THINKING GEOMETRY: A MATTER OF PHILOSOPHY. THE CASE OF HELMHOLTZ AND POINCARÉ[1]

Abstract

The controversy between Euclidean and non-Euclidean geometry arose new philosophical and scientific insights which were relevant to the later development of natural science. Here we want to consider Poincaré and Helmholtz's positions as two of the most important and original ones who contributed to the subsequent development of the epistemology of natural sciences. Based in these conceptions, we will show that the role of philosophy is still important for some aspects of science[2].

Keywords: Geometry, Helmholtz, Poincaré, epistemology, natural science.

It is well known that the development of non-Euclidean geometries prompted a philosophical debate about the real or unreal status of these geometries. This debate involved thinkers from all areas, not only philosophers but also scientists, especially mathematicians and physicists. This controversy gave place to the famous proposals for empirical tests of geometry headed by Gauss, Lobachevsky and Riemann among others. But beyond the failed experimentation attempts, it is important to remember that the discussion was unavoidable in philosophy because these geometries questioned the synthetic a priori status that geometrical propositions have hold since Kant. This debate originated several philosophical conceptions. On one side there were positions who tried to defend and justify the Kantian system by keeping some a priori elements. On the other side there were those ones who wanted to undermine Kant's philosophical system. Within this controversy, there are two conceptions that deserve a special attention because of the new perspective that they present, that is, Poincaré's conventionalism and Helmholtz's neokantian empiricism[3].

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[1] I want to thank FCT (Fundação para a Ciência e a Tecnologia) for a PhD grant (SFRH/BD/4478/2008) and Project "Poincaré, philosopher of Science" (PTDC/FIL/64748/2006), CFCUL and Faculty of Philosophy from Complutense University (Madrid) for academic support.
[2] An early version of this paper was presented in the International Conference "La Filosofía y su otro", 8th-12th, November, 2010, Granada, Spain.
[3] Neither Helmholtz nor Poincaré used these denominations to describe their philosophical conceptions. However, Poincaré's philosophy is widely known as "conventionalism", and I decided to use it by the same reasons argued by Giedymin (1992). Helmholtz is a difficult thinker to classify, but I think that these two words ("neokantian empiricism") describe quite well his position since he is strongly influenced by Kantian thought but the role of experiment is essential in his conception as we will see.

The significance of these views does not lie only in his contribution to the controversy about geometry, but they also provide two new ways for thinking science which are epistemologically close, but they are not the same. What I want to show is how, starting from a reflection about the same scientific discipline and about a concrete problem, Poincaré and Helmholtz were led to two philosophical positions that nowadays are still fruitful in the epistemological discussions about natural science.

First and second section of the paper will be a short summary of both authors towards geometry; in the third we will compare the positions remarking the points of coincidence and divergence; and finally, in section four, we will present how both views go beyond geometry and extend their reasoning to the whole natural science, but especially to physics.

I. Helmholtz's neokantian empiricism towards geometry.

The two main papers in which Helmholtz ideas about geometry have been exposed, are entitled "On the facts underlying geometry"[4] and "On the origin and significance of geometrical axioms"[5], from 1868 and 1870, respectively. In those papers it is clear that geometry, at first, is not an empirical science because its purpose is not to compile empirical facts, rather it proceeds always in a deductive way. So, geometrical axioms are "propositions that geometry cannot test, but can have the certain recognition of its correctness by means of which its sense is understood"[6]. In this way geometry is the mathematical theory of space, but then, the axioms are not representing any connection with real stuff and they could be seen as given a priori in the strictest Kantian sense.

However, there is a non mathematical way of dealing with geometry. Helmholtz calls it as the "scientific" way. In this sense, the object of geometry is physical space, which for Helmholtz is an actual space, that is, it is not a construction of the mind, but something real. Then geometry becomes physical geometry with the features of a natural science. What makes Helmholtz worthy of the adjective "empiricist" is the statement that the properties of that space can be find out by empirical observation. This leads him to assert that "axioms of geometry are certainly not propositions belonging to the pure theory of space", rather they are

[4] Helmholtz (1868) "Über die Thatsächliche Grundlagen der Geometrie", *Verhandlungen des naturhistorish-medicinischen Vereins zu Heidelberg*, 4, pp. 197-202. Here we will use the English translation by M. F. Lowe, in the Paul Hertz/Moritz Schlick centenary edition of 1921.

[5] Helmholtz (1870) "Über der Ursprung und die Bedeutung der Geometrische Axiome", Lecture given in Heidelberg, 1870. We will also use the English translation by M. F. Lowe, in the Paul Hertz/Moritz Schlick centenary edition of 1921.

[6] Helmholtz (1870), p. 199.

linked with mechanical considerations because they are assertions referred to physical bodies. So, the axioms of geometry are not about special relations when applied to actual world, rather about the mechanical behavior of solid bodies in motion. As a result, the system of propositions which constitutes physical geometry can be verified or refuted by experience.

The most controversial point on Helmholtz's conception is the statement that space can be transcendental without axioms being so. This is justified by Helmholtz's own idea of the "a priori", which is a synonym of transcendental for him and this means that this a priori is not the same as Kant's[7]. To Helmholtz, the a priori has the sense that the spatiality of sensations is something completely subjective[8]. It is a kind of psychological adaptation to the regularities of the external world. This means that spatial intuition is a subjective form of intuition and space is the necessary form of external intuition because we understand as external world precisely what we perceive as spatially determined. And it is a previous form because the perception of space is linked with the possibility of our motor impulses, and the mental and bodily ability of producing those impulses has to be prior to experience. Furthermore, to impose an a priori status on geometrical axioms would imply to suppose "intuitive facts as needs of thought"[9]. In this way, Helmholtz avoids the metaphysical presupposition which asserts that the geometry of space could be a priori determined.

As Helmholtz states that geometry of space depends on physical measurement[10], he tried to reconstruct the shape of space on a muscular physiological basis[11], using then the concept of movement. As a result, muscular activity has an essential character as the ground of motor impulses which generate the displacements of solid bodies. According to Helmholtz, free mobility of bodies seems to be a fundamental condition of this kind of geometry. To him, the parts of our bodies that we freely move will serve as measurement instruments which behavior reveals regularities in our environment[12]. So, we recognize firstly the movements of our body and then the ones of the rest of rigid bodies as spatial changes, and from these modifications of our perceptual space we construct the objective space which is characterized

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[7] In Kant's *Critic of Pure Reason*, B25 and B80, we can read the meaning of transcendental and understand the difference with *a priori*, which means "completely independent from experience". To Kant, transcendental is a way of knowing objects not the knowledge of objects (Cfr. *KrV*, B25). And in B80 we can read that not every a priori knowledge should be called "transcendental".

[8] Heinzmann (2001), p. 464.

[9] Helmholtz (1870), p. 24.

[10] Cahan (1993), p. 6.

[11] Heinzmann (2001), p. 466.

[12] DiSalle, in Cahan (1993), p 507.

by the laws of the movement of the bodies and by the propagation of light. That is, by Mechanics and Optics, besides Geometry.

Consequently, although Helmholtz understood geometry as an empirical science, in some sense he also admitted its status as a deductive and formal structure which is independent of its intuitive or sensorial content.

II. Poincaré's geometrical conventionalism.

Since the 1887 paper "Sur les hypothèses fondamentales de la géométrie"[13], there are several texts where Poincaré tried to develop his position towards geometry, especially in his famous book *La Science et l'hypothèse* from 1902. The essential point in this conception is the radical criticism that he makes to the Kantian idea that Euclidean axioms are synthetic a priori judgments. The refutation of this position is based on the evident fact that we can conceive other geometries, because to Poincaré if Euclid's axioms were synthetic and a priori, they would be imposed on us and we could not imagine opposite propositions to them, nor construct a theoretical building with them[14].

Now then, geometrical axioms would not be taken from experience because we can never have experience of straight lines nor of geometrical forms. As a result, he reveals his position with the assertions that geometrical axioms are conventions. However, they are not arbitrary conventions because they form a logically coherent building (this means, a non contradictory system) and with a deductive character. In this sense, geometry will not be an empirical science, because to Poincaré, every empirical science is subject to a constant revision and to him, geometry is like an exact science which does not require further revision.

Yet, when we deal with the solid bodies from our ordinary experience we are led to choose a geometry to determine the space in which we perceive that bodies. To Poincaré, it is here where we can identify the conventional but not arbitrary feature of geometrical axioms. Because we are free, as in every conventional choice, to choose among different geometries to describe our perceptual space. Nevertheless, we are guided by relevant empirical considerations to decide between one or another geometry. The role of experience is to show if the chosen conventions are useful when we link them to the principles of mechanics. That is, if they are operating to our physic-mathematical theories. As a result it has no sense to ask

[13] Published in *Bulletin de la Société Mathématique de France*, t. 15, pp. 203-216.
[14] Cfr. Poincaré (1902), p. 74.

about the truth or falseness of geometrical axioms[15] because experience cannot confirm nor refute geometry.

According to Poincaré, the ordinary experience of our body[16] as the first measurement instrument in perceptual space provides us with the general concept of mathematical group. This concept has the status of a rule to conceptualize geometry and it completely is an a priori element, in the extent that Poincaré considers that it preexists in our mind.

So, experience plays a double role regarding geometry. On one side, experience gives us the occasion[17] to create geometry by providing us with the empirical elements (solid bodies). Later on we will make an abstraction of those empirical elements in order to construct geometry as an exact science. On the other side, experience shows us its applicability by combining it with mechanical principles when we decide which is the most suitable to the space of physical bodies. As geometry is constituted as the study of a group, we will choose the most convenient group to adapt it to the world of experience. So, experience guides us in a choice that it does not impose on us. Experience acts by helping us to determine which the most convenient geometry of space is, but never which the true one is.

Then, although Poincaré, in some sense, moves away Kantian philosophy by stating the conventional status of geometrical axioms, he still holds some a priori elements, just like Helmholtz.

III. Apriorism and Experience in Helmholtz and Poincaré.

The case of these two thinkers is a clear token of how a scientific discipline, such as geometry, constitutes itself as an object of reflection; and what makes these philosophical positions more remarkable is the fact that the constitution of geometry as an object of thinking is previous to the constitution of philosophy of science itself as an academic discipline. As shown, the positive reception of non-Euclidean geometries forced to think again the a priori principles at the post-Kantian age. However, we can see that those a priori principles were not fully erased in none of the conceptions that we are discussing.

The original motivation which caused the reflection about geometry in Helmholtz and Poincaré is completely different. Helmholtz started from the specific problem of spatial perception and that led him to the mathematical study of geometrical axioms foundation. There is only one step further since this study to the question about the true geometry of

[15] Cfr. Poincaré (1902), p. 76.
[16] Heinzmann (2001), p. 460.
[17] Heinzmann (2001), p. 458.

space. On the contrary, Poincaré's starting point, as a mathematician, is the logical foundations of geometrical axioms and the enquiry for the fundamental hypothesis in order to discover thus their applicability to experience.

With his physiological perspective, Helmholtz excluded from his conception of physical space the mathematical theory of space. He considers this theory completely a priori. Then Helmholtz decided to focus on the empirical elements of space giving place to the construction of a physical space deeply linked to mechanical determinations and which nevertheless holds an a priori element which is space itself.

Poincaré, with his mathematical insight, cannot exclude the mathematical theory of space and also he erases every aprioristic touch in the axioms. Thanks to the conventional position, the question about the truth of geometry is barred. But the relationship with the phenomenical world is not expelled, and in this relationship truth is substituted by usefulness, this being defined in terms of simplicity and convenience. The a priori element which still remains is the concept of group, which is not an a priori form of sensation but and a priori form of understanding[18].

Both, Helmholtz and Poincaré start from the concept of rigid body and the free mobility axiom as essential features of their respective geometrical views. However, the arriving point is fully dissimilar because Helmholtz is trying to construct a physical geometry based in facts. Whereas Poincaré is merely applying to the physical world a geometry whose fundamental hypothesis are conventions. So, to Helmholtz, by establishing the congruence of rigid bodies and their free mobility, the experiences to determine space became specified. If we link these specifications to the condition of tridimensionality and infinitude of space, we can then establish, in Helmholtz's view, that space is strictly Euclidean, but not in an a priori way. On the contrary, to Poincaré we will never be able to establish the geometry that governs space because this depends on convenience, and convenience depends on the conjunction of geometry with physical hypothesis. So, geometry and convenience can change if physical hypothesis do so.

The Kantian position towards space as an a priori given intuition is not fully rejected by Helmholtz. What he considers as completely refuted is the existence of a priori features such as Euclidean axioms. Neither is entirely erased kantism in Poincaré, because although there is no pre-existence of the notion of space, this notion arises *with* experience but not

[18] Poincaré (1902), p. 93.

from experience, that is, experience provides the mind with the sensations that cause the creation of geometry but the mind does so, with its own elements.

As a result, we obtain two different reflections about the same science prompted by the same problem (the rise of non-Euclidean geometry and the characterization of physical space). These two insights are caused by different reasons but show the way for two scientists, from different fields of science (such as physiology and mathematics), to the search of philosophical categories to characterize their thought. Nevertheless, none of these two thinkers will remain in the question of space.

IV. The extent of the conceptions: Helmholtz and Poincaré's philosophy of science.

As mentioned, Helmholtz had started from spatial sensations as the elements causing the problem of the constitution of physical space. The role of sensation goes on being completely essential in Helmholtz's theory of natural science. In Helmholtz's conception sensations will be constituted as a sign, originated by the contact of the senses with the external world, but not being a copy of the objects in the world, but only a symbol of these objects. This assertion is in the origin of a view about science which states that it is impossible to reach and know the nature of the phenomena and the objects in nature by means of a mere formal logical analysis of the content of our representations.

In the 1878 paper "The facts in perception"[19], Helmholtz summarizes what can be considered as his empirical knowledge theory[20]. There, he analyzes the signs that are interpreted by the mind, but whose organization is given because they are produce by external objects. However, to learn the meaning of these signs we need practice and experience. So, by means of practice we can understand the frequent relationships between sensations and then we can make up a representation of the legality of events in natural world[21]. These laws are objective and they reach the necessity and validity of that relationship.

As a result we can say that the study of physiology led Helmholtz to develop an interpretation about geometry as well as a whole conception of natural science with a

[19] Originally as "Die Tatsachen in der Wahrnehmung". Address given during the anniversary celebrations of the Friedrich Wilhelm University in Berlin, in 1878, reprinted in *Vorträge und Reden*, Vol. II, pp. 215-247, 387-406. Once again, we are using here the English translation by M. F. Lowe, in the Paul Hertz/Moritz Schlick centenary edition of 1921.
[20] Heinzmann (2001), p. 462.
[21] Heidelberger in Cahan (1993), p. 479.

preeminent role of experiment[22]. However he does not get rid of the conceptual elements provided by understanding.

Poincaré develops also a whole theory of natural science focused on the notion of convention. Nevertheless, in natural science conventions have a slightly different origin than in geometry, because in some part, they are provoked by the repetition of certain empirical facts. Science is constituted by what Poincaré names as “scientific facts” which are brute facts observed by experience but translated to scientific language. So, science establishes principles which are conventions and are the expression of the relationships between facts. Those relationships are considered invariant.

Relationships between facts constitute, to Poincaré the actual content of knowledge. So, the guide for action that we used to discover these relationships will come useful to predict new phenomena and to establish new relationships. Therefore scientific activity is conditioned by empirical facts, but also by some a priori regulative principles which do not belong to the transcendental subject. These principles are a product of evolution and natural selection. Thus, they are the non empirical element which participates in generating conventions.

All these views set out problems regarding the role of hypothesis in science, about the criterion to choice the facts, about the cognoscitive status of scientific theories, the notion of truth, or the frames that we impose on nature. In these problems the meaning interpretation exceeds the strictly scientific level and belongs fully to philosophy. That is, with Helmholtz and Poincaré we have reached a situation in which philosophy pervades completely physics. As a result, physics, in the actuality of its processes and in its meaning content cannot be omitted in or separated from philosophical analysis of general significations which is done starting from its formal propositions. That is the ground of the problem of interpretation in which the one-sided approach by philosophies alienated from actual science considerations makes someone forget the reality of the construction. In this context it makes sense to recover Michel Paty’s expression “physics as a philosophical practice”[23]. Therefore, scientific considerations force one to a philosophical reflection about their assumptions. This is the sense in which physics still today urges philosophy to think and this will last while scientific theories intend to have a meaning content.

[22] Heidelberger in Cahan (1993), p. 463.
[23] Paty (1993), p. 33.

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