)\,\mathrm{d}x$. We rephrase Theorem 1.3 in the following, equivalent, form:

Theorem 3.2. *Fix $M \in \mathbb{N}$, $M \ge 1$. For $0 < s < M$, $p \in (1,\infty)$, $r \in [1,\infty]$, $\gamma \in \mathbb{R}$, we have*

- *(i)* $\mathfrak{B}_{M,s,p}(\gamma,r) \subseteq \mathcal{T}_M$;
- *(ii)* $\dot{\mathcal{B}}^s_p(\gamma,r) = \iota_M(\mathfrak{B}_{M,s,p}(\gamma,r))$;
- *(iii)* *The map*

$$\iota_M\big|_{\mathfrak{B}_{M,s,p}(\gamma,r)} : \mathfrak{B}_{M,s,p}(\gamma,r) \to \dot{\mathcal{B}}^s_p(\gamma,r)$$

is an isomorphism of normed vector spaces; i.e., it is a bounded, bijective linear map with bounded inverse.

The rest of this section is devoted to the proof of Theorem 3.2. In what follows we denote, for $M \in \mathbb{N}$, by $\mathcal{S}_M(\mathbb{R}^d)$ the closed subspace of $\mathcal{S}(\mathbb{R}^d)$ which consists of all $f \in \mathcal{S}(\mathbb{R}^d)$ with $\int p(x)f(x)\,\mathrm{d}x = 0$ for all polynomials of degree $\le M-1$. Then clearly $\mathcal{S}_\infty = \bigcap_{M\in\mathbb{N}} \mathcal{S}_M$ (and $\mathcal{S}_\infty \equiv \mathcal{Z}$ in the notation of [56]). We denote by $\mathcal{S}'_M$ the dual space of $\mathcal{S}_M$. To prove the theorem, we introduce two maps: $\dot{\mathcal{B}}^s_p(\gamma,r) \leftrightarrow \mathfrak{B}_{M,s,p}(\gamma,r)$, which will turn out to be inverses to each other. We begin with the map $\dot{\mathcal{B}}^s_p(\gamma,r) \to \mathfrak{B}_{M,s,p}(\gamma,r)$. The following proposition is similar to results of Bourdaud [7] and Moussai [44]

[1] A priori, $\|\cdot\|_{\mathfrak{B}_{M,s,p}(\gamma,r)}$ is merely an extended semi-norm. Lemma 3.6 below shows that $\|\pi_M f\|_{\mathfrak{B}_{M,s,p}(\gamma,r)} = 0 \Leftrightarrow \pi_M f = 0$.

for the so-called realized Besov spaces, and, in fact, could be deduced from their results by interpolation arguments.

Proposition 3.3. *Fix $M \in \mathbb{N}$, $M \geq 1$. For $0 < s < M$, $1 < p < \infty$, $\gamma \in \mathbb{R}$, and $1 \leq r \leq \infty$, there is a bounded linear map*

$$\mathcal{E}_M : \dot{\mathcal{B}}^s_p(\gamma, r) \to \mathfrak{B}_{M,s,p}(\gamma, r)$$

such that $\mathcal{E}_M(\dot{\mathcal{B}}^s_p(\gamma, r)) \subseteq \mathcal{T}_M$ and ι_M is a left inverse to $\mathcal{E}_M$; i.e., $\iota_M \mathcal{E}_M$ is the identity map $\dot{\mathcal{B}}^s_p(\gamma, r) \to \dot{\mathcal{B}}^s_p(\gamma, r)$.

We need a lemma about the Littlewood-Paley decomposition for $f \in \dot{\mathcal{B}}^s_p(\gamma, \infty) \subseteq \mathcal{S}'_\infty(\mathbb{R}^d)$, for $p \in (1, \infty)$. Note that $L_k f$ is a convolution of an element of $\mathcal{S}'_\infty(\mathbb{R}^d)$ and an element of $\mathcal{S}_\infty(\mathbb{R}^d)$, and thus a C^∞-function. By the definition of $\dot{\mathcal{B}}^s_p(\gamma, \infty)$, $L_k f \in L^{p,\infty}(\mathbb{R}^d)$ with $\|2^{ks} L_k f\|_{L^{p,\infty}(\mathbb{R}^d)} \lesssim \|f\|_{\dot{\mathcal{B}}^s_p(\gamma,\infty)}$ uniformly in $k \in \mathbb{Z}$.

By Young's convolution inequality

$$\|\widetilde{L}_k\|_{L^{p,\infty} \to L^\infty} = O(2^{kd/p})$$

and from $\widetilde{L}_k L_k = L_k$ we obtain $\|L_k f\|_\infty \lesssim 2^{kd/p} \|L_k f\|_{L^{p,\infty}}$. We use this to establish convergence of the Littlewood-Paley decomposition in $\mathcal{S}_M$, under the additional condition $M > s - d/p$.

Lemma 3.4. *Let M be a nonnegative integer, $1 < p < \infty$, $N \in \mathbb{N}$. Then the following holds.*

(i) For $f \in \dot{\mathcal{B}}^s_p(\gamma, \infty)$ and $\psi \in \mathcal{S}_M$,

$$|\langle L_j f, \psi \rangle| \leq C_{N,M,\psi} 2^{j(\frac{d}{p} - s)} \min\{2^{-jN}, 2^{jM}\} \|f\|_{\dot{\mathcal{B}}^s_p(\gamma,\infty)}.$$

(ii) Let $M > s - d/p$, and $f \in \dot{\mathcal{B}}^s_p(\gamma, r)$. Then $\sum_{j\in\mathbb{Z}} L_j f$ converges in $\mathcal{S}'_M$.

Proof. Since $\psi \in \mathcal{S}$ we get $\|\widetilde{L}_j \psi\|_1 \lesssim C_{N,\psi} 2^{-jN}$ for $j \geq 0$. Using the $M - 1$ vanishing moment conditions we get $\|\widetilde{L}_j \psi\|_1 \lesssim 2^{jM}$ for $j \leq 0$.

We have

$$\begin{aligned} |\langle L_j f, \psi \rangle| &= |\langle \widetilde{L}_j L_j f, \widetilde{L}_j \psi \rangle| \leq \|\widetilde{L}_j L_j f\|_\infty \|\widetilde{L}_j \psi\|_1 \\ &\lesssim 2^{j\frac{d}{p}} \|L_j f\|_{L^{p,\infty}} \min\{2^{-jN}, 2^{jM}\} \\ &\lesssim \|f\|_{\dot{\mathcal{B}}^s_p(\gamma,\infty)} 2^{j(\frac{d}{p} - s)} \min\{2^{-jN}, 2^{jM}\} \end{aligned}$$

where the implicit constants depend on M, N, ψ. Choosing N large enough we see that $\sum_{j>0} |\langle L_j f, \psi \rangle| < \infty$. Moreover $\sum_{j\leq 0} |\langle L_j f, \psi \rangle| < \infty$ if $M > s - d/p$ and thus $\sum_{j\in\mathbb{Z}} L_j f$ converges in $\mathcal{S}'_M$. □

Proof of Proposition 3.3. We first define the map

$$\mathcal{I}_M : \mathcal{T}_M \hookrightarrow \mathcal{S}'_M(\mathbb{R}^d)$$

as taking $\pi_M u$ (with $u \in \mathcal{T}$) to the distribution $\mathcal{I}_M \pi_M u$ defined by

$$\langle \mathcal{I}_M \pi_M u, \phi \rangle = \int_{\mathbb{R}^d} u\phi$$

and observe that $\mathcal{I}_M$ is injective. Also let $f \in \cup_{r\in[1,\infty]} \dot{\mathcal{B}}^s_p(\gamma, r) = \dot{\mathcal{B}}^s_p(\gamma,\infty)$. By Lemma 3.4 $\sum_{k\in\mathbb{Z}} \mathcal{I}_M \pi_M L_k f$ converges in $\mathcal{S}'_M(\mathbb{R}^d)$ to some $U \in \widetilde{\mathcal{S}}'_M(\mathbb{R}^d)$.

We claim $U \in \mathcal{I}_M(\mathcal{T}_M)$. To see this, decompose

$$U = U_{\text{high}} + U_{\text{low}} := \sum_{k\geq 0} \mathcal{I}_M \pi_M L_k f + \sum_{k<0} \mathcal{I}_M \pi_M L_k f,$$

where the above sums converge in $\mathcal{S}'_M(\mathbb{R}^d)$. Since $f \in \dot{\mathcal{B}}^s_p(\gamma,\infty)$, we have $\|L_k f\|_{L^{p,\infty}} \lesssim 2^{-ks}$, and since $s > 0$ we see that $\sum_{k\geq 0} L_k f$ converges in $L^{p,\infty}(\mathbb{R}^d)$ and

$$U_{\text{high}} = \sum_{k\geq 0} \mathcal{I}_M \pi_M L_k f = \mathcal{I}_M \pi_M \big[\sum_{k\geq 0} L_k f \big] \in \mathcal{I}_M(\mathcal{T}_M).$$

Since $U_{\text{low}} \in \mathcal{S}'_M(\mathbb{R}^d)$, we can use the Hahn-Banach Theorem to establish the existence of an extension $U^{\text{ext}}_{\text{low}} \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\langle U^{\text{ext}}_{\text{low}}, \psi \rangle = \langle U_{\text{low}}, \psi \rangle, \quad \forall \psi \in \mathcal{S}_M(\mathbb{R}^d).$$

In particular, by the definition of U_{low}, we see that the Fourier transform of $U^{\text{ext}}_{\text{low}}$ is supported in $\{|\xi| \leq 2\}$. Schwartz's Paley-Wiener Theorem implies there exists $G \in \mathcal{T}$ with $\langle U^{\text{ext}}_{\text{low}}, \psi \rangle = \int_{\mathbb{R}^d} G\psi$, for all $\psi \in \mathcal{S}(\mathbb{R}^d)$. It follows that $U_{\text{low}} = U^{\text{ext}}_{\text{low}}\big|_{\mathcal{S}_M} = \mathcal{I}_M \pi_M G \in \mathcal{I}_M(\mathcal{T}_M)$. This completes the proof that $U \in \mathcal{I}_M(\mathcal{T}_M)$.

We now can define $\mathcal{E}_M f$; because by injectivity of $\mathcal{I}_M$ we have

$$U = \mathcal{I}_M(\mathcal{E}_M f),$$

for a unique $\mathcal{E}_M f \in \mathcal{T}_M$. The map $\mathcal{E}_M : f \mapsto \mathcal{E}_M f$ is then clearly linear. Also $U\big|_{\mathcal{S}_\infty(\mathbb{R}^d)} = f$ and therefore $\iota_M \mathcal{E}_M f = \mathcal{I}_M \mathcal{E}_M f\big|_{\mathcal{S}_\infty(\mathbb{R}^d)} = U\big|_{\mathcal{S}_\infty(\mathbb{R}^d)} = f$; that is, $\iota_M \mathcal{E}_M$ is the identity.

We still need to establish the estimate

$$\|\mathcal{E}_M f\|_{\mathfrak{B}_{M,s,p}(\gamma,r)} \lesssim \|f\|_{\dot{\mathcal{B}}^s_p(\gamma,r)}; \tag{3.1}$$

this is done using the arguments in §2. Define $g(x,k) := 2^{k(s+\frac{\gamma}{p})} \mathcal{E}_M L_k f(x)$ so that $f = \sum_{k\in\mathbb{Z}} \mathcal{I}_M 2^{-k(s+\frac{\gamma}{p})} \widetilde{L}_k g(\cdot,k)$ with convergence in $\mathcal{S}'_M$. By definition of $\dot{\mathcal{B}}^s_p(\gamma,r)$ we have $g \in L^{p,r}(\mu_\gamma)$. For all h and a.e. x,

$$\frac{\Delta^M_h \mathcal{E}_M f(x)}{|h|^{s+\frac{\gamma}{p}}} = T_{s+\frac{\gamma}{p}} g(x,h)$$

where $T_{s+\frac{\gamma}{p}}$ is as in Lemma 2.2. Then we get (3.1) from Lemma 2.2. □

We turn to the map $\mathfrak{B}_{M,s,p}(\gamma,r) \to \dot{\mathcal{B}}^s_p(\gamma,r)$.

Proposition 3.5. *For $M \in \mathbb{N}$, $s \in \mathbb{R}$, $p \in (1,\infty)$, $r \in [1,\infty]$, $\gamma \in \mathbb{R}$, there is an injective bounded linear map*

$$\mathfrak{J}_M : \mathfrak{B}_{M,s,p}(\gamma, r) \to \dot{\mathcal{B}}^s_p(\gamma, r)$$

such that

$$\mathfrak{J}_M\big|_{\mathfrak{B}_{M,s,p}(\gamma,r)\cap\mathcal{T}_M} = \iota_M\big|_{\mathfrak{B}_{M,s,p}(\gamma,r)\cap\mathcal{T}_M}. \tag{3.2}$$

The main difficulty we must contend with in Proposition 3.5 is that elements of $\mathfrak{B}_{M,s,p}(\gamma, r)$ are a priori only equivalence classes of measurable functions (not necessarily locally integrable), and so we cannot directly use any tools from distribution theory to study them. The following lemma appears to be well-known but we include a proof since we have not been able to locate a precise reference.

Lemma 3.6. *Let $M \geq 1$, and $f : \mathbb{R}^d \to \mathbb{C}$ be measurable with $\Delta^M_h f(x) = 0$ for $\mathcal{L}^{2d}$-almost every $(h, x) \in \mathbb{R}^{2d}$. Then there is a polynomial P of degree at most $M - 1$ such that $f(x) = P(x)$ almost everywhere.*

Before proving the lemma we recall some basic facts from the theory of functional equations [39] which are needed in the proof. First, we need a formula about iterated differences, attributed to Kemperman in [39, Theorem 15.1.2], see also Djoković [22] for related results. Namely, for all dimensions d, for all $N \in \mathbb{N}$, if $v^{(1)}$, ..., $v^{(N)}$ are vectors in $\mathbb{R}^d$ then

$$\begin{gathered}\Delta_{v^{(1)}} \dots \Delta_{v^{(N)}} f(x) = \sum_{(\epsilon_1,\dots,\epsilon_N)\in\{0,1\}^N} (-1)^{\epsilon_1+\dots\epsilon_N} \Delta^M_{h(\epsilon)} f(x + \tilde{h}(\epsilon)),\\ \text{where } h(\epsilon) = -\sum_{j=1}^{N} j^{-1}\epsilon_j v^{(j)}, \qquad \tilde{h}(\epsilon) = \sum_{j=1}^{N} \epsilon_j v^{(j)}.\end{gathered} \tag{3.3}$$

Next we recall that a $\mathcal{L}^d$-measurable function $f : \mathbb{R}^d \to \mathbb{C}$ is called almost polynomial of order $M - 1$ if $\Delta^M_h f(x) = 0$ for $\mathcal{L}^{2d}$-a.e. $(x, h) \in \mathbb{R}^{2d}$. It is a result of Ger [27], which we use in its form presented in [39, Theorem 17.7.2], that there exists a measurable function $P : \mathbb{R}^d \to \mathbb{C}$ such that $f(x) = P(x)$ for $\mathcal{L}^d$-a.e. x and P is a function satisfying $\Delta^M_h P(x) = 0$, *for all* $(h, x) \in \mathbb{R}^d \times \mathbb{R}^d$; such functions are called "polynomial functions" in [27], [39].

We also use a result by Ciesielski [12] (see also [39, Theorem 15.5.2]) which states that if a measurable function $g : \mathbb{R} \to \mathbb{C}$ satisfies $\Delta^M_h g(x) \geq 0$ for all $x \in \mathbb{R}$ and all $h \in \mathbb{R}$ then g is continuous; by an argument using weak derivatives this implies that a polynomial function of order $M - 1$ on the real line is actually a polynomial of degree at most $M - 1$. In proving Lemma 3.6 we could have used a d-dimensional version of this fact which could be derived from an abstract result by Kuczma [38, Theorem 3]. However we prefer to give a more direct argument based on induction on d.

Proof of Lemma 3.6. For $d = 1$ Lemma 3.6 is an immediate consequence of the above mentioned theorems by Ciesielski and Ger. Let $d \geq 2$ and as induction hypothesis, assume Lemma 3.6 in dimension $d-1$. We split variables as $x=(x',x_d)$.

Let $f:\mathbb{R}^d\to\mathbb{C}$ be almost polynomial of order $M-1$. By Ger's theorem there is a measurable function $g:\mathbb{R}^d\mapsto\mathbb{C}$ such that $f=g$ $\mathcal{L}^d$-a.e. and g is a polynomial function of order $M-1$. We therefore get $\Delta^M_{se_d}g(x)=0$ for all $x\in\mathbb{R}^d$ and all $s\in\mathbb{R}$. Thus, for all $x'\in\mathbb{R}^{d-1}$ we get from Ciesielski's theorem that the function $t\mapsto g(x',t)$ is a polynomial of degree at most $M-1$, i.e. we have

$$g(x',x_d)=\sum_{j=0}^{M-1}a_j(x')x_d^j$$

for all $x'\in\mathbb{R}^{d-1}$ and every $x_d\in\mathbb{R}$. Since $a_j(x')=j!(\frac{\mathrm{d}}{\mathrm{d}x_d})^j g(x',x_d)|_{x_d=0}$ the coefficients a_j can be realized as a pointwise limit of $\mathcal{L}^{d-1}$-measurable functions, and thus each a_j is $\mathcal{L}^{d-1}$-measurable. Since $\Delta^M_h g(x)=0$ for all (x,h) we also have by (3.3) that $\Delta^{M-k}_{(u,0)}\Delta^k_{se_d}g(x)=0$, for all $x\in\mathbb{R}^d$, $u\in\mathbb{R}^{d-1}$ and $s\in\mathbb{R}$. Letting $s\to 0$ (and using that $x_d\mapsto g(x',x_d)$ is polynomial) this implies that for $k=0,\dots,M$,

$$0=\Delta^{M-k}_{(u,0)}(\frac{\partial}{\partial x_d})^k g(x',x_d)=\sum_{j=k}^{M-1}\Delta_u^{M-k}a_j(x')c_{j,k}x_d^{j-k},$$

with $c_{j,k}=\prod_{i=1}^k(j-i+1)$. This in turn implies $\Delta_u^{M-k}a_k(x')=0$ for $k=0,\dots,M$, and all $u\in\mathbb{R}^{d-1}$. Thus, by the induction hypothesis $a_k(x')$ is almost everywhere equal to a polynomial of degree at most $M-k-1$, and we deduce that g and thus f is $\mathcal{L}^d$-a.e. equal to a polynomial of degree at most $M-1$. □

Lemma 3.7. *Fix $\gamma\in\mathbb{R}$, $p\in(1,\infty)$, $r\in[1,\infty]$. Then, if $K,L\in\mathbb{N}$ are sufficiently large, we have*

$$\iint |F(x,h)|\min\{|h|^K,|h|^{-K}\}(1+|x|)^{-L}\,\mathrm{d}x\,\mathrm{d}h\lesssim\|F\|_{L^{p,r}(\nu_\gamma)},$$

for all $F\in L^{p,r}(\nu_\gamma)$.

Proof. By interpolation it suffices to show this for $r=p$ (possibly, after increasing K). The desired bound follows if we can show $K_1,L\in\mathbb{N}$ sufficiently large, that

$$\min\{|h|^{K_1},|h|^{-K_1}\}(1+|x|)^{-L}\in L^{p'}(\nu_\gamma),$$

where $p'=\frac{p}{p-1}$. This however is elementary. □

Before we define the operator $\mathcal{J}_M$ from Proposition 3.5, we introduce some auxiliary operators. Let $j\in\mathbb{Z}$, $m\in C_0^\infty(\mathbb{R}^d\setminus\{0\})$, $w\in\mathbb{R}^d\setminus\{0\}$, and $\varepsilon>0$ be such that $\overline{B^d(w,\varepsilon)}\subset\{\xi:1/2<|\xi|<2\}$. For $\psi\in\mathcal{S}_\infty(\mathbb{R}^d)$, define $\Gamma^j_{m,w,\varepsilon}\psi(x,h)$ by

$$\big[\Gamma^j_{m,w,\epsilon}\psi\big]^\wedge(\xi,h):=2^{jd}m(-2^{-j}\xi)\widehat{\psi}(\xi)\,\mathbb{1}_{B^d(2^{-j}w,2^{-j}\epsilon)}(h),\tag{3.4}$$

where $\wedge$ denotes the Fourier transform in the $x \to \xi$ variable.

Lemma 3.8. *Let $\Omega \subset \mathcal{S}_\infty(\mathbb{R}^d)$ be a bounded set. Then for all $K, L \in \mathbb{N}$ and $\alpha \in \mathbb{N}^d$, there exists $C_{K,L,\alpha,\Omega} \geq 0$, which may depend on the fixed j, m, w and ϵ, such that*

$$|\partial_x^\alpha \Gamma^j_{m,w,\varepsilon}\psi(x,h)| \leq C_{K,L,\alpha,\Omega} 2^{-|j|} \min\{|h|^K, |h|^{-K}\}(1+|x|)^{-L}$$

for all $\psi \in \Omega$.

Proof. Equivalently, we wish to show that the set

$$\Big\{ \max\{|h|^K, |h|^{-K}\} 2^{|j|} \Gamma^j_{m,w,\varepsilon}\psi(\cdot,h) : h \in \mathbb{R}^d \setminus \{0\}, \psi \in \Omega \Big\}$$

is bounded in $\mathcal{S}(\mathbb{R}^d)$. Since $\Gamma^j_{m,w,\epsilon}\psi(x,h) = 0$ unless $|h| \approx 2^{-j}$, it suffices to show that

$$\big\{ 2^{|j|(K+1)} \Gamma^j_{m,w,\epsilon}\psi(\cdot,h) : \psi \in \Omega, h \in \mathbb{R}^d \setminus \{0\} \big\}$$

is bounded in $\mathcal{S}(\mathbb{R}^d)$. Taking the Fourier transform, this follows if we show that

$$\big\{ 2^{|j|(K+1)} 2^{jd} m(-2^{-j}\xi) \widehat{\psi}(\xi) : \psi \in \Omega \big\}$$

is bounded in $\mathcal{S}(\mathbb{R}^d)$. Using that supp $\{m(2^{-j}\cdot)\} \subset \{|\xi| \approx 2^j\}$, for $m \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$, and $\Omega \subset \mathcal{S}_\infty(\mathbb{R}^d)$ is a bounded set, this follows, completing the proof. ☐

For $b \in \mathbb{R}$, $p \in (1,\infty)$, $r \in [1,\infty]$, $\gamma \in \mathbb{R}$, $F \in L^{p,r}(\nu_\gamma)$, and $\psi \in \mathcal{S}_\infty(\mathbb{R}^d)$, set

$$\langle U^{b,j}_{m,w,\varepsilon} F, \psi \rangle := \iint |h|^b F(x,h)\, \Gamma^j_{m,w,\varepsilon}\psi(x,h) \,\mathrm{d}x\,\mathrm{d}h \tag{3.5a}$$

$$\langle U^{b}_{m,w,\varepsilon} F, \psi \rangle := \sum_{j\in\mathbb{Z}} \langle U^{b,j}_{m,w,\varepsilon} F, \psi \rangle. \tag{3.5b}$$

Lemma 3.9. *For $F \in L^{p,r}(\nu_\gamma)$, the sums and integrals in* (3.5) *converge absolutely and* (3.5b) *defines $U^b_{m,w,\varepsilon}F \in \mathcal{S}'_\infty(\mathbb{R}^d)$.*

Proof. By Lemmas 3.8 and 3.7, we have for any $K, L \in \mathbb{N}$ sufficiently large,

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}} \iint |h|^b |F(x,h)| |\Gamma^j_{m,w,\varepsilon}\psi(x,h)| \,\mathrm{d}x\,\mathrm{d}h \\
&\lesssim_{K,L} \sum_{j\in\mathbb{Z}} 2^{-|j|} \iint |F(x,h)| \min\{|h|^K, |h|^{-K}\}(1+|x|)^{-L} \,\mathrm{d}x\,\mathrm{d}h \\
&\lesssim \sum_{j\in\mathbb{Z}} 2^{-|j|} \|F\|_{L^{p,r}(\nu_\gamma)} \lesssim \|F\|_{L^{p,r}(\nu_\gamma)}.
\end{aligned}$$

This shows the absolute convergence and defines $U^b_{m,w,\varepsilon}F$ in the algebraic dual of $\mathcal{S}_\infty(\mathbb{R}^d)$.

To see that $U^b_{m,w,\varepsilon}F \in \mathcal{S}'_\infty(\mathbb{R}^d)$, let $\psi_k \in \mathcal{S}_\infty(\mathbb{R}^d)$ be such that $\psi_k \to \psi$ in $\mathcal{S}_\infty(\mathbb{R}^d)$. In particular, $\{\psi_k : k \in \mathbb{N}\}$ is a bounded set in $\mathcal{S}_\infty(\mathbb{R}^d)$ and therefore by Lemma 3.8,

$$|h|^b\big|\Gamma^j_{m,w,\varepsilon}\psi_k(x,h)\big| \lesssim 2^{-|j|}\min\big\{|h|^K, |h|^{-K}\big\}(1+|x|)^{-L},$$

with implicit constant independent of k. Combining this with Lemma 3.7, the dominated convergence theorem shows $\langle U^b_{m,w,\varepsilon}F, \psi_k\rangle \to \langle U^b_{m,w,\varepsilon}F, \psi\rangle$, completing the proof. □

Lemma 3.10. *For $b, \gamma \in \mathbb{R}$, $p \in (1,\infty)$, $r \in [1,\infty]$,*

$$U^b_{m,w,\varepsilon} : L^{p,r}(\nu_\gamma) \to \dot{\mathcal{B}}^{b-\gamma/p}_p(\gamma, r)$$

is a bounded linear transformation.

Proof. This is an application of Lemma 2.4. From the definitions (3.4) and (3.5a) we get

$$L_k U^{b,j}_{m,w,\varepsilon}F(x) = 2^{-jb}\int\limits_{|h-2^{-j}w|\le\varepsilon 2^{-j}} (2^j|h|)^b\varphi(2^{-k}D)m(2^{-j}D)[F(\cdot,h)](x)\frac{\mathrm{d}h}{2^{-jd}}$$

and thus $L_k U^{b,j}_{m,w,\varepsilon} = 0$ when $|k-j| \ge 2$. Then with $V^{b,j}_{m,w,\varepsilon}$ as in (2.7), we get for $n = -1, 0, 1$,

$$L_k U^{b,k+n}_{m,w,\varepsilon}F = 2^{-(k+n)b}V^{b,k+n}_{\widetilde{m}_n,w,\varepsilon}F, \quad \text{with } \widetilde{m}_n = \varphi(2^n\cdot)m, \tag{3.6}$$

and

$$P^b U^b_{m,w,\varepsilon}F(\cdot,k) = 2^{kb}L_k U^b_{m,w,\varepsilon}F = \sum_{n=-1}^{1} 2^{-nb}V^{b,k+n}_{\widetilde{m}_n,w,\varepsilon}F.$$

Hence

$$\begin{aligned}\|U^b_{m,w,\varepsilon}F\|_{\dot{\mathcal{B}}^{b-\gamma/p}_p(\gamma,r)} &\le \sum_{n=-1}^{1} 2^{-nb}\|V^b_{\widetilde{m}_n,w,\varepsilon}F(\cdot,\cdot+n)\|_{L^{p,r}(\mu_\gamma)}\\ &\lesssim_{b,\gamma} \sum_{n=-1}^{1}\|V^b_{\widetilde{m}_n,w,\varepsilon}F\|_{L^{p,r}(\mu_\gamma)}\end{aligned}$$

and since by Lemma 2.4 we have $\|V^b_{\widetilde{m}_n,w,\varepsilon}F\|_{L^{p,r}(\mu_\gamma)} \lesssim \|F\|_{L^{p,r}(\nu_\gamma)}$ the proof is complete. □

The following lemma has a dual version of formula (2.6) and an extension to tempered functions.

Lemma 3.11. *Let $\psi \in \mathcal{S}_\infty(\mathbb{R}^d)$. Then*

$$\sum_{\kappa=1}^{N}\int \Delta^M_{-h}\Gamma^j_{m_\kappa,w_\kappa,\varepsilon}\psi(x,h)\,\mathrm{d}h = L_j\psi(x). \tag{3.7}$$

Moreover, for $f \in \mathcal{T}$ and $\psi \in \mathcal{S}_\infty(\mathbb{R}^d)$,

$$\sum_{\kappa=1}^{N}\sum_{j\in\mathbb{Z}}\iint \Delta_h^M f(x)\,\Gamma^j_{m_\kappa,w_\kappa,\varepsilon}\psi(x,h)\;\mathrm{d}h\,\mathrm{d}x = \int f(x)\psi(x)\;\mathrm{d}x. \tag{3.8}$$

Proof. We first check (3.7), which, after taking the Fourier transform, is equivalent with

$$\sum_{\kappa=1}^{N}\int (e^{i\langle\xi,-h\rangle}-1)^M\big[\Gamma^j_{m_\kappa,w_\kappa,\varepsilon}\psi\big]^\wedge(\xi,h)\;\mathrm{d}h = \varphi(2^{-j}\xi)\widehat{\psi}(\xi). \tag{3.9}$$

Using (3.4) and (2.5a) we have

$$\begin{aligned}&\sum_{\kappa=1}^{N}\int \big(e^{i\langle\xi,-h\rangle}-1\big)^M\big[\Gamma^j_{m_\kappa,w_\kappa,\varepsilon}\psi\big]^\wedge(\xi,h)\;\mathrm{d}h\\ &=\sum_{\kappa=1}^{N}2^{jd}\int_{|h-2^{-j}w_\kappa|<2^{-j}\varepsilon}(e^{i\langle-\xi,h\rangle}-1)^M m_\kappa(-2^{-j}\xi)\widehat{\psi}(\xi)\,\mathrm{d}h\\ &=\varphi(-2^{-j}\xi)\widehat{\psi}(\xi)=\varphi(2^{-j}\xi)\widehat{\psi}(\xi),\end{aligned}$$

here we used that φ is radial. This establishes (3.9) and thus (3.7).

We now prove (3.8). In the argument that follows all integrals and sums converge absolutely by Lemmas 3.7 and 3.8. Using (3.7) we have

$$\begin{aligned}&\sum_{\kappa=1}^{N}\sum_{j\in\mathbb{Z}}\iint \Delta_h^M f(x)\,\Gamma^j_{m_\kappa,w_\kappa,\epsilon}\psi(x,h)\,\mathrm{d}h\,\mathrm{d}x\\ &=\sum_{\kappa=1}^{N}\sum_{j\in\mathbb{Z}}\int f(x)\int \Delta_{-h}^M\Gamma^j_{m_\kappa,w_\kappa,\epsilon}\psi(x,h)\,\mathrm{d}h\,\mathrm{d}x\\ &=\sum_{j\in\mathbb{Z}}\int f(x)\,L_j\psi(x)\,\mathrm{d}x=\int f(x)\psi(x)\,\mathrm{d}x,\end{aligned}$$

where the final equality uses $f \in \mathcal{T}$ and that $\sum_{j\in\mathbb{Z}} L_j\psi = \psi$, with convergence in $\mathcal{S}_\infty(\mathbb{R}^d)$, since $\psi \in \mathcal{S}_\infty(\mathbb{R}^d)$. □

We are prepared to define $\mathfrak{I}_M$. For $f \in \mathcal{M}$ with $\pi_M f \in \mathfrak{B}_{M,s,p}(\gamma,r)$ we set

$$\langle \mathfrak{I}_M(\pi_M f),\psi\rangle \coloneqq \sum_{\kappa=1}^{N}\sum_{j\in\mathbb{Z}}\iint \Delta_h^M f(x)\Gamma^j_{m_\kappa,w_\kappa,\varepsilon}\psi(x,h)\,\mathrm{d}x\,\mathrm{d}h \tag{3.10a}$$

$$= \sum_{\kappa=1}^{N}\langle U^b_{m_\kappa,w_\kappa,\varepsilon}F_b,\psi\rangle \text{ with } F_b(x,h)=\frac{\Delta_h^M f(x)}{|h|^b} \tag{3.10b}$$

where by Lemma 3.9 the sums and integrals in (3.10a) converge absolutely. Note that the definition of $\mathfrak{I}_M$ depends on M, but not on s,γ,p,r, and that (3.10b) holds for all $b \in \mathbb{R}$. We shall later use this formula with $b = s + \gamma/p$.

When $f \in \mathcal{T}$, Lemma 3.11 shows that $\langle \mathfrak{J}_M(\pi_M f), \psi \rangle$ is the standard pairing of $f \in \mathcal{T}$ with a Schwartz function in $\mathcal{S}_\infty$, i.e.

$$\langle \mathfrak{J}_M(\pi_M f), \psi \rangle = \int f(x)\psi(x)\,\mathrm{d}x, \quad \forall f \in \mathcal{T}. \tag{3.11}$$

We need to show that $\mathfrak{J}_M$ is injective on $\mathfrak{B}_{M,s,p}(\gamma, r)$. For this, we will need the following auxiliary lemma.

Lemma 3.12. *Let $p \in (1, \infty)$, $r \in [1, \infty]$, and $\gamma \in \mathbb{R}$. Suppose that $F \in L^{p,r}(\nu_\gamma)$ and $\eta \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ are such that*

$$x \mapsto Q(x) := \int F(x, h)\eta(h)\,\mathrm{d}h$$

is almost everywhere equal to a polynomial. Then $Q(x) = 0$ almost everywhere.

Proof. Let $\phi \in C_c^\infty(\mathbb{R}^d)$ be nonnegative and $\int \phi = 1$. We claim that, for all $G \in L^{p,r}(\nu_\gamma)$,

$$\lim_{|a|\to\infty} \iint G(x, h)\eta(h)\phi(x - a)\,\mathrm{d}h\,\mathrm{d}x = 0 \tag{3.12}$$

Observe that (3.12) follows by standard estimates whenever $G \in L^q(\nu_\gamma)$ for any $q \in (1, \infty)$. It then also holds for $G \in L^{p,r}(\nu_\gamma)$ since $L^{p,r}(\nu_\gamma) \subset L^{p_1}(\nu_\gamma) + L^{p_2}(\nu_\gamma)$, with $p_1 < p < p_2$.

By (3.12) we have

$$0 = \overline{\lim_{|a|\to\infty}} \Big| \iint F(x, h)\eta(h)\phi(x - a)\,\mathrm{d}h\,\mathrm{d}x \Big| = \overline{\lim_{|a|\to\infty}} \Big| \int Q(x)\phi(x - a)\,\mathrm{d}x \Big|$$

and the last expression is equal to $|c|$ if $Q(x) = c$ almost everywhere, and equal to ∞ if Q is almost everywhere equal to a nonconstant polynomial. We conclude that $Q(x) = 0$ almost everywhere. □

Lemma 3.13. *For $M \in \mathbb{N}$, $s, \gamma \in \mathbb{R}$, $p \in (1, \infty)$, and $r \in [1, \infty]$, $\mathfrak{J}_M$ is injective on $\mathfrak{B}_{M,s,p}(\gamma, r)$.*

Proof. Suppose $f \in \mathcal{M}$ is such that $\pi_M f \in \mathfrak{B}_{M,s,p}(\gamma, r)$ and $\mathfrak{J}_M \pi_M f = 0$ as an element of $\mathcal{S}'_\infty(\mathbb{R}^d)$. We wish to show $f(x) = P(x)$, almost everywhere, for some polynomial $P(x)$ of degree $\le M - 1$. In this proof, all sums and integrals converge absolutely by Lemmas 3.7 and 3.8.

Take $\psi \in \mathcal{S}_\infty(\mathbb{R}^d)$ and $\eta \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$. Then,

$$\int \eta(h')\Delta_{h'}^M \psi(x)\,\mathrm{d}h' \in \mathcal{S}_\infty(\mathbb{R}^d).$$

Thus, we have, using (3.10a) and the definition of the translation invariant operator $\Gamma^j_{m_\kappa,w_\kappa,\epsilon}$ (see (3.4)),

$$\begin{aligned} 0 &= \left\langle \mathfrak{I}_M \pi_M f, \int \eta(h')\,\Delta^M_{-h'}\psi\,\mathrm{d}h' \right\rangle \\ &= \sum_{\kappa=1}^{N}\sum_{j\in\mathbb{Z}} \iiint \Delta^M_h f(x)\,\eta(h')\,\Delta^M_{-h'}\Gamma^j_{m_\kappa,w_\kappa,\varepsilon}\psi(x,h)\,\mathrm{d}x\,\mathrm{d}h\,\mathrm{d}h' \\ &= \sum_{j\in\mathbb{Z}} \iint \Delta^M_{h'} f(x)\eta(h') \sum_{\kappa=1}^{N} \int \Delta^M_{-h}\Gamma^j_{m_\kappa,w_\kappa,\varepsilon}\psi(x,h)\,\mathrm{d}h\,\mathrm{d}x\,\mathrm{d}h' \\ &= \sum_{j\in\mathbb{Z}} \iint \Delta^M_{h'} f(x)\eta(h')\,\mathrm{d}h'\,L_j\psi(x)\,\mathrm{d}x, \end{aligned}$$

where the last equality uses (3.7). It follows from Lemma 3.7 and the fact that $\eta \in C_c^\infty(\mathbb{R}^d\setminus\{0\})$ that $\int \Delta^M_{h'} f(\cdot)\eta(h')\,\mathrm{d}h' \in \mathcal{T}$. Since $\psi \in \mathcal{S}_\infty(\mathbb{R}^d)$, we have $\sum_{j\in\mathbb{Z}} L_j\psi = \psi$ with convergence in $\mathcal{S}_\infty(\mathbb{R}^d)$. Thus,

$$\begin{aligned} 0 &= \sum_{j\in\mathbb{Z}} \iint \Delta^M_{h'} f(x)\eta(h')\,\mathrm{d}h' L_j\psi(x)\,\mathrm{d}x \\ &= \iint \Delta^M_{h'} f(x)\,\eta(h')\,\mathrm{d}h'\psi(x)\,\mathrm{d}x, \end{aligned} \tag{3.13}$$

for arbitrary $\psi \in \mathcal{S}_\infty(\mathbb{R}^d)$ and we can conclude that

$$\int \Delta^M_{h'} f(x)\eta(h')\,\mathrm{d}h' = Q(x), \text{ a.e.}$$

for some polynomial Q. By Lemma 3.12 it follows that $Q = 0$, hence

$$\int \Delta^M_{h'} f(x)\eta(h')\,\mathrm{d}h' = 0, \text{ a.e.}$$

Since $\eta \in C_0^\infty(\mathbb{R}^d\setminus\{0\})$ was arbitrary, this implies $\Delta^M_h f(x) = 0$ for almost every $(x,h)\in\mathbb{R}^{2d}$. Lemma 3.6 shows $f(x) = P(x)$, almost everywhere, for some polynomial $P(x)$ of degree $\le M-1$, completing the proof. □

Proof of Proposition 3.5. It follows immediately from the definitions that

$$\begin{cases} \pi_M f \mapsto \left((x,h)\mapsto \frac{\Delta^M_h f(x)}{|h|^{s+\gamma/p}}\right) \\ \mathfrak{B}_{M,s,p}(\gamma,r) \to L^{p,r}(\nu_\gamma) \end{cases}$$

is bounded. Lemma 3.10 shows that $U^{s+\gamma/p}_{m_\kappa,w_\kappa,\varepsilon} : L^{p,r}(\nu_\gamma) \to \dot{\mathcal{B}}^s_p(\gamma,r)$ is bounded. Composing these maps and using (3.10b) shows that

$$\mathfrak{I}_M : \mathfrak{B}_{M,s,p}(\gamma,r) \to \dot{\mathcal{B}}^s_p(\gamma,r)$$

is bounded. $\mathfrak{I}_M$ is injective by Lemma 3.13. Finally, (3.2) follows from (3.11). □

Proof of Theorem 3.2, conclusion. By Proposition 3.3,

$$\mathcal{E}_M(\dot{\mathcal{B}}^s_p(\gamma, r)) \subseteq \mathcal{T}_M \cap \mathfrak{B}_{M,s,p}(\gamma, r),$$

and so by (3.2), $\mathcal{J}_M\big|_{\mathcal{E}_M(\dot{\mathcal{B}}^s_p(\gamma,r))} = \iota_M\big|_{\mathcal{E}_M(\dot{\mathcal{B}}^s_p(\gamma,r))}$. By Proposition 3.3, ι_M is a left inverse to $\mathcal{E}_M$, and we conclude $\mathcal{J}_M\mathcal{E}_M$ is the identity map on $\dot{\mathcal{B}}^s_p(\gamma, r)$. In particular, $\mathcal{J}_M\big|_{\mathcal{T}_M\cap\mathfrak{B}_{M,s,p}(\gamma,r)} : \mathcal{T}_M \cap \mathfrak{B}_{M,s,p}(\gamma, r) \to \dot{\mathcal{B}}^s_p(\gamma, r)$ is surjective. Proposition 3.5 shows $\mathcal{J}_M : \mathfrak{B}_{M,s,p}(\gamma, r) \to \dot{\mathcal{B}}^s_p(\gamma, r)$ is injective. We conclude

$$\mathcal{T}_M \cap \mathfrak{B}_{M,s,p}(\gamma, r) = \mathfrak{B}_{M,s,p}(\gamma, r), \tag{3.14}$$

establishing part (i) of the theorem, and moreover that $\mathcal{J}_M : \mathfrak{B}_{M,s,p}(\gamma, r) \to \dot{\mathcal{B}}^s_p(\gamma, r)$ is bijective with two-sided inverse $\mathcal{E}_M$. From (3.14) and (3.2) we see that

$$\mathcal{J}_M : \mathfrak{B}_{M,s,p}(\gamma, r) \to \dot{\mathcal{B}}^s_p(\gamma, r)$$

agrees with ι_M on all of $\mathfrak{B}_{M,s,p}(\gamma, r)$. Thus,

$$\iota_M\big|_{\mathfrak{B}_{M,s,p}(\gamma,r)} : \mathfrak{B}_{M,s,p}(\gamma, r) \to \dot{\mathcal{B}}^s_p(\gamma, r)$$

is a bounded bijective map with bounded inverse $\mathcal{E}_M$. This establishes parts (ii) and (iii) of the theorem, completing the proof. □

4. Embeddings

The proof of the embeddings in Theorem 1.5 is reduced to inequalities for the operator T_a defined on functions $F : \mathbb{R}^d \times \mathbb{Z} \to \mathbb{C}$ by

$$T_a F(x, j) = 2^{ja} F_j(x), \tag{4.1}$$

with the parameters $a = \pm\gamma/p$.

Lemma 4.1. *The following hold for all $\gamma \in \mathbb{R}$, $1 < p < \infty$.*

(i) For $p \le r \le \infty$,

$$\|T_{-\gamma/p}G\|_{\ell^r(L^{p,r})} \lesssim \|G\|_{L^{p,r}(\mu_\gamma)}.$$

(ii) For $1 \le r \le p$

$$\|T_{\gamma/p}F\|_{L^{p,r}(\mu_\gamma)} \lesssim \|F\|_{\ell^r(L^{p,r})}.$$

Proof. Part (i) follows from the definitions of Lorentz spaces via the distribution function. We use a change of variable with subsequent interchange of sum and integral to write

$$\begin{aligned}
\|T_{-\gamma/p}G\|_{\ell^r(L^{p,r})} &\lesssim \Big(\sum_j \int_0^\infty \lambda^r \big[\text{meas}\{x : 2^{-j\frac{\gamma}{p}}|G(x,j)| > \lambda\}\big]^{r/p} \frac{\mathrm{d}\lambda}{\lambda}\Big)^{1/r} \\
&= \Big(\int_0^\infty \beta^r \sum_j \big[2^{-j\gamma}\text{meas}\{x : |G(x,j)| > \beta\}\big]^{r/p} \frac{\mathrm{d}\beta}{\beta}\Big)^{1/r}
\end{aligned}$$

and since $r \ge p$ we estimate an $\ell^{r/p}$-norm by an ℓ^1-norm and see that the last displayed expression is dominated by

$$\begin{aligned}&\Big(\int_0^\infty \beta^r\Big[\sum_j 2^{-j\gamma}\mathrm{meas}\{x:\,|G(x,j)|>\beta\}\Big]^{r/p}\frac{\mathrm{d}\beta}{\beta}\Big)^{1/r}\\ &=\Big(\int_0^\infty \beta^r\big[\mu_\gamma\{(x,j):\,|G(x,j)|>\beta\}\big]^{r/p}\frac{\mathrm{d}\beta}{\beta}\Big)^{\frac1r}\lesssim\|G\|_{L^{p,r}(\mu_\gamma)}.\end{aligned}$$

For part (ii) we use that $(L^{p,r}(\mu_\gamma))^* = L^{p',r'}(\mu_\gamma)$, ([33]) and $(\ell^r(L^{p,r}))^* = \ell^{r'}(L^{p',r'})$. Observe that for $1 \le r \le p$

$$\begin{aligned}&\Big|\int\sum_j T_{\gamma/p}F(x,j)G(x,j)\frac{\mathrm{d}x}{2^{j\gamma}}\Big| = \Big|\sum_j\int F(x,j)2^{-j\gamma/p'}G(x,j)\,\mathrm{d}x\Big|\\ &\lesssim\|F\|_{\ell^r(L^{p,r})}\|T_{-\gamma/p'}G\|_{\ell^{r'}(L^{p',r'})}\lesssim\|F\|_{\ell^r(L^{p,r})}\|G\|_{L^{p',r'}(\mu_\gamma)}\end{aligned}$$

where we have used part (i) for the exponents $p' \le r'$. The proof is completed by taking the supremum over all G with $\|G\|_{L^{p',r'}(\mu_\gamma)} \le 1$. □

Lemma 4.2. *Let $1 < p < \infty$, $\gamma \ne 0$.*
(i) For $p \le r \le \infty$,

$$\|T_{\gamma/p}F\|_{L^{p,r}(\mu_\gamma)} \lesssim \|F\|_{L^p(\ell^r)}.$$

(ii) For $1 \le r \le p$,

$$\|T_{-\gamma/p}G\|_{L^p(\ell^r)} \lesssim \|G\|_{L^{p,r}(\mu_\gamma)}$$

Proof. The argument for part (i) has been used in proofs for endpoint multiplier theorems, our proof is essential the one from [41, Lemma 2.4] (see also [41] for further references).

Let $0 \le \theta \le 1$ and $1/r = (1-\theta)/p$. We use the complex interpolation formulas

$$[L^p(\ell^p), L^p(\ell^\infty)]_\theta = L^p(\ell^r), \quad [L^p(\mu_\gamma), L^{p,\infty}(\mu_\gamma)]_\theta = L^{p,r}(\mu_\gamma).$$

These imply that it suffices to prove the assertion for $r = p$ and $r = \infty$. For $r = p$ we have $\|T_{\gamma/p}F\|_{L^p(\mu_\gamma)} = \|F\|_{L^p(\ell^p)}$.

For $r = \infty$ the conclusion $T_{\gamma/p} : L^p(\ell^\infty) \to L^{p,\infty}(\mu_\gamma)$ follows from

$$\begin{aligned}\mu_\gamma\{(x,j):|T_{\gamma/p}F(x,j)|>\lambda\} &= \int_{\mathbb{R}^d}\sum_{\substack{j:\\ 2^{j\gamma/p}|F_j(x)|>\lambda}} 2^{-j\gamma}\,\mathrm{d}x\\ &\le\int_{\mathbb{R}^d}\sum_{\substack{j:\\ 2^{-j\gamma/p}<\lambda^{-1}\sup_k|F_k(x)|}} 2^{-j\gamma}\,\mathrm{d}x\lesssim\int_{\mathbb{R}^d}\frac{(\sup_k|F_k(x)|)^p}{\lambda^p}\,\mathrm{d}x;\end{aligned}$$

here we used $\gamma \ne 0$.

For part (ii) we use that $(L^{p,r}(\mu_\gamma))^* = L^{p',r'}(\mu_\gamma)$, see [33] and $(L^p(\ell^r))^* = L^{p'}(\ell^{r'})$. Observe that for $F \in L^{p'}(\ell^{r'})$

$$\int \sum_j 2^{-j\gamma/p} G_j(x) F_j(x)\,\mathrm{d}x = \int \sum_j G_j(x)\, T_{\gamma/p'} F(x,j) 2^{-j\gamma}\,\mathrm{d}x$$
$$\lesssim \|G\|_{L^{p,r}(\mu_\gamma)} \|T_{\gamma/p'}F\|_{L^{p',r'}(\mu_\gamma)} \lesssim \|G\|_{L^{p,r}(\mu_\gamma)} \|F\|_{L^{p'}(\ell^{r'})}$$

where we have used part (i). Now part (ii) follows by taking the sup over all F with $\|F\|_{L^{p'}(\ell^{r'})} \le 1$. □

Proof of Theorem 1.5. Apply Lemma 4.1 and Lemma 4.2 with $F(x,j) = 2^{js} L_j f(x)$ and $G(x,j) = 2^{j(s+\frac{\gamma}{p})} L_j f(x)$. □

5. Non-embeddings

We prove Theorem 1.6. Proposition 5.1 covers part (i) and (ii) of the theorem, in the range $\gamma \ge -d$, and Proposition 5.2 covers the same parts for the range $\gamma < -d$. Proposition 5.3 covers part (iii) of Theorem 1.6. We begin with some definitions to build the examples.

If $\gamma \ge -d$ and $k > 0$, or if $\gamma < -d$ and $k < 0$, define

$$\mathfrak{N}_\gamma(k) := \lfloor 2^{k(d+\gamma)} \rfloor.$$

Let $\{n_{i,k}\}$ be a double indexed set in $\mathbb{Z}$, with $1 \le i \le \mathfrak{N}_\gamma(k)$, which is separated in the sense that for every k

$$i_1 \ne i_2 \implies |n_{i_1,k} - n_{i_2,k}| \ge 2^{10|k|}.$$

Let $\eta \in \mathcal{S}$ such that

$$|\eta(x)| \approx 1 \text{ for } |x| \le 1. \tag{5.1a}$$

$$\operatorname{supp}(\widehat{\eta}) \subset \Big\{\xi \in \widehat{\mathbb{R}}^d : \frac{15}{16} \le |\xi| \le \frac{17}{16}\Big\} \tag{5.1b}$$

and let

$$\eta_{i,k}(x) = \eta(2^k(x - n_{i,k}e_1)). \tag{5.2}$$

By (1.2) we have

$$\eta_{i,k} = L_k \eta_{i,k} \tag{5.3a}$$

and

$$L_\ell \eta_{i,k} = 0 \text{ if } \ell \ne k. \tag{5.3b}$$

Define for $k \in \mathbb{Z}$

$$f_{\gamma,k}(x) = 2^{-k\gamma/p} \sum_{i=1}^{\mathfrak{N}_\gamma(k)} \eta_{i,k}. \tag{5.4}$$

Proposition 5.1. *Let $f_{\gamma,k}$ be as in* (5.4)*. Let $s \in \mathbb{R}$. Assume $\gamma \geq -d$, and define*[2]

$$F_{\gamma,N}(x) = \sum_{k=N+1}^{2N} 2^{-ks} f_{\gamma,k}(x). \tag{5.5}$$

(i) Then $1 < p \leq r \leq \infty$

$$\|F_{\gamma,N}\|_{\dot{\mathcal{B}}^s_p(\gamma,\infty)} = \|F_{\gamma,N}\|_{\mathcal{B}^s_p(\gamma,\infty)} \gtrsim N^{1/p} \tag{5.6}$$

$$\|F_{\gamma,N}\|_{\dot{\mathcal{B}}^s_p(\beta,r)} = \|F_{\gamma,N}\|_{\mathcal{B}^s_p(\beta,r)} \lesssim N^{1/r}, \quad \textit{for } \beta \neq \gamma, \tag{5.7}$$

$$\|F_{\gamma,N}\|_{\dot{B}^s_{p,r}} = \|F_{\gamma,N}\|_{B^s_{p,r}} \lesssim N^{1/r} \tag{5.8}$$

(ii) If $p < \infty$ then $F_\gamma = \sum_{\ell\geq 1} \ell 2^{-\ell/p} F_{\gamma,2^\ell}$ belongs to $\bigcap_{r>p} \dot{B}^s_{p,r}$ and to $\bigcap_{\substack{r>p\\ \beta\neq\gamma}} \dot{\mathcal{B}}^s_p(\beta,r)$, but not to $\dot{\mathcal{B}}^s_p(\gamma,\infty)$.

Also F_γ belongs to $\bigcap_{r>p} B^s_{p,r}$ and to $\bigcap_{\substack{r>p\\ \beta\neq\gamma}} \mathcal{B}^s_p(\beta,r)$, but not to $\mathcal{B}^s_p(\gamma,\infty)$.

Proof. Let $r \geq p$. We begin with the upper bound for the $\dot{\mathcal{B}}^s_p(\beta,r)$ quasi-norm of $F_{\gamma,N}$ for $\beta \neq \gamma$. Let

$$E_{\gamma,\beta,k}(\lambda) = \Big\{ x \in \mathbb{R}^d : \Big| \sum_{i=1}^{\mathfrak{N}_\gamma(k)} \eta_{i,k}(x) \Big|^p > \lambda^p 2^{k(\gamma-\beta)} \Big\}.$$

Note that from (5.3a) we get

$$\|F_{\gamma,N}\|_{\dot{\mathcal{B}}^s_p(\beta,r)} = \Big(r \int_0^\infty \Big[\sum_{k=N+1}^{2N} \lambda^p 2^{-k\beta} \operatorname{meas} E_{\gamma,\beta,k}(\lambda) \Big]^{r/p} \frac{\mathrm{d}\lambda}{\lambda} \Big)^{1/p}.$$

In what follows we will use, for $M > d + |\gamma|$, the estimate

$$|\eta(x)| \leq C_M (1 + |x|)^{-M}. \tag{5.9}$$

Split $\eta_{i,k} = \vartheta_{i,k} + \varepsilon_{i,k}$ where

$$\vartheta_{i,k} = \eta_{i,k} \mathbb{1}_{\{|x-n_{i,k}e_1| \leq 2^k\}}, \qquad \varepsilon_{i,k} = \eta_{i,k} - \vartheta_{i,k}.$$

Note for later reference $\|\varepsilon_{i,k}\|_p \lesssim_M 2^{-k\frac{d}{p}} 2^{2k(\frac{d}{p}-M)}$ and therefore

$$\Big\| \sum_{i=1}^{\mathfrak{N}_\gamma(k)} \varepsilon_{i,k} \Big\|_p \lesssim 2^{k(d+\gamma)} 2^{-k\frac{d}{p}} 2^{2k(\frac{d}{p}-M)}. \tag{5.10}$$

[2]The definitions in (5.5), (5.14) depend on s but we do not include the subscript s to keep the notation as simple as possible.

Let

$$E^{(1)}_{\gamma,\beta,k}(\lambda) = \Big\{x \in \mathbb{R}^d : \Big|\sum_{i=1}^{\mathfrak{N}_\gamma(k)} \vartheta_{i,k}(x)\Big|^p > (\frac{\lambda}{2})^p 2^{k(\gamma-\beta)}\Big\}$$

$$E^{(2)}_{\gamma,\beta,k}(\lambda) = \Big\{x \in \mathbb{R}^d : \Big|\sum_{i=1}^{\mathfrak{N}_\gamma(k)} \varepsilon_{i,k}(x)\Big|^p > (\frac{\lambda}{2})^p 2^{k(\gamma-\beta)}\Big\}.$$

Then

$$E_{\gamma,\beta,k}(\lambda) \subset E^{(1)}_{\gamma,\beta,k}(\lambda) \cup E^{(2)}_{\gamma,\beta,k}(\lambda). \tag{5.11}$$

Finally set, for $i = 1, \dots, \mathfrak{N}_\gamma(k)$,

$$E^{(1,i)}_{\gamma,\beta,k}(\lambda) = \{x \in \mathbb{R}^d : |\vartheta_{i,k}(x)|^p > (\frac{\lambda}{2})^p 2^{k(\gamma-\beta)}\}.$$

Observe that for fixed k the sets supp $(\vartheta_{i,k})$ are disjoint and therefore the sets $E^{(1)}_{\gamma,\beta,k}(\lambda)$ are the disjoint union of the sets $E^{(1,i)}_{\gamma,\beta,k}(\lambda)$, $i = 1, \dots, \mathfrak{N}_\gamma(k)$. Now from (5.9) we get

$$E^{(1,i)}_{\gamma,\beta,k}(\lambda) \subset \Big\{x : |x - n_{i,k}e_1| \le 2^{-k}\Big(\frac{2^p C_M^p}{\lambda^p 2^{k(\gamma-\beta)}}\Big)^{\frac{1}{Mp}}\Big\}$$

and therefore we get for the Lebesgue measure

$$\operatorname{meas} E^{(1)}_{\gamma,\beta,k}(\lambda) \le c_d \mathfrak{N}_\gamma(k) 2^{-kd}\Big(\frac{2^p C_M^p}{\lambda^p 2^{k(\gamma-\beta)}}\Big)^{\frac{d}{Mp}}.$$

Hence, using the definition of $\mathfrak{N}_\gamma(k)$

$$\lambda^p 2^{-k\beta} \operatorname{meas} E^{(1)}_{\gamma,\beta,k}(\lambda) \le c_d (2C_M)^{\frac{d}{M}} (\lambda^p 2^{k(\gamma-\beta)})^{1-\frac{d}{Mp}}$$

Using (5.9) we also see that for $k = N+1, \dots, 2N$

$$E^{(1)}_{\gamma,\beta,k}(\lambda) = \emptyset \text{ for } (\lambda/2)^p 2^{k(\gamma-\beta)} > 2C_M^p.$$

Hence we get, for $\gamma > \beta$ and $r < \infty$,

$$\Big(\int_0^\infty \Big[\sum_{\substack{N+1\le k\le 2N \\ \lambda^p 2^{k(\gamma-\beta)} \le 2C_M^p}} \lambda^p 2^{-k\beta} \operatorname{meas} E^{(1)}_{\gamma,\beta,k}(\lambda)\Big]^{r/p} \frac{\mathrm{d}\lambda}{\lambda}\Big)^{1/r} \le I + II$$

where

$$I = \Big(\int_0^{C_M(2^{-2N(\gamma-\beta)})^{1/p}} \Big[\sum_{N+1\le k\le 2N} \widetilde{C}_M (\lambda^p 2^{k(\gamma-\beta)})^{1-\frac{d}{Mp}}\Big]^{r/p} \frac{\mathrm{d}\lambda}{\lambda}\Big)^{1/r}$$

$$II = \Big(\int_{C_M(2^{-2N(\gamma-\beta)})^{1/p}}^{2C_M(2^{-N(\gamma-\beta)})^{1/p}} \Big[\sum_{\substack{N+1\le k\le 2N \\ \lambda^p 2^{k(\gamma-\beta)} \le 2^p C_M^p}} \widetilde{C}_M (\lambda^p 2^{k(\gamma-\beta)})^{1-\frac{d}{Mp}}\Big]^{r/p} \frac{\mathrm{d}\lambda}{\lambda}\Big)^{1/r}.$$

We estimate

$$I \lesssim \Big(\int_0^{C_M (2^{-2N(\gamma-\beta)})^{1/p}} (\lambda^p 2^{2N(\gamma-\beta)})^{(1-\frac{d}{Mp})\frac{r}{p}} \frac{\mathrm{d}\lambda}{\lambda} \Big)^{1/r} \lesssim 1$$

$$II \lesssim \Big(\int_{C_M (2^{-2N(\gamma-\beta)})^{1/p}}^{2C_M (2^{-N(\gamma-\beta)})^{1/p}} \frac{\mathrm{d}\lambda}{\lambda} \Big)^{1/r} \lesssim N^{1/r}$$

and it follows that $\|F_{N,\gamma}\|_{\dot{\mathcal{B}}^s_p(\beta,r)} \lesssim N^{1/r}$ provided that $\beta < \gamma$.

The calculation for $\gamma < \beta$ is very similar, except the integration is over $\lambda \in [0, C_M (2^{1+2N(\beta-\gamma)})^{1/p}]$ and the corresponding integrals for the parts I and II are extended from 0 to $C_M (2^{N(\beta-\gamma)})^{1/p}$ and from $C_M (2^{N(\beta-\gamma)})^{1/p}$ to $C_M (2^{1+2N(\beta-\gamma)})^{1/p}$, respectively. Again the first term gives an $O(1)$ contribution and the second one an $O(N^{1/r})$ contribution. Summarizing we get

$$\Big(\int_0^\infty \Big[\sum_{N<k\le 2N} \lambda^p 2^{-k\beta} \mathrm{meas}\, E^{(1)}_{\gamma,\beta,k}(\lambda) \Big]^{r/p} \frac{\mathrm{d}\lambda}{\lambda} \Big)^{1/r} \lesssim N^{1/r}. \tag{5.12}$$

A similar (and easier) calculation shows that one has the corresponding bound when $r = \infty$ as long as $\beta \neq \gamma$. We now estimate the error term; we show in fact the stronger inequality

$$\Big(\int_0^\infty \Big[\sum_{N<k\le 2N} \lambda^p 2^{-k\beta} \mathrm{meas}\, E^{(2)}_{\gamma,\beta,k}(\lambda) \Big]^{r/p} \frac{\mathrm{d}\lambda}{\lambda} \Big)^{1/r} \lesssim 2^{-N} \tag{5.13}$$

for $r \ge p$. We discretize the integral in λ, use the embedding $\ell^p \hookrightarrow \ell^r$, then the change of variables $\sigma = \lambda 2^{k(\gamma-\beta)/p}$ and then the formula for the L^p-norm via the distribution function to estimate the left hand side of (5.13) by

$$\begin{aligned}
&\lesssim \Big(\sum_{N<k\le 2N} \int_0^\infty \lambda^p 2^{-k\beta} \mathrm{meas}\, E^{(2)}_{\gamma,\beta,k}(\lambda) \frac{\mathrm{d}\lambda}{\lambda} \Big)^{1/p} \\
&\lesssim \Big(\sum_{N<k\le 2N} \int_0^\infty 2^{-k\gamma} \sigma^p \mathrm{meas}\big\{ x : \big| \sum_{i=1}^{\mathfrak{N}_\gamma(k)} \varepsilon_{i,k}(x) \big| > \sigma \big\} \frac{\mathrm{d}\sigma}{\sigma} \Big)^{1/p} \\
&\lesssim \Big(\sum_{N<k\le 2N} 2^{-k\gamma} \Big\| \sum_{i=1}^{\mathfrak{N}_\gamma(k)} \varepsilon_{i,k} \Big\|_p^p \Big)^{1/p} \lesssim 2^{-N},
\end{aligned}$$

here we used (5.10) with M large. This finishes the proof of $\|F_{\gamma,N}\|_{\dot{\mathcal{B}}^s_p(\gamma,r)} \lesssim N^{1/r}$ and since the Fourier transform of $F_{\gamma,N}$ is supported where $|\xi| \gg 1$ we may replace $\dot{\mathcal{B}}^s_p(\beta,r)$ with $\mathcal{B}^s_p(\gamma,r)$. Thus (5.7) is now proved, and this inequality also yields $\|F_\gamma\|_{\dot{\mathcal{B}}^s_p(\beta,r)} \lesssim 1$.

We now give the proof of (5.8). The proof is similar to the above but simpler. We use (5.3a) to write

$$\|F_{\gamma,N}\|_{\dot{B}^s_{p,r}} = \Big(\sum_{k=N+1}^{2N} \|f_{\gamma,k}\|_p^r \Big)^{1/r} = I_1 + II_1$$

where

$$I_1 = \Big(\sum_{k=N+1}^{2N} \Big\| 2^{-k\gamma/p} \sum_{i=1}^{\mathfrak{N}_\gamma(k)} \vartheta_{i,k} \Big\|_p^r \Big)^{1/r}$$

$$II_1 \lesssim \Big(\sum_{k=N+1}^{2N} \Big\| 2^{-k\gamma/p} \sum_{i=1}^{\mathfrak{N}_\gamma(k)} \varepsilon_{i,k} \Big\|_p^r \Big)^{1/r}$$

Using the disjointness of support property of the $\vartheta_{i,k}$ we have

$$\|2^{-k\gamma/p} \sum_{i=1}^{\mathfrak{N}_\gamma(k)} \vartheta_{i,k}\|_p \approx 1, \quad N < k \le 2N$$

and hence $I_1 \approx N^{1/r}$ (with the obvious modification if $r = \infty$). For II_1 we use (5.10) for sufficiently large M and see that $|II_1| \lesssim 2^{-N}$, and (5.8) follows. We also have $\|F_\gamma\|_{\dot{B}^s_{p,r}} \lesssim 1$.

We conclude by proving the lower bound (5.6). We have for $\lambda \ll 1$

$$\|F_{\gamma,N}\|^p_{\dot{\mathcal{B}}^s_p(\gamma,\infty)} \ge \sum_{N<k\le 2N} \lambda^p 2^{-k\gamma} \text{meas}\Big\{x : \big|\sum_{i=1}^{\mathfrak{N}_\gamma(k)} \eta_{i,k}(x)\big| > \lambda\Big\} \ge I_2^p - II_2^p$$

where

$$I_2^p = \lambda^p \sum_{N<k\le 2N} 2^{-k\gamma} \text{meas}\Big\{x : \big|\sum_{i=1}^{\mathfrak{N}_\gamma(k)} \vartheta_{i,k}(x)\big| > 2\lambda\Big\}$$

$$II_2^p = \lambda^p \sum_{N<k\le 2N} 2^{-k\gamma} \text{meas}\Big\{x : \big|\sum_{i=1}^{\mathfrak{N}_\gamma(k)} \varepsilon_{i,k}(x)\big| > \lambda\Big\}.$$

By the support properties of the $\vartheta_{i,k}$ and by (5.1a) we have for sufficiently small λ

$$\text{meas}\Big\{x : \big|\sum_{i=1}^{\mathfrak{N}_\gamma(k)} \vartheta_{i,k}(x)\big| > 2\lambda\Big\} \gtrsim \mathfrak{N}_\gamma(k) 2^{-kd} \approx 2^{k\gamma}$$

and hence $I_2 \gtrsim N^{1/p}$. By (5.10) and Chebyshev's inequality

$$II_2^p \lesssim \sum_{N<k\le 2N} 2^{-k\gamma} \Big\| \sum_{i=1}^{\mathfrak{N}_\gamma(k)} \varepsilon_{i,k} \Big\|_p^p \lesssim 2^{-N}$$

and combining the two estimates we get for sufficiently large N the desired lower bound $\|F_{\gamma,N}\|_{\dot{\mathcal{B}}^s_p(\gamma,\infty)} \ge cN^{1/p}$.

Finally

$$\begin{aligned}\|F_\gamma\|_{\dot{\mathcal{B}}^s_p(\gamma,\infty)} &\gtrsim \sup_{\lambda>0}\lambda\Big(\sum_{\ell\geq 1}\sum_{2^\ell<k\leq 2^{\ell+1}} 2^{-k\gamma}\mathrm{meas}\Big\{\ell 2^{-\ell/p}\big|\sum_{i=1}^{\mathfrak{N}_\gamma(k)}\eta_{i,k}\big|>\lambda\Big\}\Big)^{1/p}\\ &\geq \sup_{\ell\geq 1}\ell 2^{-\ell/p}\sup_{\sigma>0}\sigma\Big(\sum_{2^\ell<k\leq 2^{\ell+1}} 2^{-k\gamma}\mathrm{meas}\Big\{\big|\sum_{i=1}^{\mathfrak{N}_\gamma(k)}\eta_{i,k}\big|>\sigma\Big\}\Big)^{1/p}\\ &\gtrsim \sup_{\ell\geq 1}\ell 2^{-\ell/p}\sigma_0(2^\ell)^{1/p}=\infty\end{aligned}$$

for sufficiently small $\sigma_0 \ll 1$, and we see that $F_\gamma \notin \dot{\mathcal{B}}^s_p(\gamma,\infty)$. □

The counterpart of Proposition 5.1 in the range $\gamma < -d$ is as follows.

Proposition 5.2. *Let $f_{\gamma,k}$ be as in (5.4). Let $s \in \mathbb{R}$. Assume $\gamma < -d$ and define*

$$F_{\gamma,N}(x) = \sum_{-2N<k\leq -N} 2^{-ks} f_{\gamma,k}(x) \tag{5.14a}$$

and

$$G_{\gamma,N}(x) = 2^{3N(\frac{d}{p}-s)}F_{\gamma,N}(2^{3N}x). \tag{5.14b}$$

(i) Then for $1 < p \leq r \leq \infty$,

$$\begin{aligned}\|G_{\gamma,N}\|_{\mathcal{B}^s_p(\gamma,\infty)} &= \|F_{\gamma,N}\|_{\dot{\mathcal{B}}^s_p(\gamma,\infty)} \gtrsim N^{1/p},\\ \|G_{\gamma,N}\|_{\mathcal{B}^s_p(\beta,r)} &= \|F_{\gamma,N}\|_{\dot{\mathcal{B}}^s_p(\beta,r)} \approx N^{1/r}, \quad \textit{for } \beta\neq\gamma,\\ \|G_{\gamma,N}\|_{B^s_{p,r}} &= \|F_{\gamma,N}\|_{\dot{B}^s_{p,r}} \approx N^{1/r}.\end{aligned}$$

(ii) If $p < \infty$, then $F = \sum_{\ell\geq 1}\ell 2^{-\ell/p}F_{\gamma,2^\ell}$ belongs to $\bigcap_{r>p}\dot{B}^s_{p,r}$ and to $\bigcap_{\substack{r>p\\ \beta\neq\gamma}}\dot{\mathcal{B}}^s_p(\beta,r)$ but not to $\dot{\mathcal{B}}^s_p(\gamma,\infty)$.

(iii) If $p < \infty$, then $G_\gamma = \sum_{\ell\geq 1}\ell 2^{-\ell/p}G_{\gamma,2^\ell}$ belongs to $\bigcap_{r>p}B^s_{p,r}$ and to $\bigcap_{\substack{r>p\\ \beta\neq\gamma}}\mathcal{B}^s_p(\beta,r)$ but not to $\mathcal{B}^s_p(\gamma,\infty)$.

Sketch of proof. The proof of the bounds for $F_{\gamma,N}$ is exactly analogous to the corresponding arguments in Proposition 5.1. Observe that the parameter k now varies between $-2N$ and $-N$ and since $\gamma < -d$ we now have $\mathfrak{N}_\gamma(k) = \lfloor 2^{k(d+\gamma)}\rfloor = \lfloor 2^{|k||d+\gamma|}\rfloor \geq 1$. Also notice that the Fourier transform of $G_{\gamma,N}$ is supported on large frequencies and therefore the homogeneous and inhomogeneous Besov type norms for $G_{\gamma,N}$ coincide.

To pass from estimates for $F_{\gamma,N}$ to estimates for $G_{\gamma,N}$ we just use the dilation formulas

$$\begin{aligned}2^{n(\frac{d}{p}-s)}\|f(2^n\cdot)\|_{\dot{\mathcal{B}}^s_p(\gamma,r)} &= \|f\|_{\dot{\mathcal{B}}^s_p(\gamma,r)},\\ 2^{n(\frac{d}{p}-s)}\|f(2^n\cdot)\|_{\dot{B}^s_{p,r}} &= \|f\|_{\dot{B}^s_{p,r}}.\end{aligned}$$

□

The following two lemmas show that the assumption $\gamma \neq 0$ in part (ii) of Theorem 1.5 cannot be removed. A combination of these lemmas gives a proof of Theorem 1.7.

Lemma 5.3. *Let $s \in \mathbb{R}$, and $1 < p < \infty$. There exists $f \in \bigcap_{r>p} \dot{F}^s_{p,r}$ which does not belong to $\dot{\mathcal{B}}^s_p(0,\infty)$.*

Proof. Let $\eta_\circ$ be a Schwartz function such that $\eta_\circ(x) > 1$ for $|x| < 1$ and $\widehat{\eta_\circ}$ is supported in $\{\xi : |\xi| \le 2^{-5}\}$. For $k > 2$ let

$$f_k(x) = \eta_\circ(x) e^{i2^k x_1} \frac{\log k}{k^{1/p}}$$

and $f(x) = \sum_{k>2} 2^{-ks} f_k(x)$. Then $L_k f = 2^{-ks} f_k$ and thus

$$\begin{aligned} \Big(\sum_{k>2} 2^{ksr} |L_k f(x)|^r\Big)^{1/r} &= \Big(\sum_{k>2} |f_k(x)|^r\Big)^{1/r} \\ &\lesssim |\eta_\circ(x)| \big(\sum_{k>2} k^{-r/p} |\log k|^r\big)^{1/r} \lesssim C(p,r)|\eta_\circ(x)| \end{aligned}$$

with $C(p,r) < \infty$ for $r > p$. Hence $f \in \dot{F}^s_{p,r}$ for all $r > p$.

For $\lambda \ll 1$ we have

$$\begin{aligned} &\lambda \mu_0\{(x,k) : |P^s f(x,k)| > \lambda\}^{1/p} \\ &\ge \lambda \Big(\sum_{\substack{k>2 \\ k^{-1/p}\log k > \lambda}} \operatorname{meas}\{x : |x| < 1/4\}\Big)^{1/p} \\ &\ge c\lambda \Big(\sum_{2<k<c(\frac{\log \lambda^{-1}}{\lambda})^p} 1\Big)^{1/p} \ge c \log \lambda^{-1} \end{aligned}$$

so that f does not belong to $\dot{\mathcal{B}}^s_p(0,\infty)$. ∎

Lemma 5.4. *Let $s \in \mathbb{R}$, and $1 < p < \infty$. There exists $g \in \dot{\mathcal{B}}^s_p(0,1)$ which does not belong to $\bigcup_{r<p} \dot{F}^s_{p,r}$.*

Proof. As in the proof of Lemma 5.3 let $\eta_\circ$ be a Schwartz function such that $\eta_\circ(x) > 1$ for $|x| < 1$ and $\widehat{\eta_\circ}$ is supported in $\{\xi : |\xi| \le 2^{-5}\}$. For $k > 2$ let

$$g_k(x) = \frac{\eta_\circ(x) e^{i2^k x_1}}{k^{1/p}[\log k]^2}$$

and $g(x) = \sum_{k>2} 2^{-ks} g_k(x)$. Then $L_k g = 2^{-ks} g_k$ and thus

$$\begin{aligned} &\Big(\sum_{2<k\le 2N} 2^{ksr} |L_k g(x)|^r\Big)^{1/r} = \Big(\sum_{2<k\le 2N} |g_k(x)|^r\Big)^{1/r} \\ &\ge |\eta_\circ(x)| \big(\sum_{N\le k\le 2N} k^{-r/p} |\log k|^{-2r}\big)^{1/r} \ge C(p,r) N^{1-r/p} (\log N)^{-2} |\eta_\circ(x)| \end{aligned}$$

with $C(p,r) > 0$ and $1 - \frac{r}{p} > 0$ for $r < p$. Integrating its p-th power over $\{x : |x| \le 1/2\}$ and letting $N \to \infty$ we see that $f \notin \dot{F}^s_{p,r}$ for all $r < p$.

Now let,

$$\begin{aligned} E_{\ell,m} &= \{(x,k) : 2^{\ell-1} \le |x| < 2^\ell,\ 2^{m-1} \le k < 2^m\},\ \text{for } (\ell,m) \in \mathbb{N}^2, \\ E_{0,m} &= \{(x,k) : |x| < 1,\ 2^{m-1} \le k < 2^m\},\ \text{for } m \in \mathbb{N}. \end{aligned}$$

Then $\mu_0(E_{\ell,m}) \approx 2^{m+\ell d}$ and

$$|g_k(x)| \lesssim_N 2^{-m/p} m^{-2} 2^{-\ell N} \text{ if } (x,k) \in E_{\ell,m},$$

for any $(\ell,m) \in (\mathbb{N} \cup \{0\}) \times \mathbb{N}$. Therefore

$$|P^s g(x,k)| = |g_k(x)| \lesssim_N \sum_{\ell=0}^{\infty} \sum_{m=1}^{\infty} 2^{-\ell(N-d/p)} m^{-2} \frac{\mathbb{1}_{E_{\ell,m}}(x,k)}{\mu_0(E_{\ell,m})^{1/p}}.$$

Choosing $N > d/p$ we see that $P^s g \in L^{p,1}(\mu_0)$ and since $\widehat{g}(\xi) = 0$ for $|\xi| \le 1$ we obtain $g \in \dot{\mathcal{B}}^s_p(0,1)$. □

6. Proof of Theorem 1.9

We use a result in [10], namely for $\gamma \in \mathbb{R} \setminus [-1,0]$

$$[\mathcal{Q}_{1,1+\gamma} f]_{L^{1,\infty}(\nu_\gamma)} \lesssim \|f\|_{\dot{BV}(\mathbb{R}^d)}. \tag{6.1}$$

Since $|\mathcal{Q}_{1,(1+\gamma)/p} f|^p \le |\mathcal{Q}_{1,1+\gamma} f|\, (2\|f\|_{V^\infty})^{p-1}$, we have

$$\|f\|_{\dot{\mathcal{B}}^{1/p}_p(\gamma,\infty)} \simeq [\mathcal{Q}_{1,\frac{1+\gamma}{p}} f]_{L^{p,\infty}(\nu_\gamma)} \lesssim \|f\|_{V^\infty}^{1-\frac{1}{p}} [\mathcal{Q}_{1,1+\gamma} f]_{L^{1,\infty}(\nu_\gamma)}^{\frac{1}{p}},$$

which combined with (6.1) gives

$$\|f\|_{\dot{\mathcal{B}}^{1/p}_p(\gamma,\infty)} \lesssim \|f\|_{V^\infty}^{1-\frac{1}{p}} \|f\|_{\dot{BV}}^{\frac{1}{p}} \tag{6.2}$$

for every $\gamma \in \mathbb{R} \setminus [-1,0]$ and $1 < p < \infty$.

We can interpret inequality (6.2) as an imbedding result for the real interpolation space $[V^\infty, \dot{BV}]_{\theta,1}$ with $\theta = \frac{1}{p}$, and get

$$\|f\|_{\dot{\mathcal{B}}^{1/p}_p(\gamma,\infty)} \lesssim \|f\|_{[V^\infty,\dot{BV}]_{\frac{1}{p},1}}. \tag{6.3}$$

Indeed if, for $f \in V^\infty \cap \dot{BV}$, we set $J(f,t) = \max\{\|f\|_{V^\infty}, t\|f\|_{\dot{BV}}\}$ we have $\|f\|_{V^\infty}^{1-\frac{1}{p}} \|f\|_{\dot{BV}}^{\frac{1}{p}} \le t^{-\frac{1}{p}} J(f,t)$ for all t. Hence if $f \in V^\infty \cap \dot{BV}$ and $f = \sum_\nu f_\nu$ where $f_\nu \in V^\infty \cap \dot{BV}$ (with the sum converging in $V^\infty + \dot{BV}$), then we get from (6.2) $\|f\|_{\dot{\mathcal{B}}^{1/p}_p(\gamma,\infty)} \lesssim \sum_\nu \|f_\nu\|_{\dot{\mathcal{B}}^{1/p}_p(\gamma,\infty)} \le \sum_\nu 2^{-\nu/p} J(f_\nu, 2^\nu)$ and by taking the inf over all such decompositions and applying the equivalence of the $K_{\theta,1}$ and $J_{\theta,1}$ methods of interpolation ([4, Theorem 3.3.1]) we obtain (6.3). □

Remark. Concerning Remark 1.14, by the reiteration theorem ([4, Theorem 3.5.3]) we have $\big[[V^\infty, \dot{BV}]_{\frac{1}{p_0},1}, [V^\infty, \dot{BV}]_{\frac{1}{p_1},1}\big]_{\theta,\infty} = [V^\infty, \dot{BV}]_{\frac{1}{p},\infty}$, provided that $1 < p_0 < p < p_1 < \infty$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Hence $[V^\infty, \dot{BV}]_{\frac{1}{p},\infty}$ embeds only into $[\dot{\mathcal{B}}^{1/p_0}_{p_0}(\gamma,\infty), \dot{\mathcal{B}}^{1/p_1}_{p_1}(\gamma,\infty)]_{\theta,\infty}$ (a weaker conclusion than embedding into $\dot{\mathcal{B}}^{1/p}_p(\gamma,\infty)$).

7. Harmonic and caloric extensions

In what follows let ψ be a sufficiently well behaved integrable function with $\int \psi(x)\,\mathrm{d}x = 0$, specifically we will take ψ as one of $\psi_{(1)}$, $\psi_{(2,j)}$, $\psi_{(3)}$, $\psi_{(4,j)}$ where

$$\begin{aligned} &\widehat{\psi}_{(1)}(\xi) = |\xi| e^{-|\xi|}, \qquad \widehat{\psi}_{(2,j)}(\xi) = i\xi_j e^{-|\xi|}, \\ &\widehat{\psi}_{(3)}(\xi) = -|\xi|^2 e^{-|\xi|^2}, \qquad \widehat{\psi}_{(4,j)}(\xi) = i\xi_j e^{-|\xi|^2}, \end{aligned} \tag{7.1}$$

or we could also take $\psi = \psi_{(5)} = \frac{\partial}{\partial s}[s^{-d}\phi(s^{-1}\cdot)]|_{s=1}$ for any $\phi \in \mathcal{S}(\mathbb{R}^d)$. Let $\psi_t = t^{-d}\psi(t^{-1}\cdot)$. Classical results on characterizations of Besov spaces ([55], [53, Chapter V.5, Proposition 7'], [57, Chapter 1.8]) yield the inequality

$$\iint_{\mathbb{R}^2_+} t^{-sp} |\psi_t * f(x)|^p \,\mathrm{d}x \frac{\mathrm{d}t}{t} \lesssim \|f\|^p_{\dot{W}^{s,p}} \tag{7.2}$$

for $f \in \dot{W}^{s,p}$, $1 \le p < \infty$ and $0 < s < 1$.

With ψ, ψ_t as above, define, for $f \in \dot{\mathcal{B}}^s_p(\gamma, r)$ with $0 < s < 1$,

$$\mathcal{K}^b f(x,t) = t^{-b}\psi_t * f_\circ(x)$$

where $f_\circ$ is any representative of f modulo constants. Recall from (1.15) that $\mathrm{d}\lambda_\gamma(x,t) = t^{\gamma-1}\,\mathrm{d}t\,\mathrm{d}x$.

Proposition 7.1. *Let $0 < s < 1$, $1 < p, r < \infty$ and $\gamma \in \mathbb{R}$. The operator $\mathcal{K}^{s+\frac{\gamma}{p}}$ is bounded from $\dot{\mathcal{B}}^s_p(\gamma, r)$ to $L^{p,r}(\lambda_\gamma)$.*

Proof. We take (s_0, p_0), (s_1, p_1) and θ such that (1.17) and (1.18) holds, and $0 < s_0 < 1$, $0 < s_1 < 1$, $1 < p_0 < \infty$ and $1 < p_1 < \infty$. Recall $s + \frac{\gamma}{p} = s_i + \frac{\gamma}{p_i}$, and observe that for $f \in \dot{B}^{s_i}_{p_i,p_i}$,

$$\iint_{\mathbb{R}^2_+} \frac{|\mathcal{K}^{s+\frac{\gamma}{p}} f(x,t)|^{p_i}}{t^{1-\gamma}} \,\mathrm{d}x\,\mathrm{d}t = \iint_{\mathbb{R}^2_+} t^{-s_i p_i} |\psi_t * f_\circ(x)|^{p_i} \,\mathrm{d}x \frac{\mathrm{d}t}{t}.$$

where $f_\circ \in \dot{W}^{s_i,p_i}$ is a representative of f modulo constants. It follows from (7.2) that $\mathcal{K}^{s+\frac{\gamma}{p}}$ is bounded from $\dot{B}^{s_i}_{p_i,p_i}$ to $L^{p_i}(\lambda_\gamma)$. The conclusion then follows by interpolation, in view of Theorem 1.15 and of the classical characterization of Lorentz spaces as interpolation spaces. □

Corollary 7.2. *Let $1 < p < \infty$, $\gamma \in \mathbb{R} \setminus [-1, 0]$. Then*

$$\mathcal{K}^{\frac{\gamma+1}{p}} : [V^\infty, \dot{BV}]_{\frac{1}{p},1} \to L^{p,\infty}(\lambda_\gamma)$$

is bounded.

Proof. Combine Proposition 7.1 for $s = 1/p$ with Theorem 1.9. $\square$

7.1. *Harmonic extensions: Proof of Corollary 1.10.* From [54, Lemma 1.17] we recall that $\widehat{\mathcal{P}f}(\xi,t) = e^{-t|\xi|}\widehat{f}(\xi)$ and therefore we are led to use the function $\psi_{(1)}$ and $\psi_{(2,\nu)}$ for $\nu = 1,\dots,d$ in (7.1), for formulas for $t\frac{\partial}{\partial t}\mathcal{P}f$ and $t\frac{\partial}{\partial x_\nu}\mathcal{P}f$, respectively. We let $\mathcal{K}^b f(x,t) = t^{1-b}\nabla\mathcal{P}f(x,t)$ and apply Corollary 7.2 to obtain

$$\|\mathcal{K}^{\frac{\gamma+1}{p}} f\|_{L^{p,\infty}(\lambda_\gamma)} \lesssim \|f\|_{[V^\infty, \dot{BV}]_{\frac{1}{p},1}}$$

and the proof of the first inequality in Corollary 1.10 is complete. For the proof of the second inequality choose $\gamma = 1$ and $p = 2$, which is the unique choice of p,γ where $\mathcal{K}^{\frac{\gamma+1}{p}}$ becomes $\nabla\mathcal{P}$ and λ_γ becomes Lebesgue measure on $\mathbb{R}^{d+1}_+$. $\square$

7.2. *Caloric extensions: Proof of Corollary 1.12.* Note that $r\frac{\partial}{\partial r}[\widehat{Uf}(\xi,r^2)]$ equals $2|r\xi|^2 e^{-|r\xi|^2}\widehat{f}(\xi)$, and taking $\psi = \psi_{(3)}$ in the definition of $\mathcal{K}^b \equiv \mathcal{K}^b_{d+1}$, we get

$$2\mathcal{K}^b_{d+1} f(x,r) = r^{1-b}\frac{\partial}{\partial r}[Uf(x,r^2)] = 2t^{1-\frac{b}{2}}\frac{\partial}{\partial t}Uf(x,t)\Big|_{t=r^2} = 2\mathcal{H}^{b/2}_{d+1} f(x,r^2).$$

We apply Corollary 7.2 with $\gamma = 2\beta$ and observe that

$$\lambda_{2\beta}(\{(x,r) : |\mathcal{K}^{\frac{2\beta+1}{p}}_{d+1} f(x,r)| > \alpha\}) = \frac{1}{2}\lambda_\beta(\{(x,t) : \mathcal{H}^{\frac{2\beta+1}{2p}}_{d+1} f(x,t) > \alpha\}).$$

For $2\beta \notin [-1,0]$ the operator $\mathcal{K}^{\frac{2\beta+1}{p}}$ maps $[V^\infty, \dot{BV}]_{1/p,1}$ to $L^{p,\infty}(\lambda_{2\beta})$, and hence $\mathcal{H}^{\frac{2\beta+1}{2p}}_{d+1}$ maps $[V^\infty, \dot{BV}]_{1/p,1}$ to $L^{p,\infty}(\lambda_\beta)$.

For $j = 1,\dots,d$ we argue similar, taking $\psi = \psi_{(4,j)}$ in the definition of $\mathcal{K}^b \equiv \mathcal{K}^b_j$. We then have $\mathcal{K}^b_j f(x,r) = r^{1-b}\frac{\partial}{\partial x_j}Uf(x,r^2)$ and again apply Corollary 7.2 with $\gamma = 2\beta$. Now for $j = 1,\dots,d$,

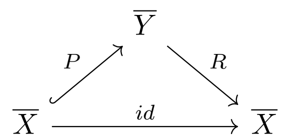
$$\mathcal{K}^b_j f(x,r) = t^{\frac{1-b}{2}}\frac{\partial}{\partial x_j}Uf(x,t)\Big|_{t=r^2} = \mathcal{H}^{b/2}_j f(x,r^2),$$

and again we see that $\mathcal{H}_j^{\frac{2\beta+1}{2p}}$ maps $[V^\infty, \dot{BV}]_{1/p,1}$ to $L^{p,\infty}(\lambda_\beta)$. This finishes the proof of part (i) of the corollary. For part (ii) we need $\mathrm{d}\lambda_\beta = \mathrm{d}x\,\mathrm{d}t$ so that we put $\beta = 1$. We then apply part (i), for the operator $\frac{\partial U}{\partial t}$ with $p = 3/2$ (so that $\frac{2\beta+1}{2p} = 1$), and for the operators $\frac{\partial U}{\partial x_j}$ with $p = 3$ (so that $\frac{2\beta+1}{2p} = \frac{1}{2}$). $\square$

8. Interpolation: Proof of Theorem 1.15

We use the standard retraction-coretraction argument (see [4, §6.4]). Recall that if $\overline{X} = (X_0, X_1)$ and $\overline{Y} = (Y_0, Y_1)$ are couples of compatible normed spaces then $P : \overline{X} \to \overline{Y}$ is a morphism of couples if P is a linear operator mapping $X_0 + X_1$ to $Y_0 + Y_1$, such that $P : X_\nu \to Y_\nu$ is a bounded linear operator for $\nu = 0$ and $\nu = 1$.

If $P : \overline{X} \to \overline{Y}$, $R : \overline{Y} \to \overline{X}$ are be morphisms of couples such that $R \circ P : \overline{X} \to \overline{X} = Id$, the identity operator on $\overline{X}$ then $\overline{X}$ is called a retract of $\overline{Y}$; R is a retraction and P is a co-retraction.



Lemma 8.1. [4] *Let $\overline{X} = (X_0, X_1)$, $\overline{Y} = (Y_0, Y_1)$ be a couples of compatible normed spaces such that $\overline{X}$ is a retract of $\overline{Y}$ with co-retraction $P : \overline{X} \to \overline{Y}$ and retraction R then*

$$[X_0, X_1]_{\theta,r} = \{f \in X_0 + X_1 : Pf \in [Y_0, Y_1]_{\theta,r}\}$$

and we have the equivalence of norms, $\|f\|_{[X_0,X_1]_{\theta,r}} \approx \|Pf\|_{[Y_0,Y_1]_{\theta,r}}$.

Lemma 8.2. *Suppose $1 \le p_0 < p_1 \le \infty$, and $\gamma, b \in \mathbb{R}$. Then there are bounded morphisms of couples*

$$P^b : (\dot{B}^{b-\frac{\gamma}{p_0}}_{p_0,p_0}, \dot{B}^{b-\frac{\gamma}{p_1}}_{p_1,p_1}) \to (L^{p_0}(\mu_\gamma), L^{p_1}(\mu_\gamma))$$
$$R_b : (L^{p_0}(\mu_\gamma), L^{p_1}(\mu_\gamma)) \to (\dot{B}^{b-\frac{\gamma}{p_0}}_{p_0,p_0}, \dot{B}^{b-\frac{\gamma}{p_1}}_{p_1,p_1})$$

Proof. The definitions of P^b, R_b will be independent of p_0, p_1 and thus one can reduce to checking the boundedness of

$$P^b : \dot{B}^{b-\frac{\gamma}{p}}_{p,p} \to L^p(\mu_\gamma) \tag{8.1}$$

$$R_b : L^p(\mu_\gamma) \to \dot{B}^{b-\frac{\gamma}{p}}_{p,p} \tag{8.2}$$

for $1 \le p \le \infty$.

Recall $L_k = \varphi(2^{-k}D)$, $\widetilde{L}_k = \widetilde{\varphi}(2^{-k}D)$ with φ as in (1.2) and $\widetilde{\varphi}$ as in (2.2), satisfying $\widetilde{L}_k = \widetilde{L}_k L_k = L_k \widetilde{L}_k$. Let P^b be as in Definition (1.1). For $F \in L^{p,r}(\mu_\gamma)$ define $F_k(x) := F(x,k)$ and $R_b F(x) = \sum_{k=0}^\infty 2^{-kb} \widetilde{L}_k F_k(x)$.

Note that $P^b : \dot{B}^{b-\gamma/p}_{p,p} \to L^p(\mu_\gamma)$ is an isometric embedding, for $1 \le p \le \infty$; moreover $R_b P^b$ is the identity on $\dot{\mathcal{B}}^{b-\frac{\gamma}{p}}_{p,p}$. It remain to show that R_b maps $L^p(\mu_\gamma)$ boundedly to $\dot{B}^{b-\frac{\gamma}{p}}_{p,p}$. Indeed we have $L_k \widetilde{L}_{k+j} = 0$ for $|j| > 2$ and thus

$$2^{kb} L_k R_b F(x) = 2^{kb} \sum_{j=-1}^{1} 2^{-(k+j)b} L_k \widetilde{L}_{k+j} F_{k+j}(x)$$

and, defining $T_jF(x,k) = L_k\widetilde{L}_{k+j}F_{k+j}(x)$ for $j=-1,0,1$, we see that

$$\|R_bF\|_{\dot{B}^{b-\frac{\gamma}{p}}_{p,p}} \lesssim \sum_{j=-1}^{1} \|T_jF\|_{L^p(\mu_\gamma)}.$$

and the boundedness of R_b follows from

$$\|T_jF\|_{L^q(\mu_\gamma)} = \Big(\sum_{k\in\mathbb{Z}} \|L_k\widetilde{L}_{k+j}F_{k+j}\|^q_{L^q} 2^{-k\gamma}\Big)^{1/q}$$

$$\lesssim \Big(\sum_{k\in\mathbb{Z}} \|F_{k+j}\|^q_{L^q} 2^{-(k+j)\gamma}\Big)^{1/q} \lesssim \|F\|_{L^q(\mu_\gamma)}, \quad j=-1,0,1. \qquad \square$$

Proof of Theorem 1.15, conclusion. Our choice of γ allows us to define

$$b := s_0 + \frac{\gamma}{p_0} = s_1 + \frac{\gamma}{p_1}.$$

We apply Lemma 8.1 with $X_\nu = \dot{B}^{b-\gamma/p_\nu}_{p_\nu,p_\nu}$, $Y_\nu = L^{p_\nu,r}(\mu_\gamma)$, $\nu = 0,1$ and $P = P^b$, $R = R_b$, as in Lemma 8.2. We then use the standard interpolation formula $[L^{p_0}, L^{p_1}]_{\theta,r} = L^{p,r}$ for $(1-\theta)/p_0 + \theta/p_1 = 1/p$, see [4], and the definition $\dot{B}^s_p(\gamma,r)$ via the operator P_b. $\square$

Proof of Corollary 1.16. (1.20a) follows from Theorem 1.15 by the reiteration theorem for the real method. (1.20b) for general q_0, q_1 follows since for $1 \le r_i \le \infty$ and $\gamma \ne 0$ given by (1.17) we have by part (ii) of Theorem 1.5

$$\dot{\mathcal{B}}^{s_i}_{p_i}(\gamma,1) \hookrightarrow \dot{F}^{s_i}_{p_i,1} \hookrightarrow \dot{F}^{s_i}_{p_i,r_i} \hookrightarrow \dot{F}^{s_i}_{p_i,\infty} \hookrightarrow \dot{\mathcal{B}}^{s_i}_{p_i}(\gamma,\infty). \qquad \square$$

Remark. Asekritova and Kruglyak [2] obtained real interpolation results for triples of the Besov spaces $(B^{s_0}_{p_0,p_0}, B^{s_1}_{p_1,p_1}, B^{s_2}_{p_2,p_2})_{\vec{\theta},r}$, with $\sum_{i=0}^2 \theta_i = 1$ (or more generally $(\ell+1)$-tuples of such spaces with $\ell \ge 2$). Under the crucial additional assumption that the three points points $(\frac{1}{p_i}, s_i)$, $i=0,1,2$ *do not lie on a line* the interpolation spaces is identified with the Besov-Lorentz space $B^s_r(L^{p,r})$ where $(\frac{1}{p}, s) = \sum_{i=0}^2 \theta_i(\frac{1}{p_i}, s_i)$. The result for triples does not seem to have an implication on the interpolation of *couples* of Besov spaces (see also [1]).

Remark. One could also use more directly results on real interpolation of weighted spaces, namely the identification of $[L^{p_0}(w_0), L^{p_1}(w_1)]_{\theta,q}$ in work by Freitag [26] and by Lizorkin [42].

9. Interpolating Besov spaces through differences

In this section we provide a direct proof of (1.19) in the case $M = 1$, which is directly based on the characterization using first differences. Suppose $0 < s < 1$, $1 < p < \infty$, $1 \le r \le \infty$, for $p_0 < p < p_1$ so that $s_i := s + \gamma(\frac{1}{p} - \frac{1}{p_i})$

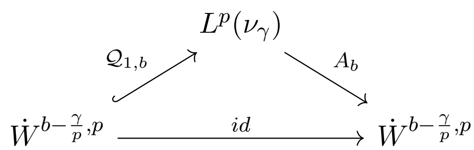
satisfy $0 < s_i < 1$ for $i = 0, 1$, and $\theta \in (0, 1)$ such that $\frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{1}{p}$. We will prove that for all functions $f : \mathbb{R}^d \to \mathbb{C}$ in $\dot{W}^{s_0,p_0} + \dot{W}^{s_1,p_1}$,

$$\|\mathcal{Q}_{1,s+\frac{\gamma}{p}} f\|_{L^{p,r}(\nu_\gamma)} \approx \|f\|_{[\dot{W}^{s_0,p_0},\dot{W}^{s_1,p_1}]_{\theta,r}} \tag{9.1}$$

The alternative proof goes by a retraction argument based on differences. One uses Lemma 8.1 once the following proposition is established.

Proposition 9.1. *Let $b \in \mathbb{R}$ with $0 < b - \gamma/p < 1$. There is a bounded operator $A_b : L^p(\nu_\gamma) \to \dot{W}^{b-\frac{\gamma}{p},p}$ such that $A_b \mathcal{Q}_{1,b}$ is the identity on $\dot{W}^{b-\frac{\gamma}{p},p}$.*

That is, we have the following retract diagram



The proof of the proposition is inspired by the metric characterization of sums $\dot{W}^{s_0,p_0} + \dot{W}^{s_1,p_1}$ due to Rodiac and the fourth named author [49].

Fix $\gamma \in \mathbb{R}$ and $1 \le p < \infty$. Fix $\phi \in C_c^\infty(\mathbb{R}^d)$ with $\int \phi = 1$ and support inside $B_{1/2}(0)$ and let

$$\psi(y) := -\phi(y)d - \langle y, \nabla\phi(y)\rangle.$$

Integration by parts shows that $\int \psi(y)\,\mathrm{d}y = 0$. For $t > 0$ define $\phi_t(y) := \frac{1}{t^d}\phi(\frac{y}{t})$ and $\psi_t(y) := \frac{1}{t^d}\psi(\frac{y}{t})$; one verifies that for all $t > 0$

$$\psi_t(y) = t\frac{d}{dt}\phi_t(y). \tag{9.2}$$

In what follows we set

$$\vartheta_t(z, y) := \phi_t(z)\psi_t(y). \tag{9.3}$$

Suppose that $F \in L^p(\nu_\gamma)$ is compactly supported in $\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$. We then define

$$A_{b,\varepsilon}F(x) = \int_\varepsilon^{1/\varepsilon} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F(y, h)|h|^b \vartheta_t(x - y - h, x - y)\,\mathrm{d}h\,\mathrm{d}y \frac{\mathrm{d}t}{t}. \tag{9.4}$$

Since ϑ is supported in $B_{1/2}(0) \times B_{1/2}(0)$ it is clear that for $\varepsilon > 0$ and for F with the above support property the integral in (9.4) converges absolutely, and defines $A_{b,\varepsilon}F$ as a smooth function. Under the additional restriction $0 < b - \frac{\gamma}{p} < 1$ the following result extends $A_{b,\varepsilon}$ to all of $L^p(\nu_\gamma)$ and establishes the existence of the limit $A_b = \lim_{\varepsilon\to 0} A_{b,\varepsilon}$ in the strong operator topology.

Lemma 9.2. *Let $b \in \mathbb{R}$ with $0 < b - \frac{\gamma}{p} < 1$. Then the following holds.*

(i) For $\varepsilon > 0$, the maps $A_{b,\varepsilon}$ extend to bounded operators

$$A_{b,\varepsilon} : L^p(\nu_\gamma) \to \dot{W}^{b-\frac{\gamma}{p},p},$$

with operator norm uniformly bounded in ε.

(ii) The operators $A_{b,\varepsilon}$ converge to a bounded operator

$$A_b : L^p(\nu_\gamma) \to \dot{W}^{b-\frac{\gamma}{p},p},$$

in the sense that $\lim_{\varepsilon\to 0} \|A_{b,\varepsilon}F - A_bF\|_{\dot{W}^{b-\frac{\gamma}{p},p}} = 0$ *for all* $F \in L^p(\nu_\gamma)$.

Proof. Let $F \in L^p(\nu_\gamma)$ and assume in addition that assume that $F \in L^p(\nu_\gamma)$ is compactly supported in $\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$. Set

$$\Delta_{h,h}\vartheta_t(u, v) = \vartheta_t(u + h, v + h) - \vartheta_t(u, v).$$

Then

$$\Delta_h A_{b,\varepsilon}F(x) = \int_\varepsilon^{1/\varepsilon} \iint_{\mathbb{R}^{2d}} F(y, z)|z|^b \Delta_{h,h}\vartheta_t(x - y - z, x - y)\,\mathrm{d}z\,\mathrm{d}y\frac{\mathrm{d}t}{t}$$

and we estimate

$$|\Delta_h A_{b,\varepsilon}F(x)| \le I(x, h) + II(x, h) + III(x, h)$$

where

$$I(x, h) := \int_{|h|}^{\infty} \iint_{\mathbb{R}^{2d}} |F(y, z)||z|^b|\Delta_{h,h}\vartheta_t(x - y - z, x - y)|\,\mathrm{d}z\,\mathrm{d}y\frac{\mathrm{d}t}{t}$$

$$II(x, h) := \int_0^{|h|} \iint_{\mathbb{R}^{2d}} |F(y, z)||z|^b|\vartheta_t(x + h - y - z, x + h - y)|\,\mathrm{d}z\,\mathrm{d}y\frac{\mathrm{d}t}{t}$$

$$III(x, h) := \int_0^{|h|} \iint_{\mathbb{R}^{2d}} |F(y, z)||z|^b|\vartheta_t(x - y - z, x - y)|\,\mathrm{d}z\,\mathrm{d}y\frac{\mathrm{d}t}{t}.$$

Setting

$$J_p(t) = \Big(\frac{1}{t^d}\int_{\mathbb{R}^d}\int_{|z|\le t} |F(y, z)|^p|z|^{bp}\,\mathrm{d}z\,\mathrm{d}y\Big)^{1/p} \tag{9.5}$$

we estimate, using Minkowski's inequality,

$$\begin{aligned}
\|I(\cdot,h)\|_p &\le \int_{|h|}^{\infty} \frac{|h|}{t^2}\Big\|\frac{1}{t^{2d}}\int_{|x+h-y|\le 2t}\int_{|z|\le t} |F(y, z)||z|^b\,\mathrm{d}z\,\mathrm{d}y\Big\|_{L^p(\,\mathrm{d}x)}\,\mathrm{d}t \\
&\le \int_{|h|}^{\infty} \frac{|h|}{t^2}\Big\|\Big(\frac{1}{t^{2d}}\int_{|x+h-y|\le 2t}\int_{|z|\le t} |F(y, z)|^p|z|^{bp}\,\mathrm{d}z\,\mathrm{d}y\Big)^{1/p}\Big\|_{L^p(\,\mathrm{d}x)}\,\mathrm{d}t \\
&\lesssim \int_{|h|}^{\infty} \frac{|h|}{t^2}J_p(t)\,\mathrm{d}t.
\end{aligned}$$

Similarly we get

$$\|II(\cdot, h)\|_p + \|III(\cdot, h)\|_p \lesssim \int_0^{|h|} \frac{1}{t}J_p(t)\,\mathrm{d}t.$$

We then have, uniformly in $\varepsilon \in (0, 1)$,

$$\begin{aligned}
\|A_{b,\varepsilon}F\|_{\dot{W}^{b-\frac{\gamma}{p},p}} &= \|\mathcal{Q}_{1,b}A_{b,\varepsilon}F\|_{L^p(\nu_\gamma)} \\
&\le \Big(\int \Big[|h|^{-b}\int_{|h|}^{\infty} \frac{|h|}{t^2}J_p(t)\,\mathrm{d}t + |h|^{-b}\int_0^{|h|} \frac{1}{t}J_p(t)\,\mathrm{d}t\Big]^p \frac{\mathrm{d}h}{|h|^{d-\gamma}}\Big)^{1/p}
\end{aligned}$$

which we estimate (using Hardy's inequalities) by

$$\Big(\int_0^\infty \Big(\int_r^\infty \frac{J_p(t)}{t^2}\,\mathrm{d}t\Big)^p \frac{\mathrm{d}r}{r^{1+(b-\frac{\gamma}{p}-1)p}}\Big)^{\frac1p} + \Big(\int_0^\infty \Big(\int_0^r \frac{J_p(t)}{t}\,\mathrm{d}t\Big)^p \frac{\mathrm{d}r}{r^{1+(b-\frac{\gamma}{p})p}}\Big)^{\frac1p}$$

$$\lesssim \Big(\int_0^\infty \Big(\frac{J_p(t)}{t^2}\Big)^p t^p \frac{\mathrm{d}t}{t^{1+(b-\frac{\gamma}{p}-1)p}}\Big)^{\frac1p} + \Big(\int_0^\infty \Big(\frac{J_p(t)}{t}\Big)^p t^p \frac{\mathrm{d}t}{t^{1+(b-\frac{\gamma}{p})p}}\Big)^{\frac1p}$$

$$\simeq \Big(\int_0^\infty J_p(t)^p \frac{\mathrm{d}t}{t^{1+(b-\frac{\gamma}{p})p}}\Big)^{\frac1p}$$

$$= \Bigg(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |F(y,z)|^p |z|^{bp} \int_{|z|}^\infty \frac{\mathrm{d}t}{t^{1+(b-\frac{\gamma}{p})p+d}}\,\mathrm{d}z\,\mathrm{d}y\Bigg)^{\frac1p} \simeq \|F\|_{L^p(\nu_\gamma)}.$$

This establishes part (i) of the lemma, first for F compactly supported in $\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$ and then, by a density argument, for general $F \in L^p(\nu_\gamma)$. The above argument also shows that $\|A_{b,\varepsilon_1}F - A_{b,\varepsilon_2}F\|_{\dot W^{b-\frac{\gamma}{p},p}} \to 0$ as $\varepsilon_1, \varepsilon_2 \to 0$ and thus $A_{b,\varepsilon}F$ converges in $\dot W^{b-\frac{\gamma}{p},p}$ to a limit $A_{b,0}F$; moreover A_b defines a bounded operator $L^p(\nu_\gamma) \to \dot W^{b-\frac{\gamma}{p},p}$. □

The proof of Proposition 9.1 is now completed by the following lemma.

Lemma 9.3. *Let $b \in \mathbb{R}$ with $0 < b - \frac{\gamma}{p} < 1$. Then $A_b\mathcal{Q}_{1,b}f = f$, for all $f \in \dot W^{b-\frac{\gamma}{p},p}$.*

Proof. Note that $\mathcal{Q}_{1,b}\colon \dot W^{b-\frac{\gamma}{p},p} \to L^p(\nu_\gamma)$ is an isometry. As $A_b : L^p(\nu_\gamma) \to \dot W^{b-\frac{\gamma}{p},p}$ is bounded, by Lemma 9.2, and since $C_c^\infty(\mathbb{R}^d)$ is dense in $\dot W^{b-\gamma/p,p}$, it suffices to prove $A_b\mathcal{Q}_{1,b}f = f$, for all $f \in C_c^\infty(\mathbb{R}^d)$.

By (9.2) and (9.3) we get for each $x \in \mathbb{R}^d$

$$\begin{aligned}
&A_{b,\varepsilon}\mathcal{Q}_{1,b}f(x) = A_{0,\varepsilon}\mathcal{Q}_{1,0}f(x)\\
&= \int_\varepsilon^{1/\varepsilon}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f(y+h)-f(y))\phi_t(x-y-h)\frac{d}{dt}[\phi_t(x-y)]\,\mathrm{d}h\,\mathrm{d}y\,\mathrm{d}t\\
&= \int_\varepsilon^{1/\varepsilon}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f(z)-f(y))\phi_t(x-z)\frac{d}{dt}[\phi_t(x-y)]\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}t\\
&= 0 - \int_\varepsilon^{1/\varepsilon}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y)\phi_t(x-z)\frac{d}{dt}[\phi_t(x-y)]\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}t
\end{aligned}$$

where we used $\int_{\mathbb{R}^d}\frac{d}{dt}[\phi_t(x-y)]\,\mathrm{d}y = \frac{d}{dt}\int_{\mathbb{R}^d}\phi_t(x-y)\,\mathrm{d}y = 0$ to integrate the term involving $f(z)$. We may now integrate in z and t in the last display, using that $\int \phi_t = 1$ to obtain for $f \in C_c^\infty(\mathbb{R}^d)$

$$A_{b,\varepsilon}\mathcal{Q}_{1,b}f(x) = \int_{\mathbb{R}^d} f(y)\left(\phi_\varepsilon(x-y) - \phi_{1/\varepsilon}(x-y)\right)\,\mathrm{d}y\,.$$

Letting $\varepsilon \to 0$ we obtain $A_b\mathcal{Q}_{1,b}f = f$ for $f \in C_c^\infty(\mathbb{R}^d)$. □

References

[1] Irina Asekritova, Nathan Kruglyak. Real interpolation of vector-valued spaces in non-diagonal case. Proc. Amer. Math. Soc. 133 (2005), no. 6, 1665–1675.

[2] ———. Interpolation of Besov spaces in the nondiagonal case (Russian). Algebra i Analiz 18 (2006), no. 4, 1–9; English translation in St. Petersburg Math. J. 18 (2007), no. 4, 511–516.

[3] Florent Autin, Gerda Claeskens, Jean-Marc Freyermuth. Hyperbolic wavelet thresholding methods and the curse of dimensionality through the maxiset approach. Appl. Comput. Harmon. Anal. 36 (2014), 239–255.

[4] Jöran Bergh, Jörgen Löfström. *Interpolation spaces. An introduction.* Grundlehren der Mathematischen Wissenschaften, No. 223. Springer-Verlag, Berlin-New York, 1976.

[5] Jöran Bergh, Jaak Peetre, On the spaces V^p $(0 < p \leq \infty)$. Boll. Un. Mat. Ital. (4) 10 (1974), 632–648.

[6] Blanca F. Besoy, Dorothee D. Haroske, Hans Triebel, Traces of some weighted function spaces and related non-standard real interpolation of Besov spaces, preprint arXiv:2009.03656.

[7] Gérard Bourdaud. Realizations of homogeneous Besov and Lizorkin-Triebel spaces. Math. Nachr. 286 (2013), no. 5-6, 476–491.

[8] Haïm Brezis, Jean Van Schaftingen, Po-Lam Yung. A surprising formula for Sobolev norms. Proc. Natl. Acad. Sci. 118 (2021), no. 8, e2025254118.

[9] ———. Going to Lorentz when fractional Sobolev, Gagliardo and Nirenberg estimates fail. arXiv:2104.09867. Calc. Var. Partial Differential Equations 60 (2021), no. 4, Paper No. 129, 12 pp.

[10] Haïm Brezis, Andreas Seeger, Jean Van Schaftingen, Po-Lam Yung. Families of functionals representing Sobolev norms. arXiv:2109.02930, preprint.

[11] Michael Christ. The extension problem for certain function spaces involving fractional orders of differentiability. Ark. Mat. 22 (1984), no. 1, 63–81.

[12] Zbigniew Ciesielski. Some properties of convex functions of higher orders. Ann. Polon. Math. 7 (1959), 1–7.

[13] Fernando Cobos, Natan Kruglyak. Exact minimizer for the couple (L^∞, BV) and the one-dimensional analogue of the Rudin-Osher-Fatemi model. J. Approx. Theory 163 (2011), no. 4, 481–490.

[14] Albert Cohen, Wolfgang Dahmen, Ronald DeVore. Adaptive wavelet methods for elliptic operator equations: convergence rates. Math. Comp. 70 (2001), no. 233, 27–75.

[15] Albert Cohen, Wolfgang Dahmen, Ingrid Daubechies, Ronald DeVore. Harmonic analysis of the space BV. Rev. Mat. Iberoamericana 19 (2003), no. 1, 235–263.

[16] Ingrid Daubechies, *Ten lectures on wavelets.* CBMS-NSF Regional Conference Series in Applied Mathematics, 61. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

[17] Ronald A. DeVore, Nonlinear approximation, in Acta Numerica, Vol. 7, Cambridge University Press, pp. 51–150.

[18] Ronald A. DeVore, Björn Jawerth, Vasil Popov. Compression of wavelet decompositions, Amer. J. Math. 114 (1992), 737–785.

[19] Ronald A. DeVore, Robert C. Sharpley. *Maximal functions measuring smoothness.* Mem. Amer. Math. Soc. 47 (1984), no. 293, viii+115 pp.

[20] Ronald A. DeVore, Guergana Petrova, Vladimir Temlyakov. Best basis selection for approximation in L_p. Found. Comput. Math. **3** (2003), 161–185.

[21] Ronald A. DeVore, Vasil A. Popov. Interpolation of Besov spaces. Trans. Amer. Math. Soc. 305 (1988), no. 1, 397–414.

[22] Dragomir Ž. Djoković. A representation theorem for $(X_1 - 1)\dots(X_n - 1)$ and its applications. Ann. Polon. Math. 22 (1969), 189–198.

[23] Óscar Domínguez, Mario Milman. New Brezis-Van Schaftingen-Yung Sobolev type inequalities connected with maximal inequalities and one parameter families of operators. Preprint arXiv 2010.15873.
[24] Michael Frazier, Björn Jawerth. Decomposition of Besov spaces. Indiana Univ. Math. J. 34 (1985), 777–799.
[25] ———, A discrete transform and decompositions of distribution spaces. J. Funct. Anal. 93 (1990), 34–170.
[26] Dietrich Freitag. Real interpolation of weighted L^p-spaces. Math. Nachr. 86 (1978), 15–18.
[27] Roman Ger. On almost polynomial functions, Colloq. Math. 24 (1971/72), 95–101.
[28] Luigi Greco, Roberta Schiattarella. An embedding theorem for BV-functions. Commun. Contemp. Math. 22 (2020), Article ID 1950032, 13 p.
[29] Rémi Gribonval, Morten Nielsen. Some remarks on non-linear approximation with Schauder bases. East J. Approx. 7 (2001), 1–19.
[30] Qingsong Gu, Po-Lam Yung. A new formula for the L^p norm. J. Funct. Anal. 281 (2021), no. 4, 109075.
[31] Markus Hansen. Nonlinear approximation rates and Besov regularity for elliptic PDEs on polyhedral domains. Found. Comput. Math. 15 (2015), no. 2, 561–589.
[32] Markus Hansen, Winfried Sickel. Best m-term approximation and Triebel–Lizorkin spaces. J. Approx. Theory 163 (2011), 923–954.
[33] Richard A. Hunt. On $L(p, q)$ spaces. Enseign. Math. (2) 12 (1966), 249–276.
[34] Tadeusz Iwaniec, Gaven Martin, Carlo Sbordone. L^p-integrability & weak type L^2-estimates for the gradient of harmonic mappings of $\mathbb{D}$. Discrete Contin. Dyn. Syst. Ser. B 11 (2009), no. 1, 145–152.
[35] Gerard Kerkyacharian, Dominique Picard. Entropy, universal coding, approximation, and bases properties. Constr. Approx. 20 (2004), 1–37.
[36] V.L. Krepkogorskiĭ. Interpolation in Lizorkin-Triebel and Besov spaces. (Russian). Mat. Sb. 185 (1994), no. 7, 63–76; translation in Russian Acad. Sci. Sb. Math. 82 (1995), no. 2, 315–326.
[37] ———. Interpolation and embedding theorems for quasinormed Besov spaces. (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. 1999, no. 7, 23–29; translation in Russian Math. (Iz. VUZ) 43 (1999), no. 7, 21–26
[38] Marek Kuczma. On measurable functions with vanishing differences. Ann. Math. Sil. No. 6 (1992), 42–60.
[39] ———. *An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality.* Second edition. Edited and with a preface by Attila Gilányi. xiv+595 pp. Birkhäuser Verlag, Basel, 2009. xiv+595 pp.
[40] G. Kyriazis. Decomposition systems for function spaces. Studia Math. 157 (2003), no. 2, 133–169.
[41] Sanghyuk Lee, Keith M. Rogers, Andreas Seeger. Square functions and maximal operators associated with radial Fourier multipliers. *Advances in analysis: the legacy of Elias M. Stein.* 273–302, Princeton Math. Ser., 50, Princeton Univ. Press, Princeton, NJ, 2014.
[42] P. I. Lizorkin. Interpolation of L^p-spaces with a weight. Trudy Mat. Inst. Steklov. 140 (1976), 201–211 (in Russian); English transl.: Proc. Steklov Inst. Math. 140 (1979), 221–232.
[43] Mariusz Mirek, Elias M. Stein, Pavel Zorin-Kranich. Jump inequalities via real interpolation. Math. Ann. 376 (2020), no. 1-2, 797–819.
[44] Madani Moussai. Characterizations of realized homogeneous Besov and Triebel-Lizorkin spaces via differences. Appl. Math. J. Chinese Univ. 2018, 33(2): 188–208.
[45] S.M. Nikol'skiĭ. *Approximation of functions of several variables and imbedding theorems.* Translated from the Russian by John M. Danskin, Jr. Die Grundlehren der

mathematischen Wissenschaften, Band 205. Springer-Verlag, New York-Heidelberg. 1975.
[46] Hoai-Minh Nguyen. Some new characterizations of Sobolev spaces. J. Funct. Anal., 237 (2006), no. 2, 689–720.
[47] Pencho P. Petrushev, Vasil A. Popov. Rational Approximation of Real Functions, in: Encyclopedia of Mathematics and its Applications, vol. 28, Cambridge University Press, Cambridge, 1987.
[48] Arkady Poliakovsky. Some remarks on a formula for Sobolev norms due to Brezis, Van Schaftingen and Yung. Preprint arXiv:2102.00557.
[49] Rémy Rodiac, Jean Van Schaftingen. Metric characterization of the sum of fractional Sobolev spaces. Studia Math., 258, (2021), 27–51.
[50] Vincent Rivoirard. Nonlinear estimation over weak Besov spaces and minimax Bayes method. Bernoulli 12 (2006), 609–632.
[51] Andreas Seeger. A note on Triebel-Lizorkin spaces. *Approximation and function spaces* (Warsaw, 1986), 391–400, Banach Center Publ., 22, PWN, Warsaw, 1989.
[52] Andreas Seeger, Walter Trebels. Embeddings for spaces of Lorentz-Sobolev type. Math. Ann. 373 (2019), no. 3-4, 1017–1056.
[53] Elias M. Stein. *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970.
[54] Elias M. Stein, Guido Weiss. *Introduction to Fourier analysis on Euclidean spaces.* Princeton Mathematical Series, No. 32. Princeton University Press, Princeton, N.J., 1971. x+297 pp.
[55] Mitchell Taibleson. On the theory of Lipschitz spaces of distributions on Euclidean n-space. I. Principal properties. J. Math. Mech. 13 (1964) 407–479.
[56] Hans Triebel. *Theory of function spaces.* Monographs in Mathematics, 78. Birkhäuser Verlag, Basel, 1983. 284 pp.
[57] ______. *Theory of function spaces. II.* Monographs in Mathematics, 84. Birkhüuser Verlag, Basel, 1992. viii+370 pp.

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